Determinants of Nonprincipal Submatrices of Positive Semidefinite Matrices

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ABSTRACT

We compute here the maximum value of the modulus of the determinant of an $m \times m$ nonprincipal submatrix of an $n \times n$ positive semidefinite matrix A, in terms of m, the eigenvalues of A, and cardinality k of the set of common row and column indices of this submatrix.

1. INTRODUCTION

The purpose of this paper is to find out how large a determinant of an $m \times m$ nonprincipal submatrix of an $n \times n$ positive semidefinite matrix A can get in terms of m, the eigenvalues of A, and the cardinality k of the set of common row and column indices of this submatrix. Clearly k gives some measure of how far the nonprincipal submatrix is from a principal submatrix.

We assume throughout that A is an $n \times n$ self-adjoint, hermitian or real symmetric matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. For any $1 \leqslant m \leqslant n-1$, denote by $Q_{m,n}$ the set of all sequences of length m with strictly increasing integer components taken from $\{1,2,\dots,n\}$. Given $\alpha, \beta \in Q_{m,n}$, we write $|\alpha \cap \beta| = k$ if $|\{\alpha(1), \alpha(2),\dots,\alpha(m)\} \cap \{\beta(1), \beta(2),\dots,\beta(m)\}| = k$.

If A is hermitian we define, following Marcus and Moore [3],

$$\rho_{k,m,\mathbf{C}}(A) = \max_{U \in U_n} \max_{\substack{\alpha,\beta \in Q_{m,n} \\ |\alpha\cap\beta| = k}} |\det U^*AU[\alpha|\beta]|.$$

Here, as usual, $A[\alpha|\beta]$ denotes the submatrix of A based on row indices in α and column indices in β; U_n denotes the group of $n \times n$ unitary matrices.

If A is real symmetric, we may define similarly

$$\rho_{k,m,\mathbf{R}}(A) = \max_{P \in \mathrm{O}_n} \max_{\substack{\alpha,\beta \in Q_{m,n} \\ |\alpha\cap\beta| = k}} |\det P^tAP[\alpha|\beta]|,$$

where O_n denotes the group of $n \times n$ real orthogonal matrices.

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Marcus and Moore [3] conjectured that $\rho_{k,m}$ is a monotone nondecreasing function of k. This conjecture was proved in [2]. The present paper may be viewed as a continuation of [2]. Here we compute the $\rho_{k,m}$ explicitly. Note (cf. [2]) that in the definition of $\rho_{k,m}$ one may choose a fixed pair $\alpha, \beta \in Q_{m,n}$; for example one may choose the sequences $\alpha_0 = (1,2,\dots,k,k+1,\dots,m)$ and $\beta_0 = (1,2,\dots,k,m+1,\dots,2m-k)$.

In Section 2 we give some preliminary results. In Section 3 we consider the special case $k=0$, namely the case where the sets of row indices and column indices of the nonprincipal submatrix are disjoint. Finally in Section 4 we consider the case $0<k<m$.

2. PRELIMINARY RESULTS

In this section we prove some results that are required in subsequent sections for the computation of the $\rho_{k,m}$. We denote by S_r the symmetric group on r letters.

LEMMA 1. *Suppose that m is a positive integer and $\lambda_1 > \lambda_2 > \cdots > \lambda_m > \lambda_{m+1} > \cdots > \lambda_{2m} > 0$. Then, for any $\sigma \in S_{2m}$, we have*

$$\prod_{i=1}^{m} (\lambda_i - \lambda_{i+m}) \geqslant \left| \prod_{i=1}^{m} (\lambda_{\sigma(2i-1)} - \lambda_{\sigma(2i)}) \right| \tag{1}$$

Proof. It suffices to consider $\sigma \in S_{2m}$ such that $\sigma(2i-1) \leqslant \sigma(2i)$ for $i=1,2,\dots,m$, and then the absolute value is not needed on the right hand side of (1). Moreover, it is clear that we may consider only $\sigma \in S_{2m}$ such that $\sigma(2i-1)=i$, $i=1,2,\dots,m$. The proof is by induction on m. For $m=1$ there is nothing to prove. We also need the case $m=2$ for the induction step, so assume now $m=2$. We only have to show that $(\lambda_1-\lambda_3)(\lambda_2-\lambda_4) \geqslant (\lambda_1 - \lambda_4)(\lambda_2-\lambda_3)$. This is true because $(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)-(\lambda_1-\lambda_4)(\lambda_2-\lambda_3) = (\lambda_1-\lambda_2)(\lambda_3-\lambda_4) > 0$.

Now suppose $m>2$ and that the result is true for $m-1$. Let $\sigma \in S_{2m}$ and such that $\sigma(2i-1)=i$, $i=1,2,\dots,m$. Suppose first that $\sigma(2)=m+1$. Then, by the induction hypothesis, $\prod_{i=2}^{m}(\lambda_i - \lambda_{i+m}) \geqslant \prod_{i=2}^{m}(\lambda_{\sigma(2i-1)} - \lambda_{\sigma(2i)})$, so (1) holds. Hence it remains to consider the case $\sigma(2) \neq m+1$. Let $j=\sigma(2)$, so $j \geqslant m+2$, by assumption on σ. Then there exists a positive integer i_0, $2 \leqslant i_0 \leqslant m$, such that $\sigma(2i_0)=m+1$. Let $l=\sigma(2i_0-1)$, so we have, by assumption on σ, $1<l \leqslant m$. By the case of $m=2$, we have $(\lambda_1-\lambda_j)(\lambda_l - \lambda_{m+1}) \leqslant (\lambda_1 - \lambda_{m+1})(\lambda_l - \lambda_j)$. Define now $\pi \in S_{2m}$ by $\pi(2)=m+1$, $\pi(2i_0)$

$= j$, and $\pi(i) = \sigma(i)$ for every i such that $i \neq 2$ and $i \neq 2i_0$. We now have, by the first case,

$$\prod_{i=1}^{m} (\lambda_i - \lambda_{i+m}) \geqslant \prod_{i=1}^{m} (\lambda_{\pi(2i-1)} - \lambda_{\pi(2i)}) \geqslant \prod_{i=1}^{m} (\lambda_{\sigma(2i-1)} - \lambda_{\sigma(2i)}) \qquad \blacksquare$$

The next lemma is a special case of a more general result due to Mirsky [4].

LEMMA 2. *Suppose S is a 2×2 real symmetric matrix with eigenvalues $\lambda_1 \geqslant \lambda_2$. Then*

$$\max_{P \in O_2} |(P^t SP)_{12}| = \frac{\lambda_1 - \lambda_2}{2}.$$

REMARK. A similar result holds if S is replaced by a 2×2 hermitian matrix and P by an arbitrary 2×2 unitary matrix.

As a consequence of Lemma 2, we get the following result.

LEMMA 3. *Let k, m, n be nonnegative integers, and suppose that $0 \leqslant k < m$ and $2m - k \leqslant n$. Let $l = m - k$. Suppose that $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. Then there exists a real symmetric matrix E which is orthogonally similar to Λ and such that*

$$|\det E[(1,3,\ldots,2l-1,2l+1,2l+2,\ldots,2l+k)|$$
$$(2,4,\ldots,2l,2l+1,2l+2,\ldots,2l+k)]|$$
$$= \frac{1}{2^l} \prod_{i=1}^{l} (\lambda_i - \lambda_{n+i-l}) \prod_{j=1}^{k} \lambda_{l+j},$$

where the last product on the right hand side is understood to be 1 if $k = 0$.

Proof. The diagonal matrix

$$D = \operatorname{diag}(\lambda_1, \lambda_{n+1-l}, \lambda_2, \lambda_{n+2-l}, \ldots, \lambda_l, \lambda_n, \lambda_{l+1}, \ldots, \lambda_{n-l})$$

is orthogonally similar to Λ, and we look on it as the direct sum $D = D_1 \oplus D_2$

$\oplus \cdots \oplus D_l \oplus \mathrm{diag}(\lambda_{l+1},\dots,\lambda_{n-l})$, where $D_i = \mathrm{diag}(\lambda_i, \lambda_{n+i-1})$, $i=1,2,\dots,l$. By Lemma 2, D_i is orthogonally similar to a matrix E_i such that $|(E_i)_{12}| = \frac{1}{2}(\lambda_i - \lambda_{n+i-l})$, $i=1,2,\dots,l$. Let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_l \oplus \mathrm{diag}(\lambda_{l+1},\dots,\lambda_{n-l})$. Then E is orthogonally similar to Λ and satisfies the required result. ■

3. COMPUTATION OF $\rho_{0,m,\mathbf{C}}(A)$ AND $\rho_{0,m,\mathbf{R}}(A)$

We first compute these quantities in case $n = 2m$. We want to show that if A is a $2m \times 2m$ self-adjoint matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{2m} \geqslant 0$, then

$$\rho_{0,m,\mathbf{C}}(A) = \frac{1}{2^m} \prod_{i=1}^{m} (\lambda_i - \lambda_{i+m}) \tag{2}$$

in case A is hermitian, while

$$\rho_{0,m,\mathbf{R}}(A) = \frac{1}{2^m} \prod_{i=1}^{m} (\lambda_i - \lambda_{i+m}) \tag{3}$$

in case A is real symmetric. In particular, if A is real symmetric, the same maximum value of the modulus of the determinant of an $m \times m$ submatrix (with no common row and column indices) is attained if we allow unitary similarities or only real orthogonal similarities. The same phenomenon will occur in the general case. Note also that by Lemma 3 it suffices to show that the right hand sides of (2), (3) provide upper bounds for $\rho_{0,m,\mathbf{C}}(A)$, $\rho_{0,m,\mathbf{R}}(A)$, respectively.

The next lemma deals with the case $m = 2$, $n = 2m = 4$.

LEMMA 4. *Suppose A is a 4×4 hermitian matrix, or real symmetric matrix, with eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$. Suppose B is a 4×4 hermitian matrix (real symmetric if A is) which is unitarily similar to A (orthogonally similar if A is real symmetric) and such that $|\det B[(12)|(34)]| = \rho_{0,2,\mathbf{C}}(A)$ or $|\det B[(12)|(34)]| = \rho_{0,2,\mathbf{R}}(A)$ in the complex or real case, respectively. Suppose moreover that B looks like*

$$B = \begin{bmatrix} b_{11} & b_{12} & \mu_1 & 0 \\ \bar{b}_{12} & b_{22} & 0 & \mu_2 \\ \mu_1 & 0 & b_{33} & b_{34} \\ 0 & \mu_2 & \bar{b}_{34} & b_{44} \end{bmatrix},$$

where μ_1, μ_2 *are nonnegative real numbers. Then*

(a) *In the real symmetric case*

$$b_{12}\mu_2 = b_{34}\mu_1 \tag{4}$$

and

$$b_{12}\mu_1 = b_{34}\mu_2. \tag{5}$$

In particular, $b_{12} = 0$ *if and only if* $b_{34} = 0$; *and if* $\mu_1 \neq \mu_2$ *then* $b_{12} = b_{34} = 0$.

(b) *In the complex hermitian case we have*

$$\mu_2 \operatorname{Re} b_{12} = \mu_1 \operatorname{Re} b_{34} \tag{6}$$

and

$$\mu_1 \operatorname{Re} b_{12} = \mu_2 \operatorname{Re} b_{34}. \tag{7}$$

Moreover, $b_{12} = 0$ *if and only if* $b_{34} = 0$; *and if* $\mu_1 \neq \mu_2$ *then* $b_{12} = b_{34} = 0$.

Proof. The proof will start in a unified way for both cases and then split into the two. It is clear from Lemma 3 and the assumptions on A and B that

$$\mu_1 > 0 \quad \text{and} \quad \mu_2 > 0.$$

Let θ be a real number, and let $c = \cos\theta$, $s = \sin\theta$. Define a 4×4 real orthogonal matrix P by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that

$$PB = \begin{bmatrix} b_{11} & b_{12} & \mu_1 & 0 \\ c\bar{b}_{12} - s\mu_1 & cb_{22} & -sb_{33} & c\mu_2 - sb_{34} \\ s\bar{b}_{12} + c\mu_1 & sb_{22} & cb_{33} & s\mu_2 + cb_{34} \\ 0 & \mu_2 & \bar{b}_{34} & b_{44} \end{bmatrix}$$

and

$$C = PBP^t = \begin{bmatrix} b_{11} & cb_{12} - s\mu_1 & sb_{12} + c\mu_1 & 0 \\ c\bar{b}_{12} - s\mu_1 & c^2 b_{22} + s^2 b_{33} & csb_{22} - csb_{33} & c\mu_2 - sb_{34} \\ s\bar{b}_{12} + c\mu_1 & csb_{22} - csb_{33} & s^2 b_{22} + c^2 b_{33} & s\mu_2 + cb_{34} \\ 0 & c\mu_2 - s\bar{b}_{34} & s\mu_2 + c\bar{b}_{34} & b_{44} \end{bmatrix}. \tag{8}$$

We now have two cases:

Case 1 (real symmetric). In this case B is a real matrix, so $\bar{b}_{12} = b_{12}$ and $\bar{b}_{34} = b_{34}$. The matrix C is orthogonally similar to A, and

$$\begin{aligned} \det C[(12)|(34)] - \rho_{0,2,\mathbf{R}}(A) &= \det C[(12)|(34)] - \mu_1\mu_2 \\ &= (sb_{12} + c\mu_1)(c\mu_2 - sb_{34}) - \mu_1\mu_2 \\ &= s\left[c(b_{12}\mu_2 - b_{34}\mu_1) - sb_{12}b_{34} - s\mu_1\mu_2\right]. \end{aligned}$$

Since θ is arbitrary, it is clear that we can choose θ close enough to 0 or π so that

$$\det C[(12)|(34)] > \rho_{0,2,\mathbf{R}}(A)$$

unless $b_{12}\mu_2 = b_{34}\mu_1$. Hence we must have

$$b_{12}\mu_2 = b_{34}\mu_1. \tag{4}$$

We now interchange the first and second rows of B and the first and second columns of B, and do likewise with third and fourth rows and columns of B. All this amounts to in B is switching the μ_1 and μ_2 in their places. Repeating the perturbation argument leading to (4), we get now in a similar way

$$b_{12}\mu_1 = b_{34}\mu_2. \tag{5}$$

The remaining claims of part (a) follow now immediately from (4) and (5) and the fact that μ_1 and μ_2 are positive.

Case 2 (*complex hermitian*). In the matrix C, defined by (8), evaluate now $\mathrm{Re}\{\det C[(12)|(34)] - \rho_{0,2,\mathbb{C}}(A)\}$. We get

$$\begin{aligned}\mathrm{Re}\{(sb_{12} + c\mu_1)(c\mu_2 - sb_{34}) - \mu_1\mu_2\}\\ = s\left[c(\mu_2 \mathrm{Re}\, b_{12} - \mu_1 \mathrm{Re}\, b_{34}) - s\,\mathrm{Re}(b_{12}b_{34}) - s\mu_1\mu_2\right].\end{aligned}$$

It is clear again that we can choose θ close enough to 0 or π so that $\mathrm{Re}\det C[(12)|(34)] > \rho_{0,2,\mathbb{C}}(A)$ unless $\mu_2 \mathrm{Re}\, b_{12} = \mu_1 \mathrm{Re}\, b_{34}$. Hence we must have

$$\mu_2 \mathrm{Re}\, b_{12} = \mu_1 \mathrm{Re}\, b_{34}. \tag{6}$$

We now interchange the first and second rows of B and the first and second columns of B, and do likewise with third and fourth rows and columns of B. This amounts to switching the μ_1 and μ_2 in their places and replacing b_{12}, b_{34} by $\bar{b}_{12}, \bar{b}_{34}$, respectively. Repeating the perturbation argument that leads to (6) yields now

$$\mu_1 \mathrm{Re}\, b_{12} = \mu_2 \mathrm{Re}\, b_{34}. \tag{7}$$

Since $\mu_1 > 0$ and $\mu_2 > 0$, (6) and (7) imply now that $\mathrm{Re}\, b_{12} = 0$ if and only if $\mathrm{Re}\, b_{34} = 0$. Also, if $\mu_1 \neq \mu_2$ we must have $\mathrm{Re}\, b_{12} = \mathrm{Re} b_{34} = 0$.

Suppose now that $b_{12} = 0$. Then $\mathrm{Re}\, b_{34} = 0$, so $b_{34} = iy_{34}$ for some $y_{34} \in \mathbb{R}$. The matrix $\mathrm{diag}(1, i, 1, i)B\,\mathrm{diag}(1, -i, 1, -i)$ is exactly

$$\begin{bmatrix} b_{11} & 0 & \mu_1 & 0 \\ 0 & b_{22} & 0 & \mu_2 \\ \mu_1 & 0 & b_{33} & y_{34} \\ 0 & \mu_2 & y_{34} & b_{44} \end{bmatrix},$$

and the perturbation arguments applied to B can also be applied here, yielding $y_{34} = 0$. Hence $b_{12} = 0$ implies $b_{34} = 0$. Similarly, $b_{34} = 0$ implies $b_{12} = 0$.

Finally, we have to show that $\mu_1 \neq \mu_2$ implies that $b_{12} = b_{34} = 0$. We have shown that it implies $\mathrm{Re}\, b_{12} = \mathrm{Re}\, b_{34} = 0$, so let $b_{12} = iy_{12}$, $b_{34} = iy_{34}$, for some $y_{12}, y_{34} \in \mathbb{R}$. Then $\mathrm{diag}(1, i, 1, i)B\,\mathrm{diag}(1, -i, 1, -i)$ is the matrix

$$\begin{bmatrix} b_{11} & y_{12} & \mu_1 & 0 \\ y_{12} & b_{22} & 0 & \mu_2 \\ \mu_1 & 0 & b_{33} & y_{34} \\ 0 & \mu_2 & y_{34} & b_{44} \end{bmatrix},$$

and the equations, corresponding to (6) and (7), that must be satisfied by this matrix are $\mu_2 y_{12} = \mu_1 y_{34}$ and $\mu_1 y_{12} = \mu_2 y_{34}$. Since $\mu_1 \neq \mu_2$, we conclude that $y_{12} = y_{34} = 0$. ■

Lemma 5. *Suppose that A is a real symmetric 4×4 matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$. Then*

$$\rho_{0,2,\mathbf{R}}(A) = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{4}.$$

Proof. We may assume that $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, for the general case will follow easily by the standard continuity argument. It also suffices to prove

$$\rho_{0,2,\mathbf{R}}(A) \leqslant \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{4},$$

because of Lemma 3.

There exists a real symmetric 4×4 matrix S which is orthogonally similar to A and such that

$$|\det S[(12)|(34)]| = \rho_{0,2,\mathbf{R}}(A).$$

By the singular value decomposition, there exist real orthogonal 2×2 matrices P_1, P_2 such that

$$P_1 S[(12)|(34)] P_2^t = \operatorname{diag}(\mu_1, \mu_2),$$

where $\mu_1 > 0$, $\mu_2 > 0$, and $\mu_1 \mu_2 = \rho_{0,2,\mathbf{R}}(A)$. Let $P = P_1 \oplus P_2$ and $B = PSP^t$. The matrix B satisfies the conditions of Lemma 4, and it is easy to check that $B[(12)|(34)] = \operatorname{diag}(\mu_1, \mu_2)$. We now have two cases.

Case 1. Suppose that $\mu_1 \neq \mu_2$. Lemma 4 implies that $b_{12} = b_{34} = 0$, so

$$B = \begin{bmatrix} b_{11} & 0 & \mu_1 & 0 \\ 0 & b_{22} & 0 & \mu_2 \\ \mu_1 & 0 & b_{33} & 0 \\ 0 & \mu_2 & 0 & b_{44} \end{bmatrix},$$

and it is the direct sum of

$$\begin{bmatrix} b_{11} & \mu_1 \\ \mu_1 & b_{33} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_{22} & \mu_2 \\ \mu_2 & b_{44} \end{bmatrix}.$$

Denote the eigenvalues of the first matrix by α_1, α_2 and of the second matrix by α_3, α_4. Then there exists a permutation $\pi \in S_4$ such that $\alpha_i = \lambda_{\pi(i)}$, $i = 1,2,3,4$. The result now follows immediately from Lemma 1, Lemma 2 and the fact that $\mu_1\mu_2 = \rho_{0,2,\mathbf{R}}(A)$.

Case 2. It remains to consider the case $\mu_1 = \mu_2$. There exists a real orthogonal 2×2 matrix V_1 such that

$$V_1 \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} V_1^t$$

is a diagonal matrix. Define $V = V_1 \oplus V_1$ and $C = VBV^t$. It is easy to see that C looks like

$$C = \begin{bmatrix} c_{11} & 0 & \mu_1 & 0 \\ 0 & c_{22} & 0 & \mu_1 \\ \mu_1 & 0 & c_{33} & c_{34} \\ 0 & \mu_1 & c_{34} & c_{44} \end{bmatrix}.$$

The matrix C satisfies the conditions of Lemma 4, so we must have $c_{34} = 0$. The proof proceeds now as in case 1. ■

LEMMA 6. *Suppose that A is a 4×4 hermitian matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$. Then*

$$\rho_{0,2,\mathbf{C}}(A) = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{4}.$$

Proof. The proof is completely analogous to the proof of Lemma 5. ■

THEOREM 1. *Let m be a positive integer. Suppose A is a real symmetric $2m \times 2m$ matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{2m} \geqslant 0$. Then*

$$\rho_{0,m,\mathbf{R}}(A) = \frac{1}{2^m} \prod_{i=1}^{m} (\lambda_i - \lambda_{i+m}).$$

Proof. The result is true for $m=1$ and $m=2$ by Lemma 2 and Lemma 5. We may also assume $\lambda_1 > \lambda_2 > \cdots > \lambda_{2m} > 0$, for the general case follows by a standard continuity argument. Also, by Lemma 3, it suffices to prove

$$\rho_{0,m,\mathbf{R}}(A) \leqslant \frac{1}{2^m}\prod_{i=1}^{m}(\lambda_i - \lambda_{i+m}).$$

By an argument similar to the one given at the beginning of the proof of Lemma 5, there exists a $2m \times 2m$ real symmetric matrix B which is orthogonally similar to A and satisfies

$$B\big[(1,2,\ldots,m)|(m+1,m+2,\ldots,2m)\big] = \operatorname{diag}(\mu_1,\mu_2,\ldots,\mu_m)$$

with

$$\mu_i > 0, \quad i = 1,2,\ldots,m, \quad \text{and} \quad \prod_{i=1}^{m}\mu_i = \rho_{0,m,\mathbf{R}}(A).$$

Suppose that the number of distinct μ's is k. We may further assume that $\{1,2,\ldots,m\}$ is partitioned into k nonempty subsets $I_1 = \{1,2,\ldots,r_1\}$, $I_2 = \{r_1 + 1,\ldots,r_1+r_2\},\ldots,$ $I_k = \{r_1+r_2+\cdots+r_{k-1}+1,\ldots,m\}$ such that if i and j are any elements of $\{1,2,\ldots,m\}$, then $\mu_i = \mu_j$ if and only if there exists s, $1 \leqslant s \leqslant k$, such that I_s contains i and j.

We now form the following partition of B into blocks:

$$B = \begin{array}{c}
\begin{array}{cccccccc} r_1 & r_2 & & & r_1 & r_2 & & r_k \end{array} \\
\left[\begin{array}{cccccccc}
B_{11} & B_{12} & \cdots & B_{1k} & \mu_1 I_{r_1} & 0 & \cdots & 0 \\
B_{21} & B_{22} & \cdots & B_{2k} & 0 & \mu_{r_1+1} I_{r_2} & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
B_{k1} & B_{k2} & \cdots & B_{kk} & 0 & 0 & \cdots & \mu_m I_{r_k} \\
\mu_1 I_{r_1} & 0 & \cdots & 0 & B_{k+1,k+1} & B_{k+1,k+2} & \cdots & B_{k+1,2k} \\
0 & \mu_{r_1+1} I_{r_2} & \cdots & 0 & B_{k+2,k+1} & B_{k+2,k+2} & \cdots & B_{k+2,2k} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \mu_m I_{r_k} & B_{2k,k+1} & B_{2k,k+2} & \cdots & B_{2k,2k}
\end{array}\right]
\begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_k \\ r_1 \\ r_2 \\ \vdots \\ r_k \end{array}
\end{array}$$

For any l, $1 \leqslant l \leqslant k$, there exists an $r_l \times r_l$ orthogonal matrix P_l such that $P_l B_{ll} P_l^t$ is a diagonal matrix. Let P be the $2m \times 2m$ real orthogonal matrix defined by

$$P = P_1 \oplus P_2 \oplus \cdots \oplus P_k \oplus P_1 \oplus P_2 \oplus \cdots \oplus P_k,$$

and let $E = PBP^t$. If we partition E conformably with B we get

$$E = \begin{array}{cc} \begin{matrix} r_1 & r_2 & \cdots & r_k & r_1 & r_2 & \cdots & r_k \end{matrix} & \\ \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1k} & \mu_1 I_{r_1} & 0 & \cdots & 0 \\ E_{21} & E_{22} & \cdots & E_{2k} & 0 & \mu_{r_1+1} I_{r_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ E_{k1} & E_{k2} & \cdots & E_{kk} & 0 & 0 & \cdots & \mu_m I_{r_k} \\ \mu_1 I_{r_1} & 0 & \cdots & 0 & E_{k+1,k+1} & E_{k+1,k+2} & \cdots & E_{k+1,2k} \\ 0 & \mu_{r_1+1} I_{r_2} & \cdots & 0 & E_{k+2,k+1} & E_{k+2,k+2} & \cdots & E_{k+2,2k} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mu_m I_{r_k} & E_{2k,k+1} & E_{2k,k+2} & \cdots & E_{2k,2k} \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_k \\ r_1 \\ r_2 \\ \vdots \\ r_k \end{matrix} \end{array}$$

where $E_{11}, E_{22}, \ldots, E_{kk}$ are diagonal matrices. Let

$$E_{12} = E\big[\,(1,2,\ldots,m)|(m+1, m+2,\ldots,2m)\big]$$

and

$$E_{21} = E\big[\,(m+1, m+2,\ldots,2m)|(1,2,\ldots,m)\big].$$

Then $E_{12} = E_{21}$ is a diagonal matrix.

We want to show now that the only nonzero entries of E off the main diagonal are the main diagonal elements of E_{12} and E_{21}. For that purpose pick any two indices i and j such that $1 \leqslant i < j \leqslant m$. Let F be the 4×4 principal submatrix of E based on indices $i, j, i+m, j+m$. Then F looks like

$$F = \begin{bmatrix} e_{ii} & e_{ij} & \mu_i & 0 \\ e_{ij} & e_{jj} & 0 & \mu_j \\ \mu_i & 0 & e_{i+m,i+m} & e_{i+m,j+m} \\ 0 & \mu_j & e_{i+m,j+m} & e_{j+m,j+m} \end{bmatrix}$$

We note also that

$$e_{ij}=0 \qquad \text{if} \quad \mu_i=\mu_j, \tag{9}$$

because $E_{11}, E_{22},\dots,E_{kk}$ are diagonal matrices.

Let θ be an arbitrary real number; let $c=\cos\theta$ and $s=\sin\theta$. Define now a $2m\times 2m$ real orthogonal matrix Q as follows: its rows are the rows of the identity matrix I_{2m}, except row j, which has c in the jth place and $-s$ in the $(i+m)$th place, and row $i+m$, which has s in the jth place and c in the $(i+m)$th place. Let $G=QEQ^t$, and denote $G_{12}=G[(1,2,\dots,m)|(m+1,m+2,\dots,2m)]$. Because E_{12} is a diagonal matrix and because of the way Q is defined, the only places where nonzero off diagonal elements of G_{12} can possibly occur are in row j and in column i. Hence, because $m\geqslant 3$, $\det G_{12}$ is still the product of its entries on the main diagonal. But the elements on the main diagonal of G_{12} coincide with those of E_{12} (in the same positions, of course), except the jth and ith main diagonal entries. The jth and ith entries on the main diagonal of G_{12} are exactly the elements in (1,3) and (2,4) position, respectively, in the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & c & -s & 0\\ 0 & s & c & 0\\ 0 & 0 & 0 & 1\end{bmatrix} F \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & c & s & 0\\ 0 & -s & c & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Now, exactly as in the proof of Lemma 4 (the real case), this fact and the fact that $\det G_{12}\leqslant \rho_{0,m,\mathbf{R}}(A)$ for any real θ imply

$$e_{ij}\mu_j=e_{i+m,\,j+m}\mu_i.$$

Similarly (interchange rows i and j of E, and columns i and j of E; also interchange rows $i+m$ and $j+m$ of E, and columns $i+m$ and $j+m$ of E), one gets

$$e_{ij}\mu_i=e_{i+m,\,j+m}\mu_j.$$

The conclusions are now exactly as in Lemma 4; namely, if $\mu_i\neq\mu_j$ then $e_{ij}=e_{i+m,\,j+m}=0$. But the same result holds true also if $\mu_i=\mu_j$, for then we have in addition (9), which implies now $e_{i+m,\,j+m}=0$.

Since i and j are arbitrary, we have shown that the only nonzero entries of E outside the main diagonal are the main diagonal entries of E_{12} and E_{21}. Hence E is the direct sum of the m 2×2 matrices

$$\begin{bmatrix} e_{11} & \mu_1\\ \mu_1 & e_{m+1,\,m+1}\end{bmatrix},\quad \begin{bmatrix} e_{22} & \mu_2\\ \mu_2 & e_{m+2,\,m+2}\end{bmatrix},\dots,\quad \begin{bmatrix} e_{m,m} & \mu_m\\ \mu_m & e_{2m,2m}\end{bmatrix}.$$

Denote the eigenvalues of the first, second,…,mth matrix by α_1, α_2; $\alpha_3, \alpha_4; \dots; \alpha_{2m-1}, \alpha_{2m}$ respectively. Then there exists a permutation $\pi \in S_{2m}$ such that $\alpha_i = \lambda_{\pi(i)}$. The result now follows immediately from Lemma 1, Lemma 2, and the fact that $\prod_{i=1}^{m} \mu_i = \rho_{0,m,\mathbf{R}}(A)$. ∎

THEOREM 2. *Let m be a positive integer. Suppose that A is a $2m \times 2m$ hermitian matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{2m} \geqslant 0$. Then*

$$\rho_{0,m,\mathbf{C}}(A) = \frac{1}{2^m} \prod_{i=1}^{m} (\lambda_i - \lambda_{i+m}).$$

Proof. The proof is analogous to the proof of Theorem 1, except that the hermitian part of the proof of Lemma 4 is used here. ∎

We finish this section by computing $\rho_{0,m,\mathbf{R}}(A)$ and $\rho_{0,m,\mathbf{C}}(A)$ for an $n \times n$ matrix A, where, of course, $n \geqslant 2m$.

THEOREM 3. *Suppose m and n are positive integers, and $n \geqslant 2m$. Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. Then*

$$\rho_{0,m,\mathbf{R}}(A) = \frac{1}{2^m} \prod_{i=1}^{m} (\lambda_i - \lambda_{n+i-m}). \tag{10}$$

Proof. By Lemma 3, it suffices to show that the right hand side of (10) is an upper bound for $\rho_{0,m,\mathbf{R}}(A)$. So suppose

$$\rho_{0,m,\mathbf{R}}(A) = \left|\det B\left[(1,2,\dots,m)|(m+1,m+2,\dots,2m)\right]\right|,$$

where B is orthogonally similar to A. Let C be the $2m \times 2m$ principal submatrix of B based on indices $1,2,\dots,2m$.

We claim that $\rho_{0,m,\mathbf{R}}(C) = \rho_{0,m,\mathbf{R}}(A)$. Indeed, we just showed that $\rho_{0,m,\mathbf{R}}(C) \geqslant \rho_{0,m,\mathbf{R}}(A)$. Now, let P be any $2m \times 2m$ real orthogonal matrix. Define $Q = P \oplus I_{n-2m}$. It is clear that $PCP^t[(1,2,\dots,m)|(m+1,m+2,\dots,2m)] = QBQ^t[(1,2,\dots,m)|(m+1,m+2,\dots,2m)]$. Hence $\rho_{0,m,\mathbf{R}}(C) \leqslant \rho_{0,m,\mathbf{R}}(A)$, and we can conclude that $\rho_{0,m,\mathbf{R}}(A) = \rho_{0,m,\mathbf{R}}(C)$. Denote the eigenvalues of C by $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{2m} \geqslant 0$. Since C is a $2m \times 2m$ positive semidefinite matrix, Theorem 1 implies now

$$\rho_{0,m,\mathbf{R}}(C) = \frac{1}{2^m} \prod_{i=1}^{m} (\alpha_i - \alpha_{i+m}).$$

The interlacing inequalities for C and A imply that

$$\lambda_i \geqslant \alpha_i \geqslant \lambda_{i+n-2m}, \qquad i=1,2,\dots,2m.$$

Hence,

$$\rho_{0,m,\mathbf{R}}(A)=\rho_{0,m,\mathbf{R}}(C)\leqslant \frac{1}{2^m}\prod_{i=1}^{m}(\lambda_i-\lambda_{n+i-m}). \qquad \blacksquare$$

THEOREM 4. *Suppose m and n are positive integers and $n \geqslant 2m$. Let A be an $n\times n$ hermitian matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$; then*

$$\rho_{0,m,\mathbf{C}}(A)=\frac{1}{2^m}\prod_{i=1}^{m}(\lambda_i-\lambda_{n+i-m}).$$

Proof. Exactly like the proof of Theorem 3. $\blacksquare$

4. COMPUTATION OF $\rho_{k,m,\mathbf{R}}(A)$ AND $\rho_{k,m,\mathbf{C}}(A)$ FOR $0<k<m$

We now turn to the computation of $\rho_{k,m,\mathbf{R}}(A)$ (in the real symmetric case) and $\rho_{k,m,\mathbf{C}}(A)$ (in the hermitian case) for $0<k<m$. It is quite remarkable that this computation depends at a crucial point on the case $k=0$. This point was observed in [2], and will be discussed in the proof.

THEOREM 5. *Let k, m, n be positive integers, and suppose that $0<k<m$ and $2m-k\leqslant n$. Suppose A is a real $n\times n$ symmetric matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. Then*

$$\rho_{k,m,\mathbf{R}}(A)=\frac{1}{2^{m-k}}\prod_{i=1}^{m-k}\left(\lambda_i-\lambda_{n+i-(m-k)}\right)\prod_{j=1}^{k}\lambda_{m-k+j}. \tag{11}$$

Proof. We may assume that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0$, for the general case follows by the usual continuity argument. Also, by Lemma 3, it suffices to show that the right hand side of (11) is an upper bound for $\rho_{k,m,\mathbf{R}}(A)$. We use here the following observation, which is proved as part of the proof of Theorem 2 in [2] (note that we assumed there that $k\leqslant m-2$, but the following observation is true also for $k=m-1$; also, the hermitian case is discussed there, but the real symmetric analogue holds as well): Suppose that

$$\rho_{k,m,\mathbf{R}}(A)=\left|\det B\left[(1,2,\dots,m)\,|\,(1,2,\dots,k,m+1,\dots,2m-k)\right]\right|,$$

where B is a matrix which is orthogonally similar to A. Let C be the $(2m-k)\times(2m-k)$ principal submatrix of B based on the indices $1,2,\dots,m, m+1,\dots,2m-k$. Then

$$\rho_{0,m-k,\mathbf{R}}(C^{-1})=\frac{\rho_{k,m,\mathbf{R}}(A)}{\det C}.$$

Denote the eigenvalues of C by

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{2m-k} > 0.$$

We write $l=m-k$. It follows now from Theorem 3 (with $n=2m-k$) that

$$\rho_{0,l,\mathbf{R}}(C^{-1})=\frac{1}{2^l}\left(\frac{1}{\alpha_{2m-k}}-\frac{1}{\alpha_l}\right)\left(\frac{1}{\alpha_{2m-k-1}}-\frac{1}{\alpha_{l-1}}\right)\cdots\left(\frac{1}{\alpha_{2m-k-l+1}}-\frac{1}{\alpha_1}\right).$$

Since

$$\rho_{k,m,\mathbf{R}}(A)=\rho_{0,m-k,\mathbf{R}}(C^{-1})\det C=\rho_{0,m-k,\mathbf{R}}(C^{-1})\prod_{i=1}^{2m-k}\alpha_i,$$

we get

$$\rho_{k,m,\mathbf{R}}(A)=\frac{1}{2^l}\prod_{j=1}^{l}(\alpha_j-\alpha_{2m-k-l+j})\prod_{i=l+1}^{m}\alpha_i.$$

The interlacing inequalities for C and A imply that

$$\lambda_i \geqslant \alpha_i \geqslant \lambda_{i+n-(2m-k)}.$$

Since $l=m-k$, we get

$$\rho_{k,m,\mathbf{R}}(A)\leqslant\frac{1}{2^{m-k}}\prod_{j=1}^{l}(\lambda_j-\lambda_{n+j-(m-k)})\prod_{i=1}^{k}\lambda_{m-k+i},$$

completing the proof. ∎

THEOREM 6. *Let k, m, n be positive integers, and suppose that $0<k<m$ and $2m-k\leqslant n$. Suppose A is an $n\times n$ hermitian matrix with eigenvalues*

$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. *Then*

$$\rho_{k,m,\mathrm{C}}(A) = \frac{1}{2^{m-k}} \prod_{i=1}^{m-k} \left(\lambda_i - \lambda_{n+i-(m-k)}\right) \prod_{j=1}^{k} \lambda_{m-k+j}.$$

Proof. Same as Theorem 5. ■

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Received 27 July 1982; revised 16 September 1982