A Key to Fuzzy-Logic Inference

K. K. Thornber
NEC Research Institute, Inc., Princeton, New Jersey

ABSTRACT

Classically, whether to effect inference, one uses a small set of axioms and modus ponens, or a set of rules of inference including modus ponens, one is going beyond what can be derived with the explicit operations of logic alone. Carrying this concept over to fuzzy logic we construct a fuzzy modus ponens and other rules of inference that include modus tollens and reductio ad absurdum. These in turn are based on (and greatly facilitated by) a choice for the operation of implication that preserves the (logic) symmetry implicit in its definition. Extensions including conditional quantification, cut rules (single, multiple, and implicitory), and fuzzy mathematical induction are sketched. As an example, a fuzzy-logic treatment of the Yale shooting problem is discussed. The results suggest that the implicit processes of inference, as distinct from the explicit processes of decision (control) theory and systems theory, can be effected in fuzzy logic if, as in classical logic, one ventures outside the scope of (fuzzy) logic operations.

KEYWORDS: ***fuzzy modus ponens / tollens / reductio ad absurdum, fuzzy rules of inference / cut rules, fuzzy conditional quantification, fuzzy mathematical induction, Yale shooting problem***

1. INTRODUCTION

The objective of fuzzy-logic inference is to obtain some properties of the fuzzy sets[1] $B_1, B_2, \ldots$ from similar properties of the fuzzy sets

Address correspondence to K. K. Thornber, NEC Research, Institute, Inc., 4 Independence Way, Princeton, NJ 08540.

Received October 1, 1991; accepted May 27, 1992.

[1]Notation here is a bit problematic. Because membership functions are used so extensively and their combinations are so closely tied to the corresponding logic operations, Goguen's [2] definition of a fuzzy set as a function $A{:}X \rightarrow [0,1]$ was utilized, X being the universe. Those preferring the "ordered-pair" notation, or the "fraction" ($\mu_A(x)/x$) notation, which as fuzzy sets are also called A [3] can read the results presented here as fuzzy sets in which for each $x \in X$ the value of the grade of membership of the resulting set is determined by the indicated operations being taken between the grades of membership of the sets indicated as variables.

International Journal of Approximate Reasoning 1993; 8:105–121

655 Avenue of the Americas, New York, NY 10010 0888-613X/93/$5.00

$A_1, A_2, \ldots$ when these sets can be related according to an inference scheme

$$A_1, A_2, \ldots \to B_1, B_2, \ldots \tag{1}$$

governed by a collection of rules of inference. In classical logic this can be accomplished in a variety of equivalent procedures based on combinations of both logic operations and *modus ponens*. Extensions of these procedures to fuzzy logic have been less successful: many lack contraposition $(A \Rightarrow B) = (B' \Rightarrow A')$; none reproduce *reductio ad absurdum* (*rada*) $(A \Rightarrow B, A \Rightarrow B') \to A'$. There are, of course, applications where such properties may be inappropriate, even undesirable; however, even the lack of a law of excluded middle should not prevent the formulation of a method of fuzzy–logic inference that captures these fundamentally essential features of classical logic. (As this is in *sharp contrast* to methods of fuzzy inference developed for the control of fuzzy systems, the reader must keep in mind that even though some of our terminology is similar, our concepts, motivation, and goals are very different.)

In approaching inference it is natural to try do derive conclusions from premises using logic operations. For example, to construct a *modus ponens* (ie, assuming the major premise $A \Rightarrow B$ and the minor premise A are valid to conclude B is valid)

$$(A, A \Rightarrow B) \to B \tag{2}$$

one could try

$$A \wedge (A \Rightarrow B) \tag{3}$$

or even

$$(A \wedge (A \Rightarrow B)) \Rightarrow B \tag{4}$$

However, in classical logic $(A \wedge (A \Rightarrow B)) = (A \wedge B) \neq B$, while $((A \wedge (A \Rightarrow B)) \Rightarrow B) = 1$. Thus, while both expressions are understandable, (the first saying that to obtain B, we must have a valid A, the second indicating that this expression is a tautology, valid even if A = B′), neither can really be said to represent the heart of *modus ponens*. On the other hand, the classical statement, "If A is true and if A implies B is true, then B is true," is simple, direct, and, if B can be extracted (detached) from $A \Rightarrow B$, effective. But, most importantly, it involves an operation *outside* of the logic operations (and ($\wedge$), or ($\vee$), negation (′), etc.), but of course, not outside of logic. Let us now capture this important aspect in fuzzy logic.

2. MODUS PONENS

To achieve *modus ponens* for fuzzy logic, we must be able to derive the fuzzy set[2] B from the fuzzy sets A and A ⇒ B. (In other words, we are *not* given B initially.) The logic operation A ⇒ B we represent as A′ ∨ B, because this form preserves the meaning of implication: when true, if A is true then B is true. (See Appendix A for a discussion of this point.) For A ∨ B we choose max (A, B), for A ∧ B we choose min (A, B), for A′ we choose 1 − A, a common choice since the beginning of fuzzy logic [1–3], and one that satisfies all the standard properties of logic connectives except the laws of excluded middle and noncontradiction. In what follows A ∨ B means the logic OR if reference is to logic, set union if reference is to set theory and max(A, B) if reference is to fuzzy logic. Similarly A ∧ B: AND, intersection, min(A, B); A′: negation, complement, 1 − A.

Returning finally to *modus ponens*, given (A ⇒ B) = max(A′, B), clearly if (A ⇒ B) > A′ then B = (A ⇒ B). Thus if we know (A ⇒ B) and A, we calculate A′ = 1 − A and compare (A ⇒ B) with A′. If the former is strictly larger, than we infer that B = (A ⇒ B). We write this as follows:

$$(A, A \Rightarrow B) \rightarrow B = (A \Rightarrow B) >_w A' \tag{5}$$

where the "$>_w$" symbol (as in X $>_w$ Y) reminds us that the preceeding equality is valid whenever X > Y. Note that for (A ⇒ B) = A′, B(≤ A′) is (otherwise) undeterminate. (The case (A ⇒ B) < A′ is impossible.) Put another way, for B > A′, which includes all the most useful region (1/2 < (A,B) ≤ 1), and in fact half of all possible values for (A,B), $0 \le A' < B \le 1$, B can be readily extracted from A ⇒ B. In other words this has a strong intuitive ring: the larger A, the smaller A ⇒ B need be to trigger inference, or the smaller A, the larger A ⇒ B must be to permit inference: B = (A ⇒ B) $>_w$ A′ captures this inverse relationship.

The advantages of the above procedure include retaining contraposition ((A ⇒ B) = (B′ ⇒ A′)), due to retention of (A ⇒ B) = (A′ ∨ B), the dependence of the result on the nature of A as well as A ⇒ B (see *modus tollens* below), and the retention of *reductio ad absurdum*. Concerning the latter, consider the case where we are given A ⇒ B and A ⇒ B′. We find

[2]All fuzzy sets are, of course, functions of elements that need not be the same when taken in combination. For example, we can have A(x) ⇒ B(x) or A(x) ⇒ B(y) depending on the problem of interest. In the former case the fuzzy set $A \Rightarrow B$ is a function which takes X → [0, 1], in the latter it is a function which takes X, Y → [0, 1]. None of the results presented depend on this difference. For sake of brevity we occasionally write A_x for A(x), etc. to indicate the variable while simplifying the notation. Subscripts normally associated with indices, eg, A_1, A_2, A_k, A_l refer to different fuzzy sets.

at once that

$$A' = min(A \Rightarrow B, A \Rightarrow B') >_w min((A \Rightarrow B)', (A \Rightarrow B')'), \quad (6)$$

a statement of *rada*. One can derive this result by recognizing that (6) has the form

$$A' = (A \Rightarrow B) \wedge (A \Rightarrow B') \quad (7a)$$

that expands into

$$A' = (A \Rightarrow (B \wedge B')) = (A' \vee (B \wedge B')) \quad (7b)$$

so that for $A' > \min(B, B')$ one obtains *rada*. (The whenever inequality in (6) ensures that $A' > \min(B, B')$. Thus the crisper (ie, closer to 1) B or B′ becomes, the wider the range of A′ ($\min(B, B') < A' \leq 1$) that can be inferred using proof by contradiction (*rada*) (6). (Note that (7a) also follows from taking the conjunction of the given quantities. Recalling (3) we remember that this is not always a useful procedure.)

There are, of course, occasions where $A \Rightarrow B$ and B′ are known. By analogy to *modus ponens* above, we can construct *modus tollens*

$$(B', A \Rightarrow B) \rightarrow A' = (A \Rightarrow B) >_w B \quad (8)$$

While $(A \Rightarrow B) > 1/2$ and $B' > 1/2$ yield $A' = (A \Rightarrow B)$, in fact (8) can be used to recover A′ from the region $0 \leq B < A' \leq 1$ complementary to that covered by *modus ponens*.

3. RULES OF INFERENCE

In logic inference can be formulated either in the form of axioms plus *modus ponens* or in the form of a set of rules of inference including *modus ponens*. For fuzzy logic, the latter are much more useful, as we shall see when we apply the rules to derive the standard axioms in fuzzy form. Thomason [4], as well as other authors [5–7], label each rule of inference in symbolic logic by a logic operation and either the word "introduction" or "elimination," or with quantifiers "instantiation" or "generalization." Extending these fuzzy logic, McCawley [8] changed "elimination" to "exploitation." Our rules differ significantly from his, however, because he chose to interpret $A \Rightarrow B$ as $A \alpha B$. Here we shall simply use the symbol for the operator and the first letter of the appropriate noun.

$\Rightarrow$ E: (implication elimination) This is simply our *modus ponens* ((5) above).

$\Rightarrow$ I: (implication introduction) Here we taken what we know about A and B to construct $(A \Rightarrow B) = (A' \vee B)$. (This, of course, is not the only source of fuzzy functions of the form $A \Rightarrow B$).

$\sim$ E: (negation elimination) A″ = 1 − A′ = 1 − (1 − A) = A.

$\sim$ I: (negation introduction) This is *rada* (6), ie, reductio ad absurdum or proof by contradiction, or *modulus tollens* (8), whichever is appropriate.

$\wedge$E: (conjunction elimination) In logic given that $\wedge_k A_k$ is true (1), one infers that A_k is true (1) for each k. In fuzzy logic one is given $\min_k(A_k) = A_{min}$ and infers $A_k \geq A_{min}$ for each k. However, by analogy with $\vee$E (see below), based on *modus ponens*, we can infer

$$(\vee_k(A_k \Rightarrow M), \wedge_l A_l) \rightarrow M = \vee_k(A_k \Rightarrow M) >_w (\wedge_l A_l)', \quad (9)$$

while based on *modus tollens* we can infer

$$(\vee_k(M \Rightarrow B_k), \wedge_l B_l') \rightarrow M' = \vee_k(M \Rightarrow B_k) >_w (\wedge_l B_l')'. \quad (10)$$

(Here $\wedge_k(\vee_k)$ means minimum (max) over the index k: $A_1, A_2, A_2 \ldots$)

$\wedge$ I: (conjunction introduction) Here we take $(A_1, A_2 \ldots)$ and form $A_{min} = \min(A_1, A_2, \ldots) = \wedge_k A_k$.

$\vee$E: (disjunction elimination) In logic this is one of the more interesting rules of inference. Using *modus ponens/tollens* it is readily extended to fuzzy logic:

$$(\wedge_k(A_k \Rightarrow M), \vee_l A_l) \rightarrow M = \wedge_k(A_k \Rightarrow M) >_w (\vee_l A_l)', \quad (11)$$

$$(\wedge_k(M \Rightarrow B_k), \vee_l B_l') \rightarrow M' = \wedge_k(M \Rightarrow B_k) >_w (\vee_l B_l')'. \quad (12)$$

$\vee$ I: (disjunction introduction) Here we take $(A_1, A_2, \ldots)$ and form $A_{max} = \max(A_1, A_2, \ldots) = \vee_k A_k$.

$\equiv$ E: (equivalence elimination)

$$(\vee_k(A_k \equiv B), \vee_l A_l, \wedge_l A_l) \rightarrow (\vee_l A_l) >_w B = \vee_k(A_k \equiv B) >_w (\wedge_l A_l)' \quad (13a)$$

$$(\wedge_k(A_k \equiv B), \vee_l A_l, \wedge_l A_l) \rightarrow (\wedge_l A_l) >_w B = \wedge_k(A_k \equiv B) >_w (\vee_l A_l)' \quad (13b)$$

$$(\vee_k(A_k \equiv B), \wedge_l A_l, \vee_l A_l) \rightarrow (\wedge_l A_l)' >_w B' = \vee_k(A_k \equiv B) >_w (\vee_l A_l) \quad (14a)$$

$$(\wedge_k(A_k \equiv B), \wedge_l A_l, \vee_l A_l) \rightarrow (\vee_l A_l)' >_w B' = \wedge_k(A_k \equiv B) >_w (\wedge_l A_l) \quad (14b)$$

≡ I: (equivalence introduction) With the choice of (min, max) for (∧, ∨) as above, we find that

$$(A \equiv B) = (A \wedge B) \vee (A' \wedge B') = (A \vee B') \wedge (A' \vee B) \quad (15)$$

⊕ E: (exclusive-OR elimination) We include this for the sake of completeness. They are ≡ E with (B ↔ B′).

$$(\vee_k(A_k \oplus B), \wedge_l A_l, \vee_l A) \to$$
$$(\wedge_l A_l)' >_w B = \vee_k(A_k \oplus B) >_w (\vee_l A_l) \quad (16a)$$
$$(\wedge_k(A_k \oplus B), \wedge_l A_l, \vee_l A) \to$$
$$(\vee_l A_l)' >_w B = \wedge_k(A_k \oplus B) >_w (\wedge_l A_l) \quad (16b)$$
$$(\vee_k(A_k \oplus B), \vee_l A_l, \wedge_l A_l) \to$$
$$(\vee_l A_l) >_w B' = \vee_k(A_k \oplus B) >_w (\wedge_l A_l)' \quad (17a)$$
$$(\wedge_k(A_k \oplus B), \vee_l A_l, \wedge_l A_l) \to$$
$$(\wedge_l A_l) >_w B' = \wedge_k(A_k \oplus B) >_w (\vee_l A_l)', \quad (17b)$$

⊕ I: (exclusive OR introduction) As with ≡ I, we find that

$$(A \oplus B) = (A \wedge B') \vee (A' \wedge B) = (A \vee B) \wedge (A' \vee B'). \quad (18)$$

4. EXTENSIONS

These rules can be extended in a number of ways. If we interpret $(\forall_x A_x)$ as $(\wedge_x A_x)$ and $(\exists_y B_y)$ as $(\vee_y B_y)$ then ∀E, ∀I, ∃E, ∃I follow at once by analogy with ∧E, ∧I, ∨E, ∨I above. (As explained,[2] the x and y here refer to elements of the universe; $A_x = A(x)$.)

Thus

$$\forall E\colon \left(\exists_x(A_x \Rightarrow M), \forall_y A_y\right) \to M = \exists_x(A_x \Rightarrow M) >_w (\forall_y A_y)' \quad (19)$$

$$\left(\exists_x(M \Rightarrow B_x), \forall_y B_y'\right) \to M' = \exists_x(M \Rightarrow B_x) >_w (\forall_y B_y')' \quad (20)$$

$$\exists E\colon \left(\forall_x(A_x \Rightarrow M), \exists_y A_y\right) \to M = \forall_x(A_x \Rightarrow M) >_w (\exists_y A_y)' \quad (21)$$

$$\left(\forall_x(M \Rightarrow B_x), \exists_y B_y'\right) \to M' = \forall_x(M \Rightarrow B_x) >_w (\exists_y B_y')' \quad (22)$$

Of greater interest are the representations of the conditional quantifiers. If we interpret $\forall_x f_x, g_x$ ie, for all elements x such that f_x is satisfied in a fuzzy sense, then g_x is also satisfied, as $\wedge_x(f_x \Rightarrow g_x) = \wedge_x(f_x' \vee g_x)$, and $\exists_x f_x, g_x$ as $\bigvee_x (f_x \wedge g_x)$, and noting that

$$(\forall_x f_x, g_x)' = \exists_x f_x, g_x' \tag{23}$$

we obtain at once the following conditional quantifier inferences:

$$\forall_c E\colon \left(\exists_x f_x, (g_x \Rightarrow B), \forall_y f_y, g_y\right) \rightarrow$$
$$\exists_x f_x >_w B = \exists_x f_x, (g_x \Rightarrow B) >_w (\forall_y f_y, g_y)' \tag{24}$$
$$\left(\exists_x f_x, (A \Rightarrow g_x), \forall_y f_y, g_y'\right) \rightarrow$$
$$\exists_x f_x >_w A' = \exists_x f_x, (A \Rightarrow g_x) >_w (\forall_y f_y, g_y')' \tag{25}$$
$$\exists_c E\colon \left(\forall_x f_x, (g_x \Rightarrow B), \exists_y f_y, g_y\right) \rightarrow B = \forall_x f_x, (g_x \Rightarrow B) >_w (\exists_y f_y, g_y)' \tag{26}$$

$$\left(\forall_x f_x, (A \Rightarrow g_x), \exists_y f_y, g_y'\right) \rightarrow A' = \forall_x f_x, (A \Rightarrow g_x) >_w (\exists_y f_y, g_y')' \tag{27}$$

As in (19)–(22), (24) and (26) generalize *modus ponens*, (25) and (27) *modus tollens*.

In symbolic logic one often expresses hypotheses in the following form [6]

$$H = ((A_1 \wedge A_2 \wedge \ldots \wedge A_n) \Rightarrow B), \tag{28a}$$

the negation of which is

$$H' = A_1 \wedge A_2 \wedge \ldots \wedge A_n \wedge B' \tag{28b}$$

If H' is true (the set $(A_1, A_2, \ldots, A_n, B')$ consistent), then H is false (not valid); if H' is false (the set inconsistent), then H is true (valid). With our intepretation of the operation of fuzzy implication, we can manipulate (28ab) much as is done in logic to obtain analogous results.

Examples of this include the following. If

$$H_{12} = ((A_1 \wedge A_2) \Rightarrow (B_1 \vee B_2)), \tag{29a}$$

then by a straightforward manipulation

$$H_{12} = ((A_1 \wedge B_2') \Rightarrow (B_1 \vee A_2')) \tag{29b}$$

In this manner, one obtains a number of implications having the same (known) fuzzy truth value. Of particular interest is to write (28a) as

$$H = ((A_2 \wedge \ldots \wedge A_n \wedge B') \Rightarrow A_1') \tag{30}$$

In logic, if one also has $A_2 \wedge \ldots \wedge A_n \Rightarrow A_1$, then using *modus tollens* it follows from (30) that

$$H = (A_2 \wedge \ldots \wedge A_n \Rightarrow B) \tag{31}$$

Hence, within a set propositions implied by other propositions can be omitted in hypotheses such as (28a). In fuzzy logic, owing to the continuous nature of the operations, it is necessary that $((A_2 \wedge \ldots \wedge A_n) \Rightarrow B) > ((A_2 \wedge \ldots \wedge A_n) \Rightarrow A_1')$ Expression (31) then remains valid since from (30)

$$\begin{aligned} H &= \left(\bar{A}_2 \Rightarrow (B \vee A_1')\right) = \left(\bar{A}_2 \Rightarrow B\right) \vee \left(\bar{A}_2 \Rightarrow A_1'\right) \\ &= \left(\bar{A}_2 \Rightarrow B\right) >_w \left(\bar{A}_2 \Rightarrow A_1'\right), \end{aligned} \tag{32}$$

where $\bar{A}_2 = A_2 \wedge \ldots \wedge A_n$.

In proving theorems, more easily proved lemmas are often used in order to shorten the proof. This technique can be formalized using the cut rule [9]

$$(A_1 \Rightarrow (B_1 \vee R_{12}), (A_2 \wedge R_{12}) \Rightarrow B_2) \rightarrow ((A_1 \wedge A_2) \Rightarrow (B_1 \vee B_2)) \tag{33}$$

where R_{12} is the cut formula (lemma of A_1) of this inference, the conclusion (result) of which does not involve R_{12} explicitly. Longer chains of reasoning are readily envisioned

$$\begin{aligned} &(A_1 \Rightarrow (B_1 \wedge R_{12}), R_{12} \wedge A_2 \Rightarrow B_2, A_2 \Rightarrow B_2 \vee R_{23}, R_{23} \wedge A_3 \Rightarrow B_3) \\ &\quad \rightarrow (A_1 \wedge A_2 \wedge A_3 \Rightarrow B_1 \vee B_2 \vee B_3) \end{aligned} \tag{34}$$

Note that even in fuzzy logic the two intermediate terms can be combined as

$$(R_{12} \wedge A_2 \Rightarrow B_2) \vee (A_2 \Rightarrow B_2 \vee R_{23}) = (R_{12} \wedge A_2 \Rightarrow B_2 \vee R_{23}), \tag{35}$$

which has the *same* structure as (33).

In order to determine the nature of the inference (33) in fuzzy logic, we consider the simpler (actually equivalent) inference

$$(u \vee v, u' \vee w) \rightarrow (v_{\vee} w) \tag{36}$$

Clearly $(v \vee w) \leq max(u \vee v, u' \vee w) = max(u, v, u', w)$. If $(u \vee v) > u > (u' \vee w)'$ then we would have equality. However, merely from $(u \vee v)$ and $(u' \vee w)$, we cannot guarantee both these strict inequalities. Now, so long as $(u \vee v) > (u' \vee w)'$, it follows that

$$min(u \vee v, u' \vee w) \leq (v \vee w) \leq max(u \vee v, u' \vee w). \tag{37}$$

This is appropriate, as the conjunction of the antecedent implications in (36) would be expected to play a key role here. If, in addition to (36), we also have

$$(t \vee v, t' \vee w) \rightarrow (v \vee w), \tag{38}$$

with $(t \vee v) > (t' \vee w)'$, then the limits in (37) can be tightened to

$$\begin{aligned} &max(min(u \vee v, u' \vee w), min(t \vee v, t' \vee w)) \\ &\quad \leq (v \vee w) \leq max((u \wedge t) \vee v, (u \vee t)' \vee w) \end{aligned} \tag{39}$$

These results, (37) and (39), differ from our previous results in that here we have not explicitly identified the consequent of the inference in terms of the antecedent, we have only bounded it. Because this is indeed a fuzzy logic, this limitation should not seem too unnatural.

However, if $(u \vee v) > (t \vee v)$ and $(u' \vee w) < (t' \vee w)$, then[3] $v = (t \vee v)$ and $w = (u' \vee w)$, and inequalities in $(v \vee w)$ can be avoided altogether. Alternatively, $v = (u \vee v) <_w (t \vee v)$ and $w = (t' \vee w) <_w (u' \vee w)$. (The other two choices of inequalities are not possible.) If both pairs are equal, then, of course, nothing is gained. If say $(u \vee v) = (t \vee v)$ while $(u' \vee w) \lesseqgtr (t' \vee w)$, then $v = (u \vee v)$ while w cannot be uniquely ascertained; similarly $w = (u' \vee w) <_w (t' \vee w)$, $(u \vee v) \lesseqgtr (t \vee v)$.). In this manner, the existence of a second (and different) inner structure (a second cut) significantly sharpens our results for cut rules. For the present we shall consider only a single cut.

We can now return to the cut rule (33). If we identify $A_1' \vee B_1$ with v, $A_2' \vee B_2$ with w, and R_{12} with u, then $v \vee w = (A_1 \wedge A_2) \Rightarrow (B_1 \vee B_2)$, and with these entries (37) yields

$$\begin{aligned} &min(A_1 \Rightarrow (B_1 \vee R_{12}), (A_2 \wedge R_2) \Rightarrow B_2) \leq (A_1 \wedge A_2) \Rightarrow (B_1 \vee B_2) \leq \\ &\quad \leq max(A_1 \Rightarrow (B_1 \vee R_{12}), (A_2 \wedge R_{12}) \Rightarrow B_2) \end{aligned} \tag{40}$$

so long as $(A_1 \Rightarrow (B_1 \vee R_{12})) > (A_2 \wedge R_{12} \Rightarrow B_2)'$. Similarly (34) works out to be

$$\begin{aligned} &min(A_1 \Rightarrow (B_1 \wedge R_{12}), R_{12} \wedge A_2 \Rightarrow B_2 \vee R_{23}, R_{23} \wedge A_3 \Rightarrow B_3) \\ &\quad \leq (A_1 \wedge A_2 \wedge A_3 \Rightarrow B_1 \vee B_2 \vee B_3) \leq \\ &\qquad max(A_1 \Rightarrow B_1 \wedge R_{12}, R_{12} \wedge A_2 \Rightarrow B_2 \vee R_{23}, R_{23} \wedge A_3 \Rightarrow B_3) \end{aligned} \tag{41}$$

[3]The proof is straightforward, but several alternative cases must be considered and systematically rejected.

provided $(A_1 \Rightarrow (B_1 \vee R_{12})) > ((R_{12} \wedge A_2) \Rightarrow (B_2 \vee R_{23}))'$ and $((R_{12} \wedge A_2) \Rightarrow (B_2 \vee R_{23})) > ((R_{23} \wedge A_3) \Rightarrow B_3)'$. (The inequalities in (40) and (41) are sharp, however many A and B are considered). Longer chains can be similarly analyzed and the bounds tightened with alternative lemmas.

A common variant of the cut rule (33) is the following minor extension:

$$\begin{aligned}(A_1 &\Rightarrow B_1 \vee R_1, R_2 \wedge A_2 \Rightarrow B_2)\\ &\rightarrow (A_1 \wedge (R_1 \Rightarrow R_2) \wedge A_2) \Rightarrow (B_1 \vee B_2)\end{aligned} \tag{42}$$

where, of course, $R_1 \Rightarrow R_2$ is understood to be a third antecedent. This can be decomposed and analyzed in much the same manner as (34), with the result that

$$\begin{aligned}&min(A_1 \Rightarrow B_1 \vee R_1, R_1 \Rightarrow R_2, R_2 \wedge A_2 \Rightarrow B_2)\\ &\quad\le (A_1 \wedge (R_1 \Rightarrow R_2) \wedge A_2) \Rightarrow (B_1 \vee B_2)\\ &\quad\le max(A_1 \Rightarrow B_1 \vee R_1, R_1 \Rightarrow R_2, R_2 \wedge A_2 \Rightarrow B_2)\end{aligned} \tag{43}$$

so long as $(A_1 \Rightarrow B_1 \vee R_1) > (R_1 \Rightarrow R_2)'$ and $(R_1 \Rightarrow R_2) > (R_2 \wedge A_2 \Rightarrow B_2)'$. Various simplifications of (33) or (42) lead to exact results. For example

$$(A_1 \wedge A_2 \Rightarrow R \vee B, S \wedge A_1 \wedge A_2 \Rightarrow B) \rightarrow (A_1 \wedge (R \Rightarrow S) \wedge A_2 \Rightarrow B) \tag{44}$$

according to

$$\begin{aligned}&(A_1 \wedge (R \Rightarrow S) \wedge A_2 \Rightarrow B)\\ &\quad = min(A_1 \wedge A_2 \Rightarrow R \vee B, S \wedge A_1 \wedge A_2 \Rightarrow B)\end{aligned} \tag{45}$$

while $A_2 \Rightarrow (B_2 \vee (A_1 \Rightarrow B_1)) = (A_1 \wedge A_2) \Rightarrow (B_1 \vee B_2)$, again owing to our choice of $A \Rightarrow B = A' \vee B$.

As an extension of $\vee E$, one can encounter this *modus ponens* like inference

$$(\{A_j \Rightarrow B_j\}, \vee_i A_i) \rightarrow \vee_i B_i \tag{46}$$

By analogous reasoning we find

$$(\vee_i A_i)' <_w min_k(A_k \Rightarrow B_k) \le \vee_i B_i \le max_k(A_k \Rightarrow B_k) \tag{47}$$

While $\max_k(A_k \Rightarrow B_k) = max_k(A_k') \vee max_l(B_l)$, so that if one knew $\wedge_k A_k$, one could write

$$\wedge_l B_l = \vee_k(A_k \Rightarrow B_k) >_w (\wedge_k A_k)', \tag{48}$$

this would not be too useful, as some A_k can be 0. Rather, it is the inequality

$$\wedge_k(A_k \Rightarrow B_k) \leq \vee_l B_l$$

which lies at the heart of (47). The equivalent expression using *modus tollens* is

$$(\{A_i \Rightarrow B_i\}, \vee_j B_j') \to \vee_j A_j' \tag{49}$$

and

$$(\vee_k B_k') <_w min_k(A_k \Rightarrow B_k) \leq \vee_j A_j' \leq max_l(A_l \Rightarrow B_l). \tag{50}$$

Once again one could have

$$\vee_j A_j' = max_l(A_l \Rightarrow B_l) >_w \vee_l B_l,$$

but as above this form has minimal utility.

Another common inference in logic (transitivity) has the form

$$(A \Rightarrow X, X_1 \Rightarrow X_2, \ldots, X_n \Rightarrow B) \to A \Rightarrow B \tag{51}$$

We can use our fuzzy modus ponens (5) to pass from A to B so long as the following inequalities are satisfied:

$$A' < (A \Rightarrow X_1), (A \Rightarrow X_1)' < (X_1 \Rightarrow X_2), \ldots, (X_{n-1} \Rightarrow X_n)' < (X_n \Rightarrow B) \tag{52}$$

From these we infer $X_s = (X_{s-1} \Rightarrow X_s)$ and $B = (X_n \Rightarrow B)$. Similarly using modus tollens (8) and

$$B' > (X_n \Rightarrow B)', (X_n \Rightarrow B) > (X_{n-1} \Rightarrow X_n)', \ldots, (X_1 \Rightarrow X_2) > (A \Rightarrow X)'$$

we infer $X_r' = (X_r \Rightarrow X_{r+1})$ and $A' = (A' \Rightarrow X_1)$. (Note that the intermediate inequalities are the same as those above.) Should B and C be joined by $Y_1 \ldots Y_m$ similarly satisfying

$$B' < (B \Rightarrow Y_1), (B \Rightarrow Y_1)' < (Y_1 \Rightarrow Y_2), \ldots, (Y_{m-1} \Rightarrow Y_m)' < (Y_m \Rightarrow C),$$

then this meshes with series (52) and $C = (Y_m \Rightarrow C)$.

These results illustrate a number of interesting features. So long as the intermediate inequalities are satisfied, the derived implication

$$(A \Rightarrow_d B) = (A \Rightarrow X_1, X_n \Rightarrow B), \tag{53}$$

now represented by *two* quantities, does *not* depend upon the actual values of $(X_1 \Rightarrow X_2), \ldots, (X_{n-1} \Rightarrow X_n)$. In other words, the successive influences do *not* dilute the inference. While (53) can *not* be reduced to the simpler form $(A \Rightarrow B) = (A' \vee B)$, it does enable us to preserve the most important feature of the classical result. In (51) if A is true, then X_1 is true, if X_1 is true, then X_2 is true, ..., if X_n is true then B is true: hence if A is true, B is true, and (51) indeed simplifies to $A \Rightarrow B$. Similarly, if $(A \Rightarrow X_1) > A'$, then $B = (X_n \Rightarrow B)$ summarizes (52). (It is this encapsulation that is not possible with the simpler form $A' \vee B$.) If we are working with quantified quantities, then $(\forall_x(X_s \Rightarrow X_{s+1})_x) > (\forall_y(X_{s-1} \Rightarrow X_s)_y)'$ guarantees $(X_s \Rightarrow X_{s+1})_x > (X_{s-1} \Rightarrow X_s)'_x$ for each x, and (53) becomes

$$(A \Rightarrow_d B) = \big(\forall_x(A \Rightarrow X_1)_x, \forall_y(X_n \Rightarrow B)_y\big)$$

Thus for each $x \in X$, $A_x > (\wedge_y(A \Rightarrow X_1)_y)'$, $B_x \geq \wedge_y(X_n \Rightarrow B)_y$, using the minimization interpretation of universal quantification introduced above. Finally $\wedge_x(A \Rightarrow X_1)_x > (\wedge_x A_x)'$ yields $\wedge_x B_x = \wedge_x(X_n \Rightarrow B)_x$.

Perhaps the most counterintuitive result here is that some of the $(X_s \Rightarrow X_{s+1})$ can in principle be less than $1/2$ without affecting the overall inference. (Clearly if each exceeds $1/2$, all inequalities are immediately satisfied.) Thus all that is required of $(X_s \Rightarrow X_{s+1})$ is that it exceeds $(X_{s-1} \Rightarrow X_s)'$ and $(X_{s+1} \Rightarrow X_{s+2})'$ which can be quite small. In general, no two adjacent implications can be less than $1/2$. However, given strong implications on either side, this model permits a weak implication in between representing an "intuitive leap" (a tunneling-in-inference-space) phenomenon. (Mathematically this results from making full use of the $B > 1 - A$ region of the $0 \leq A \leq 1, 0 \leq B \leq 1$ domain over which $A \Rightarrow B$ is defined.) Should a weak implication enter, the system could search for a stronger connective, as would a person in a similar situation. However, one would still have the result, found initially more quickly and simply but so it would seem with less rigor. Of course, if appropriate, a lower limit (eg, $1/2$) can be imposed *ad hoc* on the individual implications. In any event one is not restricted to a monotonicity increasing sequence of $A, X_1, X_2, \ldots, X_n, B$, though such could be imposed if warranted.

Closely related to the above is fuzzy mathematical induction: $(F_1, F_{n-1}) \rightarrow F_n$, or more precisely, $(F_1, F_n \Rightarrow F_{n+1}) \rightarrow F_{n+1}$. So long as F_n and $(F_n \Rightarrow F_{n+1})$ remain above $1/2$, then $F_{n+1} > 1/2$, and all is well. However, should $(F_n \Rightarrow F_{n+1})$ drop below $1/2$, induction would cease. To avoid this deadlock, one should switch to $G_{n+1} = F'_{n+1}$ so that $(F_n \Rightarrow G_{n+1}) > 1/2$. Should $(G_m \Rightarrow G_{m+1}) < 1/2$ for some $m > n + 1$, one simply switches back to $F_{m+1} = G'_{m+1}$ so that $(G_m \Rightarrow F_{m+1}) > 1/2$. One can continue in this manner to all countable m, n. (Clearly from the foregoing, implications less than 1/2 can still be used for inference. The point here is that in the spirit of induction, one demands that certain conditions be true for all n, and in this formulation the greater-than-1/2 is one of these.)

Example

Pearl [10] has presented a probabilistic treatment of a simplified version of the Yale shooting problem [11]. Using our cut-rule results of the previous section, a fuzzy–logic treatment can also be effected. In this problem a gun is loaded at t_0, (represented by LD_0), shot at someone at $t_1 > t_0(SH_1)$, who dies by $t_2 > t_1$ (AL_2'), despite the normal tendency to be (otherwise) alive at $t_2(AL_2)$ given being alive at $t_1(AL_1)$. The state LD_1 also enters. Thus

$$AL_1 \wedge SH_1 \wedge LD_1 \Rightarrow AL_2' \tag{54a}$$

$$LD_0 \Rightarrow LD_1 \tag{54b}$$

$$AL_1 \wedge (SH_1 \wedge LD_1)' \Rightarrow AL_2 \tag{54c}$$

summarize the given possibilities, and combining (54ab) according to (40), LD_1 being the cut expression, we find at once that

$$\begin{aligned} & min(AL_1 \wedge SH_1 \wedge LD_1 \Rightarrow AL_2', LD_0 \Rightarrow LD_1) \\ & \leq (AL_1 \wedge SH_1 \wedge LD_0 \Rightarrow AL_2') \\ & \leq max(AL_1 \wedge SH_1 \wedge LD_1 \Rightarrow AL_2', LD_0 \Rightarrow LD_1) \end{aligned} \tag{55}$$

so long as $(AL_1 \wedge SH_1 \wedge LD_1 \Rightarrow AL_2') > (LD_0 \Rightarrow LD_1)'$.

Under some conditions these bounds can be tighter than the corresponding probabilistic result. [10] (Expression (54c) does not really enter because as a fuzzy function its values can be determined from those of (54a)).

5. CONCLUSIONS

Fuzzy *modus ponens* remains an active area of research [12–17]. The purpose here was simply to see how far one could go by preserving as much as possible of the spirit of classical inference, keeping in mind the roles played by the operator (connective) of implication. If, by means of axioms and/or rules of inference, we have derived B from A, we write A ⇒ B (A implies B). Alternatively knowing A ⇒ B, and A, we infer B, written (A, A ⇒ B) → B. But keeping in mind that even in classical logic we cannot express inference purely in terms of logic operations, we carry that principle over to fuzzy logic. We also preserve the character (symmetry) of the implication operation. The result has been a number of rules of inference, some, like the crutial *rada* (proof by contradiction), were not obtainable by other interpretations.

One should carefully distinguish inference and decision theory. In the latter one usually carries out *explicit* operations to determine an output that elicits a decision. In the former, by contrast, one relies on *implicit* operations. For example *modus ponens* goes into $A \Rightarrow B$ to note that if $1 \Rightarrow 0$ is 0 and $1 \Rightarrow 1$ is 1, then $A = 1$ and $A \Rightarrow B = 1$ means that $B = 1$; values of $A \Rightarrow B$ for $A = 0$ are *not* needed for this inference. This going inside of, this implicit-function dependence (given x and f(x, y), find y), is what inference is all about.

ACKNOWLEDGMENTS

The author wishes to thank D. Kahng for calling attention to inference in fuzzy logic; J. A. Giordmaine for supporting this work; and C. W. Gear and W. D. Smith for critically reading the manuscript.

APPENDIX A: CHOICE OF IMPLICATION

A word about the operation of implication is necessary. We shall use $(A \Rightarrow B) = (A' \vee B)$, as if it were just another logic connective. Some authors have modified $A \Rightarrow B$ in order to emphasize one or more aspects of inference, eg, $(A \Rightarrow B) = (A \alpha B)$. However, we must bear in mind that even though the result of an inference in classical logic can be expressed in the form, "When A is true, B is true," which can be represented by the operation of implication, $A \Rightarrow B$, the two, inference and implication, are two very different concepts. In addition, and as a technical point crucial for what follows, $A \alpha B = 1$, $A \le B$, $= B$, $A > B$ has an inappropriate, logic symmetry in representing $A \Rightarrow B$. To be sure it reduces to the classical $A' \vee B$ for $(A, B) = \{0, 1\}$.[2] But it is discontinuous along $A = B$ in the very region $((A, B) > 1/2)$ where $A \Rightarrow B$ is most useful. Indeed, one cannot infer B from A and $A \Rightarrow B$ for $A < B$ since $A \alpha B = 1$ for all (A, B) in this region. In addition, choosing $A \alpha B$ at once precludes both contraposition and *reductio ad absurdum*. Contraposition is excluded since $B \alpha A$ (Godel implication) cannot be determined from $A \alpha B$ for $B > A$. ($B \alpha A = A$ for $B > A$, but A cannot be had from $A \alpha B$.) Rada is excluded since $A \alpha B$ and $A \alpha B'$ cannot be solved for A' for the same reason, neither ever equals A or A'. To be sure one can infer B from $A \alpha B$ for $A > B$,

$(A, A \alpha B) \rightarrow B = A \alpha B <_w A$, according to the spirit of this paper; however, if $A < B$, inference is impossible as $A \alpha B = 1$, even for A and B close to 1. Also, lacking contraposition, there is no modus tollens.

The choice $(A \Rightarrow B) = (1 \wedge (1 - A + B))$, which, in contrast with $A \alpha B$, does correspond to $A' \vee B$ in the choice of fuzzy logic operations bounded sum and bounded product for union and intersection, respectively, and which does preserve contraposition, also precludes *reductio ad absurdum*. Again, the logic symmetry is inappropriate since $(1 \wedge (1 - A + B)) = 1$ for $A \leq B$ and $(1 - A + B)$ for $A \geq B$. Though continuous across $A = B$, it embodies two distinct functional dependences in the region $((A, B) > 1/2)$ where only one type of dependence would be expectted. Again B cannot be inferred from A and $A \Rightarrow B$ for $A < B$. In this, the Lukasiewicz form, for $A > B$ one can effect an inference in the form $(A, 1 - A + B) \rightarrow B = (1 - A + B) - A' <_w A$, and similarly for modus tollens $A' = (1 - A + B) - B <_w B'$. However, neither inference is possible for $B > A$, although both B and A can be close to 1. In chaining the consequent can only decrease, approaching 0 for long chains. This is most undesirable. Concerning rada, a degenerate form is in fact possible, but only for $A > B > A'$, in precisely the quadrant of (A, B) where rada is least expected: specifically one finds $A' = ((1 - A + B) + (1 - A + B') - 1)/2$. (N.B. $A' < 1/2$.) In logic $(A \Rightarrow B, A \Rightarrow B') \rightarrow A' = 1$.

It is surprising, therefore, that by choosing $(A \Rightarrow B) = (A' \vee B) = \max(A', B)$, not only does one satisfy all the basic operational properties of logic, except excluded middle, but one also obtains a number of properties of inference not generally obtained with other choices. To be sure Lucasiewicz preserves excluded middle but even so loses a meaningful rada as well as idempotency, absorption, distributivity, equivalence, and symmetrical difference. $(A \Rightarrow B) = \max(A', B)$ maintains rada, as well as all these other properties of logic, even without the excluded middle.

APPENDIX B: DEGENERATE FORMS

This inference $(A, A \Rightarrow B) \rightarrow B = (A \Rightarrow B) >_w A'$ will yield the fuzzy set B only for those elements $x \in X$ for which B(x) exceeds $A'(x)$. (In much the same way, in logic $A = 1$ and $(A \Rightarrow B) = 1$ are necessary to infer $B = 1$; the other combinations do not permit inferences.) Consider the degenerate form $(A, A \Rightarrow A) \rightarrow A = (A \Rightarrow A) >_w A'$. One might conclude that unless $A > 1/2$, so that $A > A'$, one could not infer A. However, since one is given the antecedent A to begin with, no inference is necessary. Hence, $A \Rightarrow A$ is not expected to play a significant role in inference, and can be ignored. It does, of course, equal $\max(A, A')$. But in general, as in logic, structures of the form $(A \wedge B) \Rightarrow (A \vee C)$ are to be avoided because they tend to stifle inferences. (In logic this structure always equals one.)

One might conclude on intuitive grounds that (A $\Rightarrow$ A) = 1—if I know A, I know A. But this is equivalent to postulating excluded middle, and, if all the other properties of logic are demanded, one is back to logic necessarily. Even worse is that the same intuition yields (A $\Rightarrow A'$) = 0: if I know A, I know I do not have A′. But (A $\Rightarrow A'$) $\neq$ 0 even in logic.

References

1. Zadeh, L. A., Fuzzy sets, *Inf. Control.* 8:338–353, 1965.
2. Goguen, J. A., L-Fuzzy sets, *J. Math. Analysis Applications.* 18:145–155, 1967.
3. Dubois, D., and Prade, H., *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, 1980, p. 10.
4. Thomason, R. H., *Symbolic Logic, an Introduction*, Macmillan, New York, 1970, pp. 14–94.
5. Halberstadt, W. H., *An Introduction to Modern Logic: An Elementary Textbook of Symbolic Logic*, Harper and Row, New York, 1960, pp. 205–206.
6. Smullyan, A., *Fundamentals of Logic*, Prentice-Hall, New York, 1962, pp. 20–21.
7. Gustason, W., and Ulrich, D. E., *Elementary Symbolic Logic*, Holt, Rinehart, New York, 1973, inside front and back covers.
8. McCawley, J. D., *Everything that Linguists Have Always Wanted to Know about Logic, but Were Ashamed to Ask*, University of Chicago, Chicago, 1981, pp. 360–380.
9. Gallier, J. H., *Logic for Computer Science, Foundations of Automatic Theorem Proving*, Harper and Row, New York, 1986, pp. 97–113.
10. Pearl, J., *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*, Kaufmann, San Francisco, 1988, pp. 509–516.
11. Hawks, S., and McDermott, D. V., Nonmonotonic logic and temporal projection, *Artif. Intell.* 33:379–412, 1987.
12. Bandler, W., and Kohout, L. J., Semantics of implication operators and fuzzy relational products, *Int. J. Man–Machine Studies.* 12:89–116, 1980.
13. Trillas, E., and Valverde, L., On mode and implication in approximate reasoning, in *Approximate Reasoning in Expert Systems* (M. M. Gupta et al., Eds.), North Holland, Amsterdam, 157–166, 1985.

14. Trillas, E., and Valverde, L., On implication and indistinguishability in the setting of fuzzy logic, in *Management Decision Support Systems Using Fuzzy Sets and Possibility Theory* (J. Kacprzyk and R. R. Yager, Eds.), Verlag TUV Rheinland, 198–212, 1985.

15. Smet, S. P., and Magnez, P., Implication of fuzzy logic, *J. Approx. Reasoning.* 1:327–347, 1987.

16. Dubois, D., and Prade, H., On the combination of uncertain or imprecise pieces of information in rule-based systems, a discussion in the framework of possibility theory, *J. Approx. Reasoning.* 2:65–87, 1988.

17. Martin-Clouaire, R., Semantics and computation of the generalized modus ponens: the long paper, *J. Approx. Reasoning.* 3:195–217, 1989.