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A COUNTEREXAMPLE FOR A PROBLEM OF ARHANGEL'SKII CONCERNING THE PRODUCT OF FRÉCHET SPACES

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Using the continuum hypothesis, we give a counterexample for the following problem posed by Arhangel'skii: If $X\times Y$ is Fréchet for each countably compact regular Fréchet space Y, then is X an $\langle\alpha_3\rangle$-space?

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1. Introduction

A topological space X is said to be *strongly Fréchet* [10] (=countably bi-sequential in the sense of [5]) if, for every decreasing sequence $\{A_n: n\in N\}$ accumulating at $x\in X$, there exists a convergent sequence B of X with $x\in \overline{B\cap A_n}$ for each $n\in N$, where N denotes the integers. If $A_i=A_j$ for each i and j, then such a space is said to be *Fréchet.*

It is well-known that Fréchet spaces behave quite badly with respect to product operations. In fact the product of two compact Fréchet spaces need not be Fréchet [9]. The following theorems are positive results for the product of Fréchet spaces when at least one factor space is countably compact.

1.1. Theorem [5]. *A space X is strongly Fréchet if and only if $X\times C$ is Fréchet, where $C=\{0\}\cup\{1/n: n\in N\}$ or C is the closed unit interval* $[0,1]$.

1.2. Theorem [2]. *If X is an $\langle\alpha_3$-$FU\rangle$-space, then $X\times Y$ is Fréchet for each countably compact regular Fréchet space Y.*

After proving Theorem 1.2, Arhangel'skii asked whether the converse of the above theorem is true [2, 5.19], i.e. he asked: If $X\times Y$ is Fréchet for each countably compact regular Fréchet space Y, then is X an $\langle\alpha_3\rangle$-space?

The purpose of this paper is to construct, under CH, a non-$\langle\alpha_3\rangle$-space X such that $X\times Y$ is Fréchet for each countably compact regular Fréchet space Y.

In this paper all spaces are assumed to be Hausdorff topological spaces.

2. Definition and preliminary results

Let X be a space. A collection $\mathscr{A}$ of convergent sequences of X is said to be a *sheaf* in X if all members of $\mathscr{A}$ converge to the same point of X, which is said to be the *vertex* of the sheaf $\mathscr{A}$. In this paper all sheaves are assumed to be countably infinite.

We consider the following properties of X which were introduced by Arhangel'skii [1, 2].

Let $\mathscr{A}$ be a sheaf in X with vertex $x\in X$. Then there exists a sequence B converging to x such that

$$|\{A\in\mathscr{A}\colon |A\cap B|=\aleph_0\}|=\aleph_0, \qquad (\alpha_3)$$

$$|\{A\in\mathscr{A}\colon A\cap B\neq\emptyset\}|=\aleph_0, \qquad (\alpha_4)$$

where $|A|$ denotes the cardinality of a set A.

We say B satisfies (α_i) with respect to $\mathscr{A}$ if B satisfies the property (α_i) for $i=3,4$. The class of spaces satisfying the property (α_i) for every sheaf $\mathscr{A}$ and vertex $x\in X$ is denoted by $\langle\alpha_i\rangle$. We denote by $\langle\alpha_i\text{-FU}\rangle$ the intersection of the class of Fréchet spaces and the class $\langle\alpha_i\rangle$ for $i=3,4$. For a class $\mathscr{C}$ of spaces we say an element of $\mathscr{C}$ is a $\mathscr{C}$-space. Clearly an $\langle\alpha_3\rangle$-space is an $\langle\alpha_4\rangle$-space. A w-space in the sense of Gruenhage [3] and a bisequential space are $\langle\alpha_3\text{-FU}\rangle$-spaces [6, 2].

The following two theorems show the relationship between well known spaces and $\langle\alpha_4\text{-FU}\rangle$-spaces.

2.1. Theorem [2]. *A space X is strongly Fréchet if and only if it is an* $\langle\alpha_4\text{-FU}\rangle$-space.

2.2. Theorem [8]. *Each countably compact regular Fréchet space is strongly Fréchet (hence $\langle\alpha_4\text{-FU}\rangle$).*

3. Construction of our example

We denote by βN the Stone–Čech compactification of N. For a subset A of N, we denote $A^*=\mathrm{Cl}_{\beta N}A-A$. Let F be a closed subset of N^*. We put $X=N\cup\{F\}$ and topologize as follows: Points of N are isolated. The set of the form $U\cup\{F\}$ is a basic neighborhood of F in X, where U is a subset of N with $F\subset U^*$. The following facts are well-known.

3.1. Fact. *Let Z be a non-empty zero set in N^*. Then* $\mathrm{Int}_{N^*}Z \neq \emptyset$.

3.2. Fact. *Two disjoint cozero sets in N^* have disjoint closures.*

3.3. Fact [4]. *Let $X = N \cup \{F\}$. Then X is strongly Fréchet if and only if F is regular closed in N^* and, for each zero set Z of N^*, $F \cap Z \neq \emptyset$ implies $F \cap \mathrm{Int}_{N^*}Z \neq \emptyset$.*

3.4. Lemma (CH). *Let Z be a zero set in N^* with the non-empty boundary H. Then there exist two regular closed sets F_1 and G_1 in N^* such that*

(i) $F_1 \subset Z$ *and* $G_1 \subset Z$,

(ii) $\mathrm{Bdy}_{N^*}F_1 = \mathrm{Bdy}_{N^*}G_1 = H$,

(iii) $\mathrm{Int}_{N^*}F_1 \cap \mathrm{Int}_{N^*}G_1 = \emptyset$,

(iv) *for each zero set K of N^* such that* $H \cap \mathrm{Bdy}_{N^*}K \neq \emptyset$,

$K \cap \mathrm{Int}_{N^*}F_1 \neq \emptyset$ *and* $K \cup \mathrm{Int}_{N^*}G_1 \neq \emptyset$.

After constructing F_1 and G_1, put $F = F_1 \cup (N^* - Z)$ and let $X = N \cup \{F\}$. Then we can show that X is the desired space. This lemma is proved in [7] for another purpose, but we include the proof (under CH); since details of the construction will be used later.

Proof of Lemma 3.4. We construct F_1 and G_1 by transfinite induction. Note that the cardinality of the set of all zero sets in N^* equals the cardinality of the continuum. Let $\{Z_\alpha: \alpha < \omega_1\}$ be the family of all zero sets in Z such that $H \cap \mathrm{Bd}\, y_{N^*}\, Z_\alpha \neq \emptyset$ for $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal. Let $\{W_\alpha;\ \alpha < \omega_1\}$ be a family of zero sets in Z such that $W_\alpha \subsetneqq W_\beta$ for $\alpha > \beta$, $W_\alpha = \bigcap \{W_\beta: \beta < \alpha\}$ if α is limit and $\bigcap \{W_\alpha: \alpha < \omega_1\} = H$. We choose O_1 and V_1, non-empty disjoint clopen subsets of $Z_1 \cap W_1$, and inductively we suppose that we have defined for each $\beta < \alpha$, non-empty clopen subsets O_β and V_β of N^* such that

$$\bigcup\{O_\gamma: \gamma < \beta\} \subset O_\beta \subset \mathrm{Int}_{N^*}Z, \qquad \bigcup\{V_\gamma: \gamma < \beta\} \subset V_\beta \subset \mathrm{Int}_{N^*}Z, \tag{1}$$

$$O_\beta \cap Z_\beta \neq \emptyset, \qquad V_\beta \cap Z_\beta \neq \emptyset, \qquad (O_\beta - \bigcup\{O_\gamma: \gamma < \beta\}) \cup (V_\beta - \bigcup\{V_\gamma: \gamma < \beta\}) \subset W_\beta, \tag{2}$$

$$O_\gamma \cap V_\delta = \emptyset \quad \text{for } \gamma, \delta < \alpha. \tag{3}$$

We define O_α and V_α. We first define O'_α and V'_α as follows: If α is isolated, we put $O'_\alpha = O_{\alpha-1}$ and $V'_\alpha = V_{\alpha-1}$. Assume α is a limit ordinal. Put $\hat{O}_\alpha = \bigcup\{O_\beta: \beta < \alpha\}$ and $\hat{V}_\alpha = \bigcup\{V_\beta: \beta < \alpha\}$. Then the relation

$$\bigcup\{(N^* - W_\beta) - O_\beta: \beta < \alpha\} = (N^* - W_\alpha) - \hat{O}_\alpha$$

expresses $(N^* - W_\alpha) - \hat{O}_\alpha$ is cozero set in N^*. Thus it follows that $((N^* - W_\alpha) - \hat{O}_\alpha) \cup \hat{V}_\alpha$ and $\hat{O}_\alpha$ are disjoint cozero sets in N^*. We choose O'_α any clopen subset of N^* which contains $\hat{O}_\alpha$ and which is disjoint from $((N^* - \hat{W}_\alpha) - \hat{O}_\alpha) \cup \hat{V}_\alpha$. By exchanging O_α for V_α, we can define V'_α.

Note that $Z - O'_\alpha$ and $Z - V'_\alpha$ are zero sets in N^* whose boundaries in N^* are H. Since $Z_\alpha \cap (Z - O'_\alpha) \cap (Z - V'_\alpha) \neq \emptyset$, Int_{N^*} $(Z_\alpha \cap (Z - O'_\alpha) \cap (Z - V'_\alpha)) \neq \emptyset$ by Fact 3.1. Let S_α and T_α be non-empty clopen subsets of N^* such that

$$S_\alpha \cup T_\alpha \subset \mathrm{Int}_{N^*}(Z_\alpha \cap (Z - O'_\alpha) \cap (Z - V'_\alpha) \cap W_\alpha).$$

Let $O_\alpha = O'_\alpha \cup S_\alpha$ and $V_\alpha = V'_\alpha \cup T_\alpha$. We have choosen O_α and $V_\alpha (\alpha < \omega_1)$ satisfying the conditions (1), (2) and (3). Put

$$F_1 = \mathrm{Cl}_{N^*}(\bigcup \{O_\alpha : \alpha < \omega_1\}), \qquad G_1 = \mathrm{Cl}_{N^*}(\bigcup \{V_\alpha : \alpha < \omega_1\}).$$

Then clearly F_1 and G_1 satisfy (i), (iii) and (iv). We show (ii). Let U be any clopen subset of N^* with $U \cap H \neq \emptyset$. Then $U \cap Z$ is a non-empty zero set with $U \cap Z \cap H \neq \emptyset$. Hence $U \cap F_1 \neq \emptyset$ and $U \cap G_1 \neq \emptyset$ by (iv). This implies that H is the boundary of F_1 and G_1. The proof is completed. □

It is easy to see that every zero set Z in N^* with non-empty boundary in N^* can be expressed by the form $Z = N^* - \bigcup \{T_n^* : n \in N\}$, where $\{T_n : n \in N\}$ is pairwise disjoint infinite subsets of N and $\bigcup \{T_n : n \in N\} = N$. In the arguments below, we fix such $\{T_n : n \in N\}$. We put $F = F_1 \cup (N^* - Z)$ and $X = N \cup \{F\}$. Then clearly F is a regular closed in N^* and, by (iv) and Fact 3.3, X is strongly Fréchet. For each clopen subset O in N^*, we denote by $\tilde{O}$ a subset of N with $\tilde{O}^* = O$. We note that

$$\text{if } \alpha < \beta, \quad \text{then } O_\beta - W_\alpha \subset O_\alpha \text{ by (2)} \tag{4}$$

and

$$T_n \cap \tilde{O}_\alpha \text{ is finite for } n \in N \text{ and } \alpha < \omega_1. \tag{5}$$

We also note that each T_n and $\tilde{O}_\alpha$ converges to F in X.

A subset $A \subset X$ is closed if and only if $F \in A$, or if A meets each $\tilde{O}_\alpha$ and each T_n in a finite set.

Assertion 1. *The space X does not satisfy (α_3).*

Proof. Let $\mathscr{A} = \{T_n : n \in N\}$. Then $\mathscr{A}$ is a sheaf with vertex F. Let B be any subset of N satisfying $|\{T_n \in A : |T_n \cap B| = \aleph_0\}| = \aleph_0$. We show that B is not a convergent sequence. We note that $B^* \cap Z$ is a zero set in N^* and $H \cap (B^* \cap Z) \neq \emptyset$. Choose Z_α such that $Z_\alpha = B^* \cap Z$. Then, by (2), $V_\alpha \cap Z_\alpha \neq \emptyset$. This show that we can choose an infinite subset $C \subset B$ with $C^* \subset V_\alpha \cap Z_\alpha$. C does not converge to F by (3). The proof is completed. □

Assertion 2. *Let Y be any countably compact regular Fréchet space. Then $W = X \times Y$ is Fréchet.*

Proof. Let A be a subset of W and $(p, q) \in \mathrm{Cl}_W A - A$. We choose a convergent sequence $\{w_n: n \in N\}$ in A with $\lim_{n\to\infty} w_n = (p, q)$. If $p \neq F$, the arguments are completed trivially. Therefore we can suppose $p = F$.
Put

$$S_n = (T_n \times Y) \cap A.$$

If $(F, q) \in \mathrm{Cl}_W S_n$ for some $n \in N$, then, by Theorems 2.2 and 1.1, we can choose $\{w_n: n \in N\}$ in S_n. Therefore we assume $(F, q) \notin \mathrm{Cl}_W S_n$ for every $n \in N$. Note that we can assume $A = \bigcup\{S_n: n \in N\}$. We show that there exists O_α such that

$$(F, q) \in \mathrm{Cl}_W((\tilde{O}_\alpha \times Y) \cap A). \tag{6}$$

If such O_α exists, then, since $\tilde{O}_\alpha$ converges to F in X and Y is strongly Fréchet, the arguments are completed by using Theorem 1.1. We show assertion (6) by dividing into two cases.

Case 1. $(F, q) \notin \mathrm{Cl}_W(\bigcup\{\mathrm{Cl}_W S_n \cap (\{F\} \times Y): n \in N\})$.

Let G be an open neighborhood of q in Y such that

$$(\{F\} \times \mathrm{Cl}_Y G) \cap (\bigcup\{\mathrm{Cl}_W S_n: n \in N\}) = \emptyset. \tag{7}$$

We first show that there exists an open neighborhood U_n of F in a subspace $T_n \cup \{F\}$ such that $(U_n \times G) \cap S_n = \emptyset$ for each $n \in N$. If such U_n does not exist for some $n \in N$, then there exist $m_k \in T_n$ $(m_1 < m_2 < \cdots)$ and $y_k \in G$ such that $(m_k, y_k) \in S_n$. Since $\mathrm{Cl}_Y G$ is countably compact, an accumulation point y of the set $\{y_n: n \in N\}$ exists. Then $(F, y) \in (\{F\} \times \mathrm{Cl}_Y G) \cap \mathrm{Cl}_W S_n$. This contradicts (7).

The set $T_n - U_n$ is finite for each $n \in N$ and $(\bigcup\{T_n - U_n: n \in N\})^* \subset Z$. So there exists W_α with $W_\alpha \subset Z - (\bigcup\{T_n - U_n: n \in N\})^* \subset (\bigcup\{U_n\, n \in N\})^*$. We put $E = \bigcup\{U_n: n \in N\} \cup O_\alpha \cup \{F\}$. Note $O_\beta - (\bigcup\{U_n: n \in N\})^* \subset O_\alpha$ for $\beta \geqslant \alpha$ by (4), therefore $F = F \cap W_\alpha \cup (F - W_\alpha) \subset (\bigcup\{U_n: n \in N\})^* \cup O_\alpha$. This implication shows that E is a neighborhood of F in X. Since $(F, q) \notin \mathrm{Cl}_W((\bigcup\{U_n: n \in N\} \times G) \cap A)$, $(F, q) \in \mathrm{Cl}_W(\tilde{O}_\alpha \times Y) \cap A)$. The assertion (6) is proved in this case.

Case 2. $(F, q) \in \mathrm{Cl}_W(\bigcup\{\mathrm{Cl}_W S_n \cap (\{F\} \times Y): n \in N\})$. Using the Fréchetness of Y, we choose $(F, y_k) \in \mathrm{Cl}_W S_{n_k} \cap (\{F\} \times Y)$ with $\lim_{k\to\infty}(F, y_k) = (F, q)$. Since $S_{n_k} \subset (T_{n_k} \cup \{F\}) \times Y$ and T_{n_k} converges to F, there exists $\{w_m^k: m \in N\} \subset S_{n_k}$ with $\lim_{m\to\infty} w_m^k = (F, y_k)$. We put $w_m^k = (a_m^k, b_m^k)$ for k, $m \in N$. Let G be any open neighborhood of q in Y. Then there exists $r \in N$ such that $\{y_k: k > r\} \subset G$. Since $\{(F, b_n^k): n \in N\}$ converges to (F, y_k), there exists $m_k \in N$ such that $\{(F, b_m^k): m > m_k\} \subset \{F\} \times G$ for $k > r$. Since $\{T_n: n \in N\}$ is pairwise disjoint,

$$(\bigcup\{T_{n_k} - \{a_m^k: m > m_k: k > r\})^* \cap (\bigcup\{\{a_m^k: m > m_k\}: k > r\})^* = \emptyset.$$

On the other hand

$$\mathrm{Cl}_{N^*}(\bigcup\{a_m^k: m > m_k\}^*: k > r\}) \subset (\bigcup\{\{a_m^k: m > m_k\}: k > r\})^*$$

and the boundary of $\mathrm{Cl}_{N^*}(\bigcup\{\{a_m^k: m > m_k\}^*: k > r\})$ in N^* has non-empty intersection with H. These arguments show that $Z - (\bigcup\{T_{n_k} - \{a_m^k: m > m_k: k > r\})^*$ is a zero set in N^* whose boundary in N^* meets H. Consequently there exists $\alpha(G) < \omega_1$ such that

$$Z_{\alpha(G)} = Z - (\bigcup \{T_{n_k} - \{a^k_m : m > m_k\} : k > r\})^*.$$

Since $O_{\alpha(G)} \cap Z_{\alpha(G)} \neq \emptyset$ by (2) and, by (5), $\tilde{O}_{\alpha(G)} \cap T_n$ is finite for each $n \in N$, $\tilde{O}_{\alpha(G)}$ contains an infinite set $\{a^{k_i}_{s_i} : i \in N\}$, where $k_i \neq k_j$ if $i \neq j$. We put

$$A_{\alpha(G)} = \tilde{O}_{\alpha(G)} \cap \bigcup \{\{a^k_m : m_k\} : k > r\},$$

$$B_{\alpha(G)} = \{b^k_m : a^k_m \in A_{\alpha(G)}\}.$$

As Y is countably compact Fréchet, there exist $b_{\alpha(G)} \in \mathrm{Cl}_Y B_{\alpha(G)}$ and an infinite convergent sequence $C_{\alpha(G)} \subset B_{\alpha(G)}$ such that $\lim C_{\alpha(G)} = b_{\alpha(G)}$. We note $b_{\alpha(G)} \in \mathrm{Cl}_Y G$. Hence, by the regularity of Y,

$$q \in \mathrm{Cl}_Y \{b_{\alpha(G)} : q \in G,\ G \text{ is open in } Y\}.$$

Again, using the Fréchetness of Y, we can choose $\alpha(G_i) < \omega_1$ with

$$\lim_{i \to \infty} b_{\alpha(G_i)} = q.$$

Let $\alpha = \sup\{\alpha(G_i) : i \in N\}$. We show that O_α satisfies the assertion (6). Let E and G be open neighborhoods of F in X and q in Y, respectively. There exist $b_{\alpha(G_i)} \in G$ and $C_{\alpha(G_i)}$ such that $\lim C_{\alpha(G_i)} = b_{\alpha(G_i)}$. Since G is open, $C_{\alpha(G_i)} - G$ is finite. Then $\{a^k_m : b^k_m \in C_{\alpha(G_i)} \cap G\}$ is an infinite subset of $O_{\alpha(G_i)}$. This shows

$$E \times G \cap (\tilde{O}_\alpha \times Y) \cap A \neq \emptyset.$$

The proof of the assertion (6) is completed. □

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