A new proof of existence of equilibria in infinite normal form games

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\textbf{ABSTRACT}

In this note, we prove the existence of Nash equilibria in infinite normal form games with compact sets of strategies and continuous payoffs by constructing Nash mappings.

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\textbf{1. Introduction}

In the two seminal papers [1,2], Nash introduced a concept of equilibrium points as a natural solution concept for non-cooperative games and established the existence of equilibria in all finite games; especially, the proof in [2] based directly on the Brouwer theorem is a considerable improvement over the earlier version in [1] based on Kakutani fixed point theorem. In fact, in Nash’s unpublished Ph.D. Dissertation [3], there are two interpretations of equilibrium concept for non-cooperative games, one rationalistic and one mass-action. However, the former is “quite strongly a rationalistic and idealizing interpretation”, while the latter is more realistic and appropriate.

The mass-action view, as described by Weibull in [4], suggests that “Nash equilibria could be identified as stationary, or perhaps dynamically stable, population states in dynamic models of boundedly rational strategy adaptation in large strategically interactions populations”. In detail, consider a finite \( n \)-player game \( G \): Let \( A_i \) be the pure strategy set of players position \( i \in I = \{1, \ldots, n\} \), \( S_i \) be its mixed strategy simplex, the expected payoff to player position \( i \) when a profile \( s \in S = \Pi_{i \in I} S_i \) is played be denoted \( \pi_i(s) \), while \( \pi_{i\alpha}(s) \) denotes the payoff to player \( i \) when he uses pure strategy \( \alpha \in A_i \) against the profile \( s \in S \). Now let the game be played over and over again by individuals who are randomly drawn from infinitely large populations, one population for each player position \( i \) in the game, while a population state for any time \( t \) is then formally identical with a mixed strategy profile \( s(t) \in S \). Then, if a population state \( s \) is stationary (i.e. \( s(t) = s \) for all \( t \), or \( s_{i\alpha}(t) = 0 \) for all \( i, \alpha \)) under the following dynamics:

\[ s_{i\alpha}(t) = \pi_{i\alpha}^+(s) - s_{i\alpha} \sum_{\beta \in A_i} \pi_{i\beta}^+(s) \]

where \( \pi_{i\alpha}^+(s) = \max(\pi_{i\alpha} - \pi_i(s), 0) \), then \( s \) constitutes a Nash equilibrium, that is,

\[ \pi_{i\alpha}(s) = \max_{\beta \in A_i} \pi_{i\beta}(s) \quad (\forall i \in I). \]
It is not unique that a continuous-time analogue of the iteration mapping $T : S \to S$ defined by $T_i(s) = s_i'$ for
\[
s_i' = \frac{s_{i\alpha} + \pi_{i\alpha}(s)}{1 + \sum_{\beta} \pi_{i\beta}(s)} \quad (\forall i \in I, \alpha \in A_i),
\]
now described as Nash mapping, introduced in Nash’s [2] influential existence proof for equilibrium points is nothing else than the population dynamics given above.

Accordingly, in this note, for infinite normal form games with compact sets of strategies and continuous payoffs, we construct Nash mappings and also prove the existence of Nash equilibria by the Tychonov fixed point theorem [5], while Glicksberg [6] constructed best reply correspondences and established the existence of Nash equilibria by his generalized Kakutani fixed point theorem.

2. Nash mapping and the existence of equilibria

We consider an $n$-player infinite normal form game $f$, in which each player $i$ has a compact metric space $X_i$ of pure strategies (with metric $d_i$) and a real-valued continuous payoff function $f_i$ over $X = \Pi_{i=1}^n X_i$.

For each player $i$, denote by $S_i$ the space of mixed strategies, or probability measures on $X_i$, endowed with $\omega^*$ topology, which is a compact convex set of a locally convex linear space (see [7]). And denote by $\delta_{x_i}$ the mixed strategy which assigns probability 1 to a pure strategy $x_i \in X_i$. Let $S = \Pi_{i=1}^n S_i$ be the product space of mixed strategy profiles. For each player $i$, the expect utility function on $S$ is defined as follows: for any $\mu = (\mu_1, \ldots, \mu_n) \in S$,
\[
u_i(\mu) = \int_{X_i} f_i(x_1, x_2, \ldots, x_n) d\mu_1 d\mu_2 \ldots d\mu_n.
\]

Clearly, $\nu_i$ is continuous on $S$ by Proposition 2.1 in [8]. A mixed strategy profile $\mu \in S$ is called an equilibrium of the infinite game $f$ if for each $i$,
\[
u_i(\mu) = \max_{v_i \in S_i} \nu_i(v_i, \mu_{-i}),
\]
where the symbol $-i$ denotes “all players but $i$” given a player $i$.

Given an infinite game $f$, for each $i$, for any $\mu \in S$ and for any Borel subset $B_i$ of $X_i$, define
\[
N_i(\mu)(B_i) = \frac{\mu_i(B_i) + \int_{B_i} \max\{0, \nu_i(\delta_{x_i}, \mu_{-i}) - \nu_i(\mu)\} d\nu_i}{1 + \int_{X_i} \max\{0, \nu_i(\delta_{x_i}, \mu_{-i}) - \nu_i(\mu)\} d\nu_i},
\]
where $v_i$ is a mixed profile in $S_i$ satisfying
\[
v_i((x^k_i)) = \frac{1}{2^k}, \quad k = 1, 2, \ldots
\]
for some fixed dense countable subset $D_i = \{x^k_i : k = 1, 2, \ldots\}$ of compact metric space $X_i$. Then $N_i(\mu)$ is a mixed strategy in $S_i$. If $N_f = (N_1, N_2, \ldots, N_n)$ then $N_f : S \to S$ is a well defined mapping.

Lemma 2.1. The mapping $N_f$ is continuous on $S$ and has at least one fixed point.

Proof. Since $S$ is metrizable [7], we need only to prove that for any sequence $\{\mu^k : k = 1, 2, \ldots\}$ converging to $\mu \in S$ (under $\omega^*$), $\{N_f(\mu^k) : k = 1, 2, \ldots\}$ converges to $N_f(\mu)$ (under $\omega^*$), or for each $i$, $\{N_i(\mu^k) : k = 1, 2, \ldots\}$ converges to $N_i(\mu)$ (under $\omega^*$).

Let $g_i$ be any real-valued continuous function on $X_i$. Then, for any $k$,
\[
\int_{X_i} g_i dN_i(\mu^k) = \int_{X_i} g_i d\mu^k + \int_{X_i} g_i \max\{0, \nu_i(\delta_{x_i}, \mu_{-i}) - \nu_i(\mu^k)\} d\nu_i
\]

(cf. [9]). So the sequence \{\int_{X_i} g_i dN_i(\mu^k) : k = 1, 2, \ldots\} converges to
\[
\int_{X_i} g_i d\mu + \int_{X_i} g_i \max\{0, \nu_i(\delta_{x_i}, \mu_{-i}) - \nu_i(\mu)\} d\nu_i
\]

\[
1 + \int_{X_i} \max\{0, \nu_i(\delta_{x_i}, \mu_{-i}) - \nu_i(\mu)\} d\nu_i
\]

That is, $\{N_i(\mu^k) : k = 1, 2, \ldots\}$ converges to $N_i(\mu)$ (under $\omega^*$). Therefore, $N_i$ is continuous on $S$, and thus $N_f$ is continuous on $S$.

Since $S$ is also a compact convex set of a locally convex linear space, by the Tychonov fixed point theorem [2], there exists at least one fixed point of $N_f$. 

Lemma 2.2. A mixed strategy profile $\mu$ is an equilibrium of infinite game $f$ if and only if it is a fixed point of $N_f$.

Proof. Clearly, if $\mu \in S$ is an equilibrium of the game $f$ then it is a fixed point of $N_f$. Then, we need to prove that if $\mu$ is a fixed point of $N_f$ then it is an equilibrium of the game $f$.

Assume that $\mu$ is a fixed point of $N_f$. We claim that for each $i$,

$$\int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}v_i = 0.$$ 

Suppose that it were not. Then, there would exist some $i$ such that

$$\int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}v_i > 0.$$ 

Since $\mu$ is a fixed point of $N_f$, then for any Borel subset $B_i$ of $X_i$,

$$\mu_i(B_i) = \frac{\int_{B_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}v_i}{\int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}v_i} = \frac{\sum_{k=1}^{\infty} 2^{-k} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\}}{\sum_{k=1}^{\infty} 2^{-k} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\}}.$$ 

It is clear that $\mu_i(D_i) = 1$, which also implies that if $\mu(x^k_i) > 0$ (here $x^k_i \in D_i$) then $u_i(\delta_{x_i}, \mu_{-i}) > u_i(\mu)$. Hence $u_i(\mu_i, \mu_{-i}) = \sum_{k=1}^{\infty} \mu(x^k_i)u_i(\delta_{x_i}, \mu_{-i}) > u_i(\mu)$, the desired contradiction.\footnote{1 A referee points out a mistake in our earlier proof and suggests this version.}

We now prove that $\mu$ is an equilibrium.

Since for each $i$,

$$\int_{X_i} \max\{0, u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu)\} \mathrm{d}v_i = 0,$$ 

by the definition of $v_i$, it is easy to show that for any $x^k_i \in D_i$,

$$u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu) \leq 0.$$ 

Since $D_i$ is dense in $X_i$, then for any $x_i \in X_i$, there is a sequence $\{x^m_i \in D_i : m = 1, 2, \ldots\}$ converging to $x_i$ (under $d_i$). Clearly, the corresponding mixed strategy sequence $\{\delta_{x^m_i} : m = 1, 2, \ldots\}$ converges to $\delta_{x_i}$ (under $\omega$). Since $u_i$ is continuous,

$$u_i(\delta_{x_i}, \mu_{-i}) - u_i(\mu) \leq 0.$$ 

Therefore, $u_i(\mu) = \max_{x_i \in S} u_i(v_i, \mu_{-i})$, that is, $\mu$ is an equilibrium. \hfill \Box

Theorem 2.1. There exists at least one equilibrium of infinite game $f$.

Proof. It follows from Lemmas 2.1 and 2.2.

Since the mapping $N_f$, which has properties of Lemmas 2.1 and 2.2, is similar to the mapping $T$ which is constructed by Nash [2] to establish the existence of equilibria in a finite game, we call the mapping $N_f$ a Nash mapping of the infinite game $f$. \hfill \Box

Remark 2.1. Our method to construct a Nash mapping for an infinite game is also valid to construct a Nash mapping for a finite game, which could be of benefit to comparing Nash equilibria of a finite game.

Remark 2.2. Since the Tychonov fixed point theorem is simpler than the Glicksberg fixed point theorem, our proof of Nash equilibria in infinite games given here is an improvement over the earlier version of Glicksberg in [6].\footnote{2 The same referee suggests another simpler proof: given $\epsilon > 0$, obtain using the continuity of $u_i, \delta > 0$ corresponding to $\epsilon$ and obtain a finite open cover $\{B_i(x^k_i)\}$ of $X_i$. Consider the game $G$ where players can only choose mixed strategies supported on $[x^k_i]$. This game has a Nash equilibrium $\mu_\epsilon$, and we have that $u_i(\mu_\epsilon) > u_i(\delta_{x_i}) - \epsilon$ for all $x_i \in X_i$. The sequence $\mu_\epsilon$ obtained by setting $\epsilon = 1/j$ has a convergent subsequence and its limit point is a Nash equilibrium of the original game.}

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