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Non-conformable subgraphs of non-conformable graphs

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Abstract

We show that if G and H are non-conformable graphs, with H being a subgraph of G of the same maximum degree $\Delta(G)$, and if $\Delta(G) \geqslant \lceil \frac{1}{2}|V(G)| \rceil$, then $|V(H)| = |V(G)|$. We also show that this inequality is best possible, for when $\Delta(G) = \lfloor \frac{1}{2}|V(G)| \rfloor$ there are examples of graphs G and H with $\Delta(H) = \Delta(G)$ and $|V(H)| < |V(G)|$ which are both non-conformable. We determine all such examples. Interest in this stems from the modified Conformability Conjecture of Chetwynd, Hilton and Hind, which would characterize all graphs G with $\Delta(G) \geqslant \lceil \frac{1}{2}|V(G)| \rceil$, for which the total chromatic number $\chi_T(G)$ satisfies $\chi_T(G) = \Delta(G) + 1$, in terms of non-conformable subgraphs.

Keywords: Total colouring; Conformability

1. Introduction

In this paper, all graphs are simple, so they have no loops or multiple edges. A *total colouring* of a graph is a map $\phi : V(G) \cup E(G) \to C$, where C is a set of colours, such that any two adjacent vertices have distinct colours, any two edges incident with the same vertex have distinct colours, and any vertex and incident edge have distinct colours. The *total chromatic number* $\chi_T(G)$ is the least number of colours for which a total colouring of G exists. The total chromatic number conjecture, due independently

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to Behzad [1] and Vizing [13], is that

$$\Delta(G)+1 \leqslant \chi_T(G) \leqslant \Delta(G)+2.$$

The lower bound here is trivially true, but the upper bound remains to be proved. If $\chi_T(G)=\Delta(G)+1$ then G is *Type* 1, and if $\chi_T(G) \geqslant \Delta(G)+2$ then G is *Type* 2.

One criterion that is necessary for a graph G to be Type 1 is that G be conformable. To explain conformability, let us first define a vertex colouring of G and the deficiency of G. A $(\Delta+1)$-*vertex colouring* of G is a map $\psi : V(G) \to C$, where C is a set of $\Delta(G)+1$ colours, such that any two adjacent vertices have distinct colours. The *deficiency* of G, $\mathrm{def}(G)$, is defined by the equation

$$\mathrm{def}(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v)).$$

A graph G is said to be *conformable* if it has a vertex colouring with $\Delta(G)+1$ colours in which the number of colour classes of parity different from that of $V(G)$ is at most $\mathrm{def}(G)$ [we include here the possibility that some classes are empty]. The $(\Delta+1)$-vertex colouring itself is also called conformable. If G is not conformable, then it is said to be *non-conformable*.

Let $E_{\min}(G)$ and $E_{\max}(G)$ be the smallest and greatest, respectively, of the numbers of even colour classes in $(\Delta+1)$-vertex colourings of G. Let $O_{\min}(G)$ and $O_{\max}(G)$ be defined similarly for odd colour classes. Clearly,

$$E_{\min}(G) + O_{\max}(G) = \Delta(G)+1$$

and

$$E_{\max}(G) + O_{\min}(G) = \Delta(G)+1.$$

It is also immediate that if G is conformable and $|V(G)|$ is even, then $O_{\min}(G) \leqslant \mathrm{def}(G)$, and if G is conformable and $|V(G)|$ is odd, then $E_{\min}(G) \leqslant \mathrm{def}(G)$; indeed, this could be taken as the definition of conformability.

It is easy to see that if G is Type 1 then G is conformable:

Lemma 1. *If G is non-conformable then G is Type* 2.

Proof. In any total colouring of G with $\Delta(G)+1$ colours, each colour occurs at each vertex v of degree $\Delta(G)$, either on v itself, or on an edge incident with v. Therefore if some colour occurs on a number of vertices of parity different from that of $|V(G)|$, then it contributes at least one to the deficiency of G. It follows that the number of such colours is at most $\mathrm{def}(G)$, in other words, that G is conformable. □

Let K^*_{2n+1} denote the complete graph of odd order, $2n+1$, with one edge subdivided. Chen and Fu showed [3] that K^*_{2n+1} is a Type 2 graph that is conformable, and thereby provided a counterexample to the original form [4] of the Conformability Conjecture. We call K^*_{2n+1} a *Chen and Fu graph*; there is good reason (see [6]) to suppose that the Chen and Fu graphs are the only counterexamples to the original form of the Conformability Conjecture.

The modified form of the Conformability Conjecture [6] is:

Conformability Conjecture. Let G be a graph with $\Delta(G) \geqslant \frac{1}{2}(|V(G)|+1)$. Then G is Type 2 if and only if G contains either a non-conformable subgraph H with $\Delta(H)=\Delta(G)$, or $\Delta(G)$ is even and G contains a subgraph H with $H = K^*_{\Delta(G)+1}$.

If n is even, then $K_{n,n}$ is conformable and Type 2, showing that the bound in the conjecture cannot be lowered.

There is an increasing amount of evidence to support the Conformability Conjecture; [8] and [10] are two papers which contain quite a lot of evidence for it. Thus, since total colouring is of great interest, it behoves us to gain a good understanding of non-conformability. Conformability does not actually seem to be an easy concept to deal with. However, for even order graphs with $\Delta(G) \geqslant \frac{1}{2}|V(G)|$ there is an easy and useful equivalent formulation of conformability. Let $e(G)=|E(G)|$ and let $j(G)$ be the edge independence number of G, i.e. the maximum number of edges in any matching in G. Let $\bar{G}$ be the complementary graph of G.

Lemma 2 (see Hamilton et al. [7]). *Let G be a graph of even order $2n$ and maximum degree $\Delta(G)=\Delta$.*

(a) *If $e(\bar{G}) + j(\bar{G}) \leqslant n(2n-\Delta)-1$ then G is non-conformable.*
(b) *If G is non-conformable and has maximum degree $\Delta \geqslant n-1$, then $e(\bar{G}) + j(\bar{G}) \leqslant n(2n-\Delta)-1$.*

Part (a) of the lemma is due to Yap [11], and part (b) to Hamilton et al. [7].

We have not so far noticed any similar equivalent formulation of conformability for odd order graphs. This dichotomy points up the question as to what relationship there can be, if any, between non-conformability in even order graphs and non-conformability in odd order graphs.

There are a number of questions about conformability that one would like to know the answer to, particularly when $\Delta(G) \geqslant \frac{1}{2}(|V(G)|+1)$. One of the most fundamental is: If G is non-conformable with $\Delta(G) \geqslant \frac{1}{2}(|V(G)|+1)$ and H is a subgraph of G with $\Delta(H)=\Delta(G)$, can in fact $|V(H)|$ be $<|V(G)|$? We provide a complete answer to this question in this paper.

Theorem 3. *Let G be a graph with $\Delta(G) \geqslant \lceil \frac{1}{2}|V(G)| \rceil$ and let G be non-conformable. Let H be a subgraph of G with $\Delta(H)=\Delta(G)$, and let H also be non-conformable. Then $|V(H)|=|V(G)|$.*

The inequality $\Delta(G) \geqslant \lceil \frac{1}{2}|V(G)| \rceil$ is best possible here. This is shown by Theorems 4 and 5 and the example immediately after Theorem 5.

Before discussing these, let us mention that Theorem 3 has an important consequence for the Conformability Conjecture. Call a graph *totally-critical* if it is Type 2, connected, and $\chi_T(G-e)<\chi_T(G)$ for all $e\in E(G)$. It is shown in [8] that if we assume the total chromatic number conjecture that $\chi_T(G)\leqslant\Delta(G)+2$, then the Conformability Conjecture 1 can be restated in the following way.

Conformability Conjecture 2. Let G be a graph with $\Delta(G)\geqslant\frac{1}{2}(|V(G)|+1)$. Then G is totally critical if and only if either G is non-conformable and G contains no proper non-conformable subgraph having maximum degree $\Delta(G)$, or $\Delta(G)$ is even and G is obtained by subdividing an edge of $K_{\Delta(G)+1}$.

It follows using Theorem 3 (and Lemma 9) of this paper, that the Conformability Conjecture 2 can be put in the following simpler way.

Conformability Conjecture 3. Let G be a graph with $\Delta(G)\geqslant\frac{1}{2}(|V(G)+1)$. Then G is totally critical if and only if either G is non-conformable and $G\backslash e$ is conformable ($\forall e\in E(G)$), or $\Delta(G)$ is even and G is obtained by subdividing an edge of $K_{\Delta(G)+1}$.

To see the equivalence of these latter two conjectures, note that, by Lemma 9, if G is non-conformable then $\mathrm{def}(G)\leqslant\Delta(G)-2\leqslant(\Delta(G)+1)-3\leqslant|V(G)|-3$, so that G has at least three vertices of maximum degree, and so $\Delta(G\backslash e)=\Delta(G)$ for each $e\in E(G)$. We can now apply Theorem 3 to see the equivalence.

Now let us discuss our refinements of Theorem 3.

Theorem 4. *Let G be a graph with $\Delta(G)=\lfloor\frac{1}{2}|V(G)|\rfloor$ and let G be non-conformable. Let H be a subgraph of G with $\Delta(H)=\Delta(G)$ that is also non-conformable. If $|V(G)|=4m+1$ for some integer m then $|V(H)|=|V(G)|$. If $|V(G)|=4m+3$ for some integer m then either $|V(H)|=|V(G)|$ or $|V(H)|=2m+2$.*

In the case when $|V(G)|=4m+3$ and $|V(H)|=2m+2$ we can describe exactly when G and H are both non-conformable. This is particularly interesting in view of the dichotomy mentioned above, since $|V(G)|$ and $|V(H)|$ have opposite parities. First we need some further notation.

If H is a subgraph of G with $|V(G)|=4m+3$ for some integer m, and $|V(H)|=2m+2$ and $\Delta(G)=2m+1$, let $W=V(G)\backslash V(H)$, let $G(W)$ and $\bar{G}(W)$ be the subgraphs of G and $\bar{G}$, respectively, induced by W, let $r=|E(\bar{G}(W))|$, let w be the edge independence number of $\bar{G}(W)$, and let ξ be the number of edges of G joining vertices of W to vertices of $V(H)$.

Theorem 5. *Let $|V(G)|=4m+3$, $|V(H)|=2m+2$ and $\Delta(G)=\Delta(H)=2m+1$. The following are equivalent*:

(i) *G and H are both non-conformable*;
(ii) $e(\bar{H})+j(\bar{H})\leqslant\min\{m,\xi-r-w-1\}$.

In view of Theorem 5, it is quite easy to construct examples of non-conformable graphs G with non-conformable subgraphs H, where $|V(G)|=4m+3$, $|V(H)|=2m+2$

and $\Delta(G)=\Delta(H)=2m+1$. For H, take a K_{2m+2} and remove a small number of edges, so that $e(\bar{H})+j(\bar{H})\leqslant m$. For $G(W)$ take a disjoint K_{2m+1} and remove a small number of edges with the numerical property that $2e(\bar{H})\geqslant e(\bar{H})+j(\bar{H})+r+w+1$. Finally, insert at least $e(\bar{H})+j(\bar{H})+r+w+1$ edges joining $V(H)$ to W in such a way that the maximum degree does not increase above $2m+1$.

Finally we note the following result.

Theorem 6. *Let G be a graph with $\Delta(G)=|V(G)|-2$. Then G is conformable.*

But note that if $G=K^*_{2n+1}$, the Chen and Fu graph, then $\Delta(G)=|V(G)|-2$ and G is Type 2.

2. Proof of Theorems 3, 4 and 6

To prove Theorems 3 and 4 we need a large number of subsidiary results. Some of these are of some independent interest. Theorem 6 falls out along the way.

Lemmas 7–9 give some useful preliminary facts.

Lemma 7. *Let G be a graph with deficiency* def(G). *Then* def(G) *is odd if and only if $\Delta(G)$ and $|V(G)|$ are both odd.*

Proof.

$$\begin{aligned}\operatorname{def}(G) &= \sum_{r\in V(G)}(\Delta(G)-d_G(v))\\ &= |V(G)|\Delta(G)-\sum_{v\in V(G)}d_G(v)\\ &= |V(G)|\Delta(G)-2|E(G)|\\ &\equiv |V(G)|\Delta(G)\ (\mathrm{mod}\ 2). \qquad \square\end{aligned}$$

Lemma 8. *Let G be non-conformable. Then*

$$\operatorname{def}(G)\leqslant\begin{cases}E_{\min}(G)-2 & \text{if } |V(G)| \text{ is odd},\\ O_{\min}(G)-2 & \text{if } |V(G)| \text{ is even}.\end{cases}$$

Proof. Let G be non-conformable. If $|V(G)|$ is odd then every $(\Delta+1)$-vertex colouring contains at least def$(G)+1$ even colour classes, and thus

$$\operatorname{def}(G)\leqslant E_{\min}(G)-1.$$

In the case when $|V(G)|$ is odd and $\Delta(G)$ is odd, then, by Lemma 7, def(G) is also odd. Moreover, since $|V(G)|$ is odd, any $(\Delta+1)$-vertex colouring contains an odd number of odd colour classes, and so, since $(\Delta+1)$ is even, it contains an odd number of even colour classes. Thus $E_{\min}(G)$ is odd. Therefore,

$$\operatorname{def}(G)\leqslant E_{\min}(G)-2.$$

If $|V(G)|$ is odd and $\Delta(G)$ is even, then $\mathrm{def}(G)$ is even. Any $(\Delta+1)$-vertex colouring contains an odd number of odd colour classes, and therefore it contains an even number of even colour classes. Thus $E_{\min}(G)$ is even. Therefore, again

$$\mathrm{def}(G) \leqslant E_{\min}(G) - 2.$$

The case when $|V(G)|$ is even is similar. □

Lemma 9. *Let G be a non-conformable graph with $\Delta = \Delta(G) \geqslant 2$. Then*

$$\mathrm{def}(G) \leqslant \begin{cases} \Delta - 3 & \text{if } |V(G)| \text{ is even and } \Delta \text{ odd}, \\ \Delta - 2 & \text{otherwise}. \end{cases}$$

Proof. The lemma is clearly true if G is a complete graph or an odd circuit. Therefore, we may suppose that G is not complete and is not an odd circuit.

If $|V(G)|$ is even, then by Lemma 8 $\mathrm{def}(G) \leqslant O_{\min}(G) - 2$. By Brooks' theorem [2], G has a $(\Delta+1)$-vertex colouring with at most Δ non-empty colour classes. Since the total number of odd colour classes must be even, there are therefore at most

$$\begin{array}{ll} \Delta & \text{odd colour classes if } \Delta \text{ is even} \\ \Delta - 1 & \text{odd colour classes if } \Delta \text{ is odd.} \end{array}$$

Therefore

$$\mathrm{def}(G) + 2 \leqslant \begin{cases} \Delta & \text{if } \Delta \text{ is even}, \\ \Delta - 1 & \text{if } \Delta \text{ is odd}. \end{cases}$$

Now suppose that $|V(G)|$ is odd. Then by Lemma 8, $\mathrm{def}(G) \leqslant E_{\min}(G) - 2$. Any $(\Delta + 1)$-vertex colouring has at least one odd colour class. Therefore $E_{\min}(G) \leqslant \Delta$. Therefore

$$\mathrm{def}(G) \leqslant \Delta - 2. \quad \square$$

Lemma 10, together with its corollaries, Lemmas 11 and 12, is the key inequality which leads to the proof of Theorems 3 and 4.

If H is a subgraph of G with $\Delta(G) = \Delta(H)$ and $W = V(G) \setminus V(H)$, let

$$\mathrm{def}_W(G) = \sum_{v \in W} (\Delta(G) - d_G(v)) \quad \text{and} \quad \mathrm{def}_H(G) = \sum_{v \in V(H)} (\Delta(G) - d_G(v)).$$

Lemma 10. *Let G be a graph, let H be a subgraph of G with $\Delta(H) = \Delta(G) = \Delta$. Let $W = V(G) \setminus V(H)$, and let $1 \leqslant |W| \leqslant \Delta$. Then*

$$\mathrm{def}(G) + \mathrm{def}(H) \geqslant |W|(\Delta - |W| + 1) + 2\,\mathrm{def}_H(G).$$

Moreover, if there is equality then W induces a complete subgraph in G.

Proof. Clearly $\mathrm{def}(G)=\mathrm{def}_W(G)+\mathrm{def}_H(G)$. The number of edges of G joining W to $V(H)$ is at least

$$|W|(\Delta-|W|+1)-\mathrm{def}_W(G)$$

and is exactly this only if W induces a complete subgraph in G. Since $\mathrm{def}(H)$ is the sum of $\mathrm{def}_H(G)$ and the number of edges of G joining W to $V(H)$, therefore

$$\begin{aligned}\mathrm{def}(H) &\geqslant |W|(\Delta-|W|+1)-\mathrm{def}_W(G)+\mathrm{def}_H(G)\\ &= |W|(\Delta-|W|+1)-\mathrm{def}(G)+2\,\mathrm{def}_H(G),\end{aligned}$$

with equality only if W induces a complete subgraph in G. □

Lemma 11. *Let G be non-conformable, let H be a non-conformable subgraph of G with $\Delta(H)=\Delta(G)=\Delta$, let $W=V(G)\backslash V(H)$ and let $\Delta\geqslant|W|\geqslant 1$. Then*

$$|W|(\Delta-|W|+1)+2\,\mathrm{def}_H(G)\leqslant 2\Delta-4.$$

Proof. This follows immediately from Lemmas 9 and 10. □

Lemma 12. *Let G be non-conformable, let H be a non-conformable subgraph of G with $\Delta(H)=\Delta(G)=\Delta$, let $W=V(G)\backslash V(H)$, let $\Delta\geqslant|W|\geqslant 1$, and let $x\leqslant\min\{|W|,\Delta+1-|W|\}$. Then*

$$x(\Delta+1-x)+2\,\mathrm{def}_H(G)\leqslant 2\Delta-4.$$

Proof. Since $x\leqslant|W|$ and $x\leqslant\Delta+1-|W|$, it is clear that $|W|(\Delta-|W|+1)\geqslant x(\Delta+1-x)$. Lemma 12 now follows from Lemma 11. □

Lemma 13 is a further useful fact, easily derivable from the Hajnal–Szemerédi theorem.

Lemma 13. *Let $\Delta=\Delta(G)\geqslant\lfloor\frac{1}{2}|V(G)|\rfloor$. Then*

$$E_{\min}(G)\leqslant|V(G)|-\Delta-1\leqslant E_{\max}(G)$$

and

$$O_{\min}(G)\leqslant 2\Delta-|V(G)|+2\leqslant O_{\max}(G).$$

Proof. Since $\Delta\geqslant\lfloor\frac{1}{2}|V(G)|\rfloor$, by the Hajnal–Szemerédi theorem [5], G has a $(\Delta+1)$-vertex colouring in which each colour class is a doubleton or a singleton. Let there be d doubletons and s singletons. Then

$$d+s=\Delta+1$$

and

$$2d+s=|V(G)|,$$

so that

$$d = |V| - \Delta - 1 \quad \text{and } s = 2\Delta - |V| + 2. \qquad \square$$

Lemmas 14–17 deal with various special cases of Theorem 3.

Lemma 14. *Let G be a non-conformable graph of order $2n+1$, and let H be a non-conformable subgraph of G with $\Delta(H) = \Delta(G)$. Then $|V(H)| \neq 2n$.*

Proof. Suppose $V(H) = 2n$. Since H is non-conformable and has even order, by Lemma 8

$$\operatorname{def}(H) \leqslant O_{\min}(H) - 2.$$

Similarly

$$\operatorname{def}(G) \leqslant E_{\min}(G) - 2.$$

Let $H = G \backslash w$. Consider a $(\Delta+1)$-vertex colouring of G with $E_{\min}(G)$ even colour classes. We obtain a vertex colouring of H as follows.

First suppose that w is in an odd colour class in G. Then we have $E_{\min}(G)+1$ even colour classes in H and $O_{\max}(G) - 1$ odd colour classes in H. Therefore

$$\operatorname{def}(H) \leqslant O_{\min}(H) - 2 \leqslant O_{\max}(G) - 3,$$

so

$$\operatorname{def}(G) + \operatorname{def}(H) \leqslant E_{\min}(G) - 2 + O_{\max}(G) - 3 = \Delta - 4.$$

But by Lemma 10 (with $|W| = 1$)

$$\operatorname{def}(G) + \operatorname{def}(H) \geqslant \Delta + 2\operatorname{def}_H(G),$$

a contradiction.

If w is in an even colour class in G, then we have $E_{\min}(G) - 1$ even colour classes in H and $O_{\max}(G)+1$ odd colour classes in H. Therefore

$$\operatorname{def}(H) \leqslant O_{\min}(G) - 2 \leqslant O_{\max}(G) - 1,$$

so

$$\operatorname{def}(G) + \operatorname{def}(H) \leqslant E_{\min}(G) - 2 + O_{\max}(G) - 1 = \Delta(G) - 2,$$

which gives a similar contradiction. $\square$

Lemma 15. *Let G be a non-conformable graph of order $2n$, and let H be a non-conformable subgraph of G with $\Delta(H) = \Delta(G)$. Then $|V(H)| \neq 2n-1$.*

Proof. The proof is essentially the same as that of Lemma 14, but with obvious minor modifications throughout. $\square$

Lemmas 14 and 15 together show that if H is a subgraph of G, and G and H are both non-conformable, with $\Delta(H)=\Delta(G)$, then $|V(H)|\neq|V(G)|-1$.

We are now able to prove Theorem 6. Recall that Theorem 6 says that if G is a graph with $\Delta(G)=|V(G)|-2$, then G is conformable.

Proof of Theorem 6. This is proved in [7, Corollary 16] in the case when $|V(G)|$ is even. Suppose now that $|V(G)|$ is odd. If G is Type 1 then G is conformable. On the other hand, it is shown by Yap et al. [12] that if G is Type 2 then G contains a vertex v such that $H=G\backslash v$ is non-conformable with $\Delta(H)=\Delta(G)=|V(H)|-1$. But, by Lemma 14, G itself cannot be non-conformable. □

Lemma 16. *Let G be a non-conformable graph of order $2n$ and maximum degree $\Delta\geqslant n$. Let H be a subgraph of order $2n-2$ and maximum degree Δ. Then H is conformable.*

Proof. Suppose that H is non-conformable. By Lemma 10

$$\mathrm{def}(H)+\mathrm{def}(G)\geqslant 2(\Delta-1)+2\,\mathrm{def}_H(G)$$

provided $\Delta+1-|W|\geqslant 2$, or in other words, $\Delta\geqslant 3$. Then, if $\Delta\geqslant 3$, by Lemmas 8 and 13,

$$2\Delta-|V(G)|+2\geqslant O_{\min}(G)\geqslant \mathrm{def}(G)+2.$$

Therefore

$$2\Delta-2n\geqslant \mathrm{def}(G).$$

Therefore

$$\begin{aligned}\mathrm{def}(H)&\geqslant 2\Delta-2-\mathrm{def}(G)+2\,\mathrm{def}_H(G)\\&\geqslant 2\Delta-2-(2\Delta-2n)+2\,\mathrm{def}_H(G)\\&\geqslant 2n-2+2\,\mathrm{def}_H(G).\end{aligned}$$

But by Lemma 9,

$$\Delta-2\geqslant \mathrm{def}(H)\geqslant 2n-2,$$

so that $\Delta\geqslant 2n$, which is impossible.

If $2\geqslant\Delta$ then $2\geqslant n$, so $2=4-2\geqslant|V(H)|\geqslant\Delta+1$, which is impossible if $\Delta=2$. If $\Delta=1$ then $n=1$ and the lemma is again vacuous. □

Lemma 17. *Let G be a non-conformable graph of order $2n+1$ and maximum degree $\Delta\geqslant n$. Let H be a subgraph of order $2n-1$ and maximum degree Δ. Then H is conformable.*

Proof. Suppose that H is non-conformable. By Lemma 10,

$$\operatorname{def}(H)+\operatorname{def}(G)\geqslant 2(\Delta-1)+2\operatorname{def}_H(G)$$

as in Lemma 16, provided $\Delta+1-|W|\geqslant 2$, or in other words $\Delta\geqslant 3$. Then, by Lemmas 8 and 13,

$$|V(G)|-\Delta-1\geqslant E_{\min}(G)\geqslant \operatorname{def}(G)+2.$$

Therefore

$$2n+1-\Delta-1\geqslant \operatorname{def}(G)+2,$$

so that

$$2n-\Delta-2\geqslant \operatorname{def}(G).$$

Therefore

$$\begin{aligned}\operatorname{def}(H) &\geqslant 2\Delta-2-\operatorname{def}(G)+2\operatorname{def}_H(G)\\ &\geqslant 2\Delta-2-(2n-\Delta-2)+2\operatorname{def}_H(G)\\ &\geqslant 3\Delta-2n.\end{aligned}$$

But by Lemma 9,

$$\Delta-2\geqslant \operatorname{def}(H)\geqslant 3\Delta-2n,$$

so that

$$2n\geqslant 2\Delta+2\geqslant 2n+2,$$

which is impossible.

If $2\geqslant\Delta$, then $2\geqslant n$, so $3=5-2\geqslant|V(H)|$. Since K_3 is conformable, we cannot have $\Delta=2$ and $|V(H)|=3$. Therefore $\Delta=1=n$ so $|V(G)|=3$. This case does not arise either. □

Lemmas 14–17 together show that if H is a subgraph of G, and G and H are both non-conformable, with $\Delta(H)=\Delta(G)\geqslant\lfloor\frac{1}{2}|V(G)|\rfloor$, then $|V(H)|\notin\{|V(G)|-1, |V(G)|-2\}$.

Lemma 18 actually covers most of Theorem 3; the proof is just an application of Lemma 11 (and thus of Lemma 10). After that Lemma 19 is a special result, proved elsewhere [7], and Lemma 20 deals with a final remaining case.

Lemma 18. *Let G be a non-conformable graph with $\Delta\geqslant\lfloor\frac{1}{2}|V(G)|\rfloor$. Let H be a subgraph of G with $\Delta(H)=\Delta(G)=\Delta$. Let $W=V(G)\backslash V(H)$. Let $|W|\geqslant 3$ and $\Delta+1-|W|\geqslant 3$. Then H is conformable.*

Proof. Suppose that H is non-conformable. Then from Lemma 11 it follows that

$$3(\Delta-2)+2\operatorname{def}_H(G)\leqslant 2\Delta-4,$$

so that

$$\Delta \leqslant 2.$$

But since $|W| \geqslant 3$ and $\Delta + 1 - |W| \geqslant 3$, it follows that $\Delta + 1 \geqslant 6$, so that $\Delta \geqslant 5$, a contradiction. Therefore H is conformable. □

Lemmas 14–18 together show that if H is a subgraph of G, and H are both non-conformable with $\Delta(G) = \Delta(H)$, then $\Delta + 1 - |W| \ngeqslant 3$.

We now need a development due to Hamilton et al. [8] of a basic result of Hilton [9]. Recall that $e(G) = |E(G)|$ and $j(G)$ is the edge independence number of G.

Lemma 19. *Let $|V(G)| = 2n$ and $\Delta(G) = 2n - 1$. Then G is non-conformable if and only if $e(\bar{G}) + j(\bar{G}) \leqslant n - 1$.*

We are now ready to deal with the remaining case, namely $\Delta + 1 - |W| \leqslant 2$.

Lemma 20. *Let G be a non-conformable graph with $\Delta \geqslant \lfloor \frac{1}{2}|V(G)| \rfloor$. Let H be a subgraph of G of degree $\Delta(H) = \Delta(G) = \Delta$. Let $W = V(G) \setminus V(H)$. If $\Delta + 1 - |W| \leqslant 2$, then H is conformable, except possibly in the case when $|V(G)| = 4m+3$, $|V(H)| = 2m+2$, and $|W| = \Delta = 2m + 1$.*

Proof. Let $n = \lfloor \frac{1}{2}|V(G)| \rfloor$. We first note that $|V(H)| \geqslant \Delta + 1 \geqslant n + 1$, so $|W| = |V(G) \setminus V(H)| \leqslant (2n+1) - (n+1) = n$. Therefore $2 \geqslant \Delta + 1 - |W| \geqslant (n+1) - n = 1$, so $|W| = \Delta$ or $\Delta - 1$. Suppose H is non-conformable.

First suppose that $|W| = \Delta - 1$. Suppose initially that $|V(G)| = 2n$. Then $n + 1 \leqslant \Delta + 1 \leqslant |V(H)| = |V(G)| - |W| \leqslant 2n - \Delta + 1$, so $\Delta \leqslant n$, and so $n = \Delta$ and $|V(H)| = n + 1$. Since any complete graph of odd order is conformable, it follows that $n + 1$ is even, $n + 1 = 2m + 2$, say, so we have $|V(G)| = 4m + 2$, $|V(H)| = 2m + 2$, $|W| = 2m$ and $\Delta = 2m + 1$. Since H is non-conformable it follows from Lemma 19 that $e(\bar{H}) + j(\bar{H}) \leqslant (m + 1) - 1 = m$. Therefore $e(\bar{H}) \leqslant m - 1$, and so $\operatorname{def}(H) = 2e(\bar{H}) \leqslant 2m - 2$. Therefore, by Lemma 10,

$$\operatorname{def}(G) \geqslant (\Delta - |W| + 1)|W| - (2m - 2) = 2(2m) - (2m - 2) = 2m + 2.$$

But by Lemma 9, $\operatorname{def}(G) \leqslant \Delta - 2 = 2m - 1$, a contradiction. Therefore, if $|W| = \Delta - 1$ and $|V(G)| = 2n$, H is conformable.

Next suppose that $|W| = \Delta - 1$ and $|V(G)| = 2n + 1$. Then

$$n + 1 \leqslant \Delta + 1 \leqslant |V(H)| = |V(G)| - |W| \leqslant (2n + 1) - (n - 1) = n + 2.$$

Thus, $|V(H)| = n + 1$ or $n + 2$. If $|V(H)| = n + 2$ then $|W| = n - 1$, so $\Delta = n$. But this is not possible, for by Theorem 6, all graphs with $|V| = \Delta + 2$ are conformable. If $|V(H)| = n + 1$, then $|W| = n$ and so $\Delta = n + 1$. But this is not possible, since $|V(H)| \geqslant \Delta + 1$.

Finally suppose that $|W| = \Delta$. Then $n + 1 \leqslant \Delta + 1 \leqslant |V(H)| = |V(G)| - |W| \leqslant (2n + 1) - n = n + 1$, so $|V(H)| = n + 1$, $|W| = \Delta = n$ and $|V(G)| = 2n + 1$. As any complete graph of odd order is conformable, it follows that $n+1$ is even, $n+1=2m+2$ say, so $|V(G)|=4m+3$, $|V(H)|=2m+2$ and $\Delta=|W|=2m+1$. □

Theorems 3 and 4 now follow from Lemmas 14–18 and 20.

3. Proof of Theorem 5

The proof of Theorem 5 shows one way in which the concepts of non-conformability in odd order graphs and even order subgraphs of the same maximum degree relate to each other.

Let $C=C(4m+3,2m+2,2m+1)$ be the class of all pairs of graphs (G,H) with H an induced subgraph of G, $|V(G)|=4m+3$, $|V(H)|=2m+2$ and $\Delta(H)=\Delta(G)=2m+1$.

If $(G,H)\in C(4m+3,2m+2,2m+1)$, recall that $W=V(G)\backslash V(H)$, r is the number of edges of $\bar{G}$ with both ends in W, and ξ is the number of edges of G joining $V(H)$ to $V(G)\backslash V(H)$. Also w is the largest number of independent edges of $\bar{G}$ with both ends in W, and $j(\bar{H})$ is the largest number of independent edges of $\bar{H}$.

Lemmas 21 and 22 lead to Lemma 23, which is the 'easy' half of Theorem 5, that if inequality (ii) is satisfied then G and H are both non-conformable.

Lemma 21. *Let* $(G,H)\in C(4m+3,2m+2,2m+1)$. *Then*

$$\operatorname{def}(G)=2m+1+2r+2e(\bar{H})-2\xi.$$

Proof. Since $\Delta(G)=\Delta(H)=2m+1$ and $|V(H)|=2m+2$, the vertices of H contribute $2e(\bar{H})-\xi$ to the deficiency of G.

Since $|W|=2m+1=\Delta(H)$, the vertices of W contribute $|W|+2r-\xi$ to the deficiency of G. Therefore

$$\begin{aligned}\operatorname{def}(G)&=|W|+2r+2e(\bar{H})-2\xi\\&=2m+1+2r+2e(\bar{H})-2\xi\end{aligned}$$

as asserted. □

Lemma 22. *Let* $(G,H)\in C(4m+3,2m+2,2m+1)$. *Then*

$$E_{\min}(G)\geqslant 2m+1-2j(\bar{H})-2w.$$

Proof. We may suppose that G is given a vertex colouring in which the number of even colour classes is minimal. We may suppose that there are no empty colour classes, for if there were, we could split some colour classes of size at least two, each into two smaller colour classes without increasing the number of even colour classes.

For $1\leqslant i\leqslant\Delta+1$, let c_i^* be the number of colour classes of size i. Then

$$\sum_{i=1}^{\Delta+1}c_i^*=\Delta+1=2m+2$$

and $\sum_{i=1}^{\Delta+1} ic_i^* = 4m+3$, so that $\sum_{i=2}^{\Delta+1} (i-1)c_i^* = 2m+1$ and $E_{\min}(G) = c_2^* + c_4^* + c_6^* + \cdots$. Therefore,

$$\begin{aligned} E_{\min}(G) &= c_2^* + c_4^* + c_6^* + \cdots \\ &= (2m+1) - (2c_3^* + 4c_5^* + 6c_7^* + \cdots) - (2c_4^* + 4c_6^* + 6c_8^* + \cdots) \\ &= (2m+1) - 2(c_3^* + c_4^* + 2c_5^* + 2c_6^* + 3c_7^* + 3c_8^* + \cdots) \\ &= (2m+1) - 2\sum_{i=3}^{\Delta+1} \left\lfloor \frac{1}{2}(i-1) \right\rfloor c_i^*. \end{aligned}$$

Any colour class of size i uses at least $\lfloor \frac{1}{2}(i-1) \rfloor$ independent edges of $\bar{H}$ and $\bar{W}$ (where $\bar{W}$ stands for the restriction of $\bar{G}$ to the vertex set W). Therefore

$$\sum_{i=3}^{\Delta+1} \left\lfloor \frac{1}{2}(i-1) \right\rfloor c_i^* \leqslant j(\bar{H}) + w.$$

Therefore

$$E_{\min}(G) \geqslant (2m+1) - 2j(\bar{H}) - 2w. \qquad \square$$

We are now in a position to prove half of Theorem 5, namely that (ii) implies (i).

Lemma 23. *Let* $(G,H) \in C(4m+3, 2m+2, 2m+1)$, *and let*

$$e(\bar{H}) + j(\bar{H}) \leqslant \min\{m, \xi - r - w - 1\}.$$

Then G *and* H *are both non-conformable.*

Proof. Suppose that

$$e(\bar{H}) + j(\bar{H}) \leqslant \min\{m, \xi - r - w - 1\}.$$

Since $|V(H)| = 2m+2$, which is even, and since $e(\bar{H}) + j(\bar{H}) \leqslant m$, it follows from Lemma 19 that H is non-conformable. It remains to show that G is non-conformable. By definition, G is non-conformable if and only if $\text{def}(G) < E_{\min}(G)$, so it suffices to demonstrate this inequality.

By Lemma 22,

$$E_{\min}(G) \geqslant (2m+1) - 2j(\bar{H}) - 2w.$$

Therefore, using Lemma 22 and our hypothesis,

$$\begin{aligned} \text{def}(G) &= 2m + 1 + 2r + 2e(\bar{H}) - 2\xi \\ &= 2m + 1 - 2w + 2e(\bar{H}) - 2(\xi - r - w - 1) - 2 \end{aligned}$$

$$\leqslant 2m + 1 - 2w + 2e(\bar{H}) - 2e(\bar{H}) - 2j(\bar{H}) - 2$$

$$= (2m + 1) - 2w - 2j(\bar{H}) - 2$$

$$< E_{\min}(G).$$

Thus G is non-conformable. □

The rest of the argument is devoted to the tricky task of proving the converse of Lemma 23, i.e. of showing that (i) of Theorem 5 implies (ii). The main part of the proof will be to establish Lemma 31, according to which if $(G,H) \in C(4m+3, 2m+2, 2m+1)$ and G and H are non-conformable, then G has a vertex colouring with $\Delta + 1$ colours in which c_1, c_2 and c_3 (the number of colour classes with one, two or three vertices, respectively) is given by a certain formula. En route we need to prove a succession of increasingly strong inequalities in Corollaries 25, 27, 29 and 30. The main difficulty in the proof lies in proving these increasingly strong inequalities. They proceed by a kind of 'bootstrap argument', i.e. the first of these inequalities is refined, and later the same argument that was used to produce the first refinement is used on the situation that one now knows exists because of the first refinement to produce a further refinement.

Lemma 24. *Let* $(G,H) \in C(4m+3, 2m+2, 2m+1)$ *and let both* G *and* H *be non-conformable. Then*

$$\text{def}(G) \geqslant 1 + 2r + 2j(\bar{H}) + 2\,\text{def}_H(G).$$

Proof. Since H is non-conformable and $|V(H)| = 2m+2$, which is even, it follows from Lemma 19 that $e(\bar{H}) + j(\bar{H}) \leqslant m$, so that $e(\bar{H}) \leqslant m - j(\bar{H})$.

The following two equations are evident:

$$\xi = 2e(\bar{H}) - \text{def}_H(G) \tag{1}$$

$$\text{def}_W(G) = 2m + 1 + 2r - \xi. \tag{2}$$

From (1),

$$\xi \leqslant 2m - 2j(\bar{H}) - \text{def}_H(G).$$

Therefore, from (2),

$$\text{def}_W(G) \geqslant 2m + 1 + 2r - (2m - 2j(\bar{H}) - \text{def}_H(G))$$

$$= 1 + 2r + 2j(\bar{H}) + \text{def}_H(G).$$

Since $\text{def}(G) = \text{def}_W(G) + \text{def}_H(G)$, it now follows that

$$\text{def}(G) \geqslant 1 + 2r + 2j(\bar{H}) + 2\,\text{def}_H(G)$$

as required. □

Corollary 25. *Let* $(G,H)\in C(4m+3,2m+2,2m+1)$ *and let both G and H be non-conformable. Then*

$$m\geqslant 1+r+j(\bar{H})+\mathrm{def}_H(G).$$

Remark. This is a strong bound on r.

Proof of Corollary 25. By Lemma 9, $\mathrm{def}(G)\leqslant \Delta-2=2m-1$. Therefore,

$$2m-1\geqslant 1+2r+2j(\bar{H})+2\,\mathrm{def}_H(G),$$

so

$$2m\geqslant 2+2r+2j(\bar{H})+2\,\mathrm{def}_H(G),$$

from which the corollary follows. □

However, it is possible to strengthen Corollary 25 using the lemma below and the fact shown in Lemma 8 that $\mathrm{def}(G)\leqslant E_{\min}(G)-2$.

Lemma 26. *Let* $(G,H)\in C(4m+3,2m+2,2m+1)$ *and let G and H both be non-conformable. Let G be given a vertex-colouring with* $\Delta(G)+1=2m+2$ *colours with no empty colour classes. Let* c_i *be the number of colour classes of size i. Then*

$$E_{\min}(G)\leqslant 2m+1-2\sum_{i=3}^{\Delta+1}\left\lfloor\frac{1}{2}(i-1)\right\rfloor c_i.$$

Proof. Following the argument in Lemma 22, we have

$$E_{\min}(G)\leqslant c_2+c_4+c_6+\cdots$$

with equality if and only if the number of even colour classes is minimal. It then follows from the same line of reasoning as in Lemma 22 that

$$E_{\min}(G)\leqslant 2m+1-2\sum_{i=3}^{\Delta+1}\left\lfloor\frac{1}{2}(i-1)\right\rfloor c_i$$

as required. □

Corollary 27. *Let* $(G,H)\in C(4m+3,2m+2,2m+1)$ *and let both G and H be non-conformable. Let G be given a vertex colouring with* $\Delta(G)+1=2m+2$ *colours in which there are no empty colour classes. Then*

$$m\geqslant 1+r+j(\bar{H})+\mathrm{def}_H(G)+\sum_{i=3}^{\Delta+1}\left\lfloor\frac{1}{2}(i-1)\right\rfloor c_i.$$

Proof. By Lemma 24,

$$
\begin{aligned}
1 + 2r + 2j(\bar{H}) + 2\,\mathrm{def}_H(G) &\leqslant \mathrm{def}(G) \\
&\leqslant E_{\min}(G) - 2 \\
&\leqslant 2m + 1 - \sum_{i=3}^{\Delta+1} \left\lfloor \frac{1}{2}(i-1) \right\rfloor c_i - 2.
\end{aligned}
$$

Therefore

$$2r + 2j(\bar{H}) + 2\,\mathrm{def}_H(G) + 2\sum_{i=3}^{\Delta+1} \left\lfloor \frac{1}{2}(i-1) \right\rfloor c_i + 2 \leqslant 2m,$$

from which the corollary follows. □

We shall need the following general lemma.

Lemma 28. *Let G be a graph with no isolated vertices. Then $|V(G)| \leqslant e(G) + j(G)$.*

Proof. If $e(G) = j(G)$ then G consists of $j(G)$ independent edges, so $|V(G)| = 2j(G) = e(G) + j(G)$, so the lemma is true in this case. Now we use induction on the number of edges. Suppose that $e(G) \neq j(G)$. Then $e(G) > j(G)$ and there is an edge e such that $j(G) = j(G - e)$. Then, by induction, $|V(G - e)| \leqslant e(G - e) + j(G - e)$. Clearly $e(G - e) = e(G) - 1$; also $|V(G)| \leqslant |V(G - e)| + 1$, for otherwise e would be a further independent edge. Therefore,

$$
\begin{aligned}
|V(G)| &\leqslant |V(G-e)| + 1 \\
&\leqslant e(G-e) + j(G-e) + 1 \\
&= e(G) + j(G).
\end{aligned}
$$

The lemma now follows by induction. □

Next, we show that the term $\sum_{i=3}^{\Delta+1} \lfloor \frac{1}{2}(i-1) \rfloor c_i$ in Corollary 27 can be replaced by w. The main point in the proof is to show that the vertex colouring of Corollary 27 can be chosen so that

$$\sum_{i=3}^{\Delta+1} \left\lfloor \frac{1}{2}(i-1) \right\rfloor c_i = w.$$

Corollary 29. *Let $(G, H) \in C(4m+3, 2m+2, 2m+1)$ and let both G and H be non-conformable. Then*

$$m \geqslant 1 + w + r + j(\bar{H}) + \mathrm{def}_H(G).$$

Proof. We shall choose a vertex colouring of G with $\Delta(G)+1=2m+2$ colour classes, none of which are empty. In this vertex colouring $c_3=w$, $c_2=2m+1-2w$ and $c_1=w+1$ (then $c_1+c_2+c_3=\Delta(G)+1=2m+2$ and $c_1+2c_2+3c_3=4m+3$, so $c_i=0$ $(i\geqslant 4)$). Then $\sum_{i=3}^{\Delta+1}\lfloor\frac{1}{2}(i-1)\rfloor c_i=c_3=w$. Then, by Corollary 27, $m\geqslant 1+w+r+j(\bar{H})+\mathrm{def}_H(G)$, as required.

Let Λ be the set of ξ edges of G joining a vertex of W to a vertex of $V(\bar{H})$. Since $\Delta(G)=\Delta(H)=2m+1$ and $|V(H)|=2m+2$, the number of edges of Λ that a vertex v of $V(\bar{H})$ is incident with in G is at most $d_{\bar{H}}(v)$. Consequently vertices of $V(\bar{H})$ that are not incident with any edge of $\bar{H}$ are incident in G with no edges of Λ. By Lemma 28, the number of such vertices is at most $e(\bar{H})+j(\bar{H})$. Since H is non-conformable, $e(\bar{H})+j(\bar{H})\leqslant m$. Therefore, there are at least $m+2$ vertices of $V(\bar{H})$ that are not incident in G with any edge of Δ. But, by Corollary 25, $w\leqslant r\leqslant m-1-j(\bar{H})-\mathrm{def}_H(G)\leqslant m-1$. Let M be a set of w independent edges in the subgraph of $\bar{G}$ induced by W. Then each edge $uv\in M$ can be paired with a distinct vertex $y\in W$, with $uv,uy,vy\in E(\bar{G})$. Let each such set of three vertices, independent in G, constitute one of the colour classes of size 3. Then $c_3=w$.

Let B and $\bar{B}$ be the bipartite subgraphs of G and $\bar{G}$, respectively, whose vertex sets are, on the one side, the subset, say X, of W consisting of all vertices of W that are not in one of the c_3 chosen colour classes of size three, and, on the other side, the subset, say Y, of $V(\bar{H})$ consisting of all vertices of $V(\bar{H})$ that are, again, not in one of the colour classes of size three. Let the edge sets of B and $\bar{B}$ consist of all edges of G or $\bar{G}$, respectively, with one end vertex in X, the other end vertex in Y. It is clear that $|X|=|W|-2|M|=2m+1-2w$ and that $|Y|=|V(\bar{H})|-|M|=2m+2-w$. We shall use Hall's theorem to show that $\bar{B}$ has a matching of X into Y.

Let $G(W)$ and $\bar{G}(W)$ be the subgraphs of G and $\bar{G}$, respectively, induced by the vertices of W. If $d_{\bar{G}(W)}(v)=0$ then, since $\Delta(G)=2m+1-|W|$, v is incident with at most one edge of Λ, and so $d_B(v)=2m+2-w$ or $2m+2-w-1$. The r edges of $\bar{G}(W)$ each have at least one end vertex incident with an edge of M. Therefore, if I is an arbitrary non-empty subset of X, then

$$\sum_{v\in I} d_{G(W)}(v)\geqslant |I|2m-(r-w),$$

so

$$\sum_{v\in I}(2m+1-d_{G(W)}(v))\leqslant |I|+(r-w).$$

The number of edges of B incident with I is at most $\sum_{v\in I}(2m+1-d_{G(W)}(v))\leqslant |I|+(r-w)$. Therefore the number of edges of $\bar{B}$ incident with I is at least

$$|I|(2m+2-w)-|I|-(r-w)=|I|(2m+1-w)-(r-w).$$

Let $N_{\bar{B}}(I)$ be the set of neighbours in $\bar{B}$ of the vertices of I. The number of edges of $\bar{B}$ that are incident with at least one vertex of $N_{\bar{B}}(I)$ is at most $(2m+1-2w)|N_{\bar{B}}(I)|$. Therefore

$$|I|(2m+1-w)-(r-w)\leqslant |N_{\bar{B}}(I)|(2m+1-2w).$$

Therefore

$$|N_{\bar{B}}(I)| \geqslant |I|\,\frac{(2m+1-w)}{(2m+1-2w)} - \frac{r-w}{2m+1-2w}$$
$$\geqslant |I| - \frac{r-w}{2m+1-2w}.$$

But, by Corollary 25, $r-w=(r+w)-2w \leqslant 2r-2w \leqslant 2(m-1)-2w < 2m+1-2w$, so that $(r-w)/(2m+1-2w)<1$, and so $|N_{\bar{B}}(I)|>|I|-1$, so that $|N_{\bar{B}}(I)| \geqslant |I|$. Since I was an arbitrary non-empty subset of X, it follows by Hall's theorem that there is a matching in $\bar{B}$ of X into Y.

We take the two end vertices of each edge of such a matching in $\bar{B}$ to be a vertex colour class of G of size 2. Then $c_2=2m+1-2w$.

Each vertex that is not now in one of the c_3 colour classes of size three, or in one of the c_2 colour classes of size two, is taken as a colour class of size 1. Thus $c_1=4m+3-2c_2-3c_3=4m+3-3w-2(2m+1-2w)=w+1$. Then $c_1+c_2+c_3=2m+2$ and $c_1+2c_2+3c_3=4m+3$. Then, as explained at the start of this proof, $m \geqslant 1+w+r+j(\bar{H})+\mathrm{def}_H(G)$, as required. □

Corollary 29 admits a further strengthening in which the term $\sum_{i=3}^{\Delta+1}\lfloor\frac{1}{2}(i-1)\rfloor c_i$ in Corollary 27 is replaced by $w+j(\bar{H})$, instead of just by w as in Corollary 29.

Corollary 30. *Let $(G,H)\in C(4m+3,2m+2,2m+1)$, and let both G and H be non-conformable. Then*

$$m \geqslant 1+w+r+2j(\bar{H})+\mathrm{def}_H(G).$$

Corollary 30 follows from the following lemma.

Lemma 31. *Let $(G,H)\in C(4m+3,2m+2,2m+1)$ and let G and H be non-conformable. Then G has a vertex colouring with $\Delta(G)+1=2m+2$ non-empty colour classes in which $c_1=w+j(\bar{H})+1$, $c_2=2m+1-2w-2j(\bar{H})$ and $c_3=w+j(\bar{H})$.*

Remark. Note that in Lemma 31, $c_1+c_2+c_3=2m+2=\Delta(G)+1$ and $c_1+2c_2+3c_3=4m+3=|V(G)|$.

Proof of Corollary 30 using Lemma 31. By Lemma 31 there is a vertex colouring satisfying the conditions of Corollary 27 with $c_3=w+j(\bar{H})$ and $c_i=0$ for $i\geqslant 4$. Then $\sum_{i=3}^{\Delta+1}\lfloor\frac{1}{2}(i-1)\rfloor c_i=c_3$, so Corollary 30 now follows from Corollary 27. □

The main consequence however following from Lemma 31 is:

Lemma 32. *Let $(G,H)\in C(4m+3,2m+2,2m+1)$ and let both G and H be non-conformable. Then*

$$E_{\min}(G)=2m+1-2j(\bar{H})-2w.$$

Proof. In the vertex colouring of Lemma 31 there are just $2m+1-2j(\bar{H})-2w$ even colour classes. Now apply Lemma 22. □

Proof of Lemma 31. The argument used in the proof of Corollary 29 can be used again to find w vertices in $V(H)$ that have no edges of $\bar{H}$ incident with them, and so have no vertices of Λ incident with them either, and thus can be paired off with a set of w independent edges in $\bar{G}(W)$. Then, again, if uv is one of these edges of $\bar{G}(W)$, and y is the vertex of $V(\bar{H})$ that is paired with uv, then $\{u,v,y\}$ is a triple of independent vertices in $\bar{G}$. Each such triple is taken as a colour class of size 3. This argument yields w such colour classes, and we need to find a further $j(\bar{H})$ colour classes of size 3.

The further $j(\bar{H})$ triples of independent vertices of G each consist of two vertices $r,s\in V(\bar{H})$, where rs is one of a set of $j(\bar{H})$ independent edges in $\bar{H}$, and a vertex $t\in W$, where $tr,ts\notin E(G)$. These vertices t are chosen so as not to be among the set of the at most $r+w$ non-isolated vertices of $\bar{G}(W)$, so they are disjoint from the vertices of any of the w triples chosen so far. Also recall that the vertices in $V(\bar{H})$ that are used in the w triples are all isolated vertices of $V(\bar{H})$, and so are disjoint from the vertices in the set of $j(\bar{H})$ independent edges. Thus the new set of $j(\bar{H})$ triples will be pairwise disjoint from the old set of w triples.

We choose these $j(\bar{H})$ triples iteratively. Suppose that $x\in\{0,\dots,j(\bar{H})-1\}$ of them have been chosen. Thus x vertices from W, which do not lie in the set of at most $r+w$ non-isolated vertices of $\bar{G}(W)$, are paired with x edges from a set of $j(\bar{H})$ independent edges of $\bar{H}$. In $\bar{H}$, since e is one of a set of $j(\bar{H})$ independent edges of $\bar{H}$, there are at most $e(\bar{H})-j(\bar{H})$ edges of $\bar{H}$, other than e itself, incident with the two end vertices of e. Since H is non-conformable, $e(\bar{H})+j(\bar{H})\leqslant m$, and so $e(\bar{H})-j(\bar{H})+2\leqslant m-2j(\bar{H})+2$. Therefore, the number of vertices of W available to be paired with e is at least

$$\begin{aligned}(2m+1)-(r+w)-x-(m-2j(\bar{H})+2) &= m-1-r-w+2j(\bar{H})-x\\ &\geqslant 2j(\bar{H})-x\\ &> 0\end{aligned}$$

using Corollary 29.

Thus $j(\bar{H})$ further independent triples can be chosen. We put $c_3=w+j(\bar{H})$.

Let D and $\bar{D}$ be the bipartite subgraphs of G and $\bar{G}$, respectively, whose vertex sets are, on the one side, the subset, say R, of W consisting of all vertices of W that are not in one of the c_3 chosen colour classes of size three, and, on the other side, the subset, say S, of $V(\bar{H})$ consisting of all vertices of $V(\bar{H})$ that are, again, not in one of the colour classes of size three. Let the edge sets of D and $\bar{D}$ consist of all edges of G or $\bar{G}$, respectively, with one end vertex in R and the other end vertex in S. It is clear that $|R|=2m+1-2w-j(\bar{H})$ and $|S|=2m+2-w-2j(\bar{H})$. We shall use Hall's theorem to show that $\bar{D}$ has a partial matching of size $2m+1-2w-2j(\bar{H})$.

To apply Hall's theorem it is easiest to divide the argument into two cases.

Case 1: $w \geqslant j(\bar{H})$. Let I be an arbitrary non-empty subset of R. Slightly modifying the argument used in the proof of Corollary 29, we find that the number of edges of $\bar{D}$ incident with I is at least

$$\begin{aligned}&|I|(2m+2-2j(\bar{H})-w)-|I|-(r-w)\\ &\quad =|I|(2m+1-2j(\bar{H})-w)-(r-w).\end{aligned}$$

Letting $N_{\bar{D}}(I)$ be the set of neighbours in $\bar{D}$ of the vertices of I, the number of edges of $\bar{D}$ that are incident with at least one vertex of $N_{\bar{D}}(I)$ is at most

$$(2m+1-2w-j(\bar{H}))|N_{\bar{D}}(I)|.$$

Therefore

$$|I|(2m+1-2j(\bar{H})-w)-(r-w)\leqslant(2m+1-2w-j(\bar{H}))|N_{\bar{D}}(I)|$$

so that

$$|I|\,\frac{(2m+1-2j(\bar{H})-w)}{(2m+1-2w-j(\bar{H}))}-\frac{(r-w)}{(2m+1-2w-j(\bar{H}))}\leqslant|N_{\bar{D}}(I)|.$$

Since $w \geqslant j(\bar{H})$ it follows that $(2m+1-2j(\bar{H})-w)\geqslant(2m+1-2w-j(\bar{H}))$. We also have that $r-w=(r+w)-2w\leqslant m-1-j(\bar{H})-2w<2m+1-2w-j(\bar{H})$, using Corollary 29 and the fact that $r\geqslant w$. It follows that $|I|\leqslant|N_{\bar{D}}(I)|$. Since I was an arbitrary non-empty subset of R, it follows that there is a matching of R into S, and so *a fortiori* $\bar{D}$ has a partial matching of size $2m+1-2w-2j(\bar{H})$.

Case 2: $j(\bar{H})\geqslant w+1$. This time let I be an arbitrary non-empty subset of S. Then

$$\sum_{v\in I} d_H(v)\geqslant|I|(2m+1)-(e(\bar{H})-j(\bar{H})),$$

so that

$$\sum_{v\in I}(2m+1-d_H(v))\leqslant e(\bar{H})-j(\bar{H}).$$

The number of edges of D incident with I is at most $\sum_{v\in I}(2m+1-d_H(v))\leqslant e(\bar{H})-j(\bar{H})$. The number of edges of $\bar{D}$ incident with I is therefore at least

$$|I|(2m+1-2w-j(\bar{H}))-(e(\bar{H})-j(\bar{H})).$$

The number of edges of $\bar{D}$ that are incident with at least one vertex of $N_{\bar{D}}(I)$ is at most

$$(2m+2-w-2j(\bar{H}))|N_{\bar{D}}(I)|.$$

Therefore

$$|I|(2m+1-2w-j(\bar{H}))-(e(\bar{H})-j(\bar{H}))\leqslant(2m+2-w-2j(\bar{H}))|N_{\bar{D}}(I)|.$$

Therefore

$$|I|\frac{(2m+1-2w-j(\bar{H}))}{(2m+2-w-2j(\bar{H}))}-\frac{(e(\bar{H})-j(\bar{H}))}{(2m+2-w-2j(\bar{H}))}\leqslant|N_{\bar{D}}(I)|.$$

Since $j(\bar{H})\geqslant w+1$ it follows that $2m+1-2w-j(\bar{H})\geqslant 2m+2-w-2j(\bar{H})$. We also have that $e(\bar{H})-j(\bar{H})=e(\bar{H})+j(\bar{H})-2j(\bar{H})\leqslant m-2j(\bar{H})<2m+1-2j(\bar{H})-w$, using the fact that H is non-conformable as well as Corollary 29. Therefore $|I|\leqslant|N_{\bar{D}}(I)|$. Since I was an arbitrary non-empty subset of S it follows that there is a matching of S into R. Therefore $\bar{D}$ has a partial matching of size $2m+1-2w-2j(\bar{H})$ in this case also.

We take the two end vertices of each edge of such a partial matching of size $2m+1-2w-2j(\bar{H})$ of $\bar{D}$ to be a vertex colour class of G of size two. Then $c_2=2m+1-2w-2j(\bar{H})$.

Each vertex that is not now in one of the colour classes of size two or three is taken as a colour class of size one. Thus

$$\begin{aligned}c_1&=4m+3-2c_2-3c_3\\&=4m+3-2(2m+1-2w-2j(\bar{H}))-3(w+j(\bar{H}))\\&=1+w+j(\bar{H}).\qquad\square\end{aligned}$$

Lemma 33 is the desired converse of Lemma 23.

Lemma 33. *Let $(G,H)\in C(4m+3,2m+2,2m+1)$ and let both G and H be non-conformable. Then*

$$e(\bar{H})+j(\bar{H})\leqslant\min\{m,\xi-r-w-1\}.$$

Proof. Suppose that G and H are both non-conformable. Since $|V(H)|=2m+2$, which is even, it follows by Lemma 19 that $e(\bar{H})+j(\bar{H})\leqslant m$. It remains to show that

$$e(\bar{H})+j(\bar{H})\leqslant\xi-r-w-1.$$

By Lemma 32,

$$E_{\min}(G)=2m+1-2j(\bar{H})-2w.$$

By Lemma 8, $\operatorname{def}(G)\leqslant E_{\min}(G)-2$, and by Lemma 21, $\operatorname{def}(G)=2m+1+2r+2e(\bar{H})-2\xi$. Therefore

$$2m+1+2r+2e(\bar{H})-2\xi\leqslant 2m+1-2j(\bar{H})-2w-2.$$

Rearranging this, we obtain

$$e(\bar{H})+j(\bar{H})\leqslant\xi-r-w-1$$

as required. □

Proof of Theorem 5. Theorem 5 now follows from Lemmas 23 and 33. □

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