Discrete Mathematics 23 (1978) 17–22.

CHARACTERIZATIONS OF 2-VARIEGATED GRAPHS AND OF 3-VARIEGATED GRAPHS

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Received 13 July 1976 Revised 15 August 1977

A graph is said to be k-variegated if its vertex set can be partitioned into k equal parts such that each vertex is adjacent to exactly one vertex from every other part not containing it. We prove that a graph G on $2n$ vertices is 2-variegated if and only if there exists a set S of n independent edges in G such that no cycle in G contains an odd number of edges from S. We also characterize 3-variegated graphs.

1. Introduction

For terminology we refer to [2]. We consider finite graphs without loops or multiple edges. In [1] Bednarek and Sanders gave the following characterization of 2-variegated trees.

Theorem. (Bednarek and Sanders.) *A tree T with $2n$ vertices is 2-variegated if and only if the point independence number $\beta_0(T)$ of T is n.*

They illustrated that the condition in this characterization is neither necessary nor sufficient for an arbitrary graph to be 2-variegated and posed the problem of characterizing 2-variegated graphs and in general k-variegated graphs. We prove the following characterization of 2-variegated graphs and deduce that of Bednarek and Sanders for trees as a corollary.

2. A characterization of 2-variegated graphs

Theorem 2.1. *A graph G of order $2n$ is 2-variegated if and only if there exists a set S of n independent edges in G such that no cycle in G contains an odd number of edges from S.*

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Proof. The necessity of the condition is obvious. If G is 2-variegated with $V(G)=A_1\sqcup B_1$ as a 2-variegation then we can take for S the set of n independent edges joining a vertex of A_1 to a vertex of B_1. We will call such edges in G *special edges*.

We establish the sufficiency by induction on n. The result is trivially true for $n=1$. Suppose that if in a graph H on $2n-2$ vertices, $n\geq 2$, there exists a set $\tilde{S}$ of $n-1$ independent edges such that no cycle in H contains an odd number of edges from $\tilde{S}$ then there exists a 2-variegation $V(H)=\tilde{A}_1\sqcup\tilde{B}_1$ of H with edges from $\tilde{S}$ as special edges. Let G be a graph on $2n$ vertices, $n\geq 3$ with a set S of n independent edges in G such that no cycle in G contains an odd number of edges from S. Let $S=\{e_1,\ldots,e_n\}$ with $e_l=a_lb_l$, $1\leq l\leq n$. Let $H=G-\{a_n,b_n\}$. Obviously H is on $2n-2$ vertices and $\tilde{S}=\{e_1,\ldots,e_{n-1}\}$ is such that no cycle in H contains an odd number of edges from $\tilde{S}$. By induction hypothesis H is 2-variegated with edges from $\tilde{S}$ as special edges. Let $C_1,\ldots,C_t$ be the connected components of H. Each C_i, $1\leq i\leq t$, is 2-variegated. Let $V(C_i)=A_i\sqcup B_i$, $1\leq i\leq t$, be a 2-variegation of C_i with some edges from $\tilde{S}$ as special edges. Without loss of generality we may take

$$\bigsqcup_{i=1}^{t}A_i=\{a_l,1\leq l\leq n-1\}\quad\text{and}\quad\bigsqcup_{i=1}^{t}B_i=\{b_l,1\leq l\leq n-1\}.$$

If the edge set $E(G)=E(H)\sqcup\{e_n\}$ then $V(G)=X\sqcup Y$ is a 2-variegation of G where

$$X=\{a_n\}\sqcup A\quad\text{and}\quad Y=\{b_n\}\sqcup B,\qquad A=\bigsqcup_{i=1}^{t}A_i\quad\text{and}\quad B=\bigsqcup_{i=1}^{t}B_i.$$

Before proceeding to handle the case $E(G)\neq E(H)\sqcup\{e_n\}$ we make the following observation. If H is a 2-variegated connected graph of order $2l$ with $V(H)=A_1\sqcup B_1$ as a 2-variegation and $\tilde{S}$, the set of l independent edges between A_1 and B_1 then every xy-path, $x,y\in A_1$ (or $x,y\in B_1$) contains an even number of edges from $\tilde{S}$ and if $x\in A_1$, $y\in B_1$, say, then every xy-path contains an odd number of edges from $\tilde{S}$.

Now suppose $E(G)\neq E(H)\sqcup\{e_n\}$. Let us suppose, without loss of generality, that $aa_n\in E(G)$, $a\in A_i$ for some i, $1\leq i\leq t$. We then claim that $ba_n\notin E(G)$, $b\in B_i$. Otherwise the cycle $a_nP(a,b)\,a_n$ where $P(a,b)$ is a path from a to b in the connected component C_i, contains an odd number of edges from S, a contradiction. Similarly, b_n is not joined to any vertex in A_i. Since $C_1,\ldots,C_t$ are connected components of H, there are no edges of G between C_i and C_j, $i\neq j$, $1\leq i,j\leq t$. It now follows that, if necessary by calling A_j as B_j and B_j as A_j, we can take, without loss of generality, $V(G)=X\sqcup Y$ as a 2-variegation of G, $X=\{a_n\}\sqcup A$, $Y=\{b_n\}\sqcup B$ with S as the n independent edges between X and Y. The proof is now complete.

Let G be a graph. A set of points in G is *independent* if no two of them are adjacent. The largest number of points in such a set is called the *point independence number* of G and is denoted by $\beta_0(G)$ or β_0. Analogously, an *independent set of lines* of G has no two of its lines adjacent and the maximum cardinality of such a set is the *line independence number* $\beta_1(G)=\beta_1$ of G. A point and a line are said to *cover* each other if they are incident. A set of points which covers all the lines of a graph G is called a *point cover* for G, while a set of lines which covers all the points is a *line cover*. The smallest number of points in any point cover for G is called its *point covering number* and is denoted by $\alpha_0(G)$ or α_0. Similarly, $\alpha_1(G)$ or α_1 is the smallest number of lines in any line cover of G and is called its *line covering number.*

Corollary 2.2. (Bednarek and Sanders.) *A tree T with $2n$ vertices is 2-variegated if and only if the point independence number $\beta_0(T)$ of T is n.*

Proof. We know that for a nontrivial connected graph on p vertices $\alpha_0+\beta_0=\alpha_1+\beta_1=p$ [2, p. 95] and that for a connected bipartite graph $\beta_1=\alpha_0$ [2, p. 96]. Consequently for a connected bipartite graph on $2n$ vertices $\beta_0=n$ if and only if $\beta_1=n$.

Remarks 2.3. For an arbitrary 2-variegated graph G there is no uniqueness of 2-variegation as in the case of a 2-variegated tree. For example consider the 4-cycle $abcd$; if $\{a,b\}\sqcup\{c,d\}$ is a 2-variegation then $\{a,d\}\sqcup\{b,c\}$ is another 2-variegation. However, it can be easily shown that if G is a connected 2-variegated graph relative to a set S of n independent edges, then the partition is unique to the set S; that is if $V(G)=A_1\sqcup B_1=A_2\sqcup B_2$ and S is the set of n independent edges between A_1 and B_1 as well as between A_2 and B_2 then either (i) $A_1=A_2$, $B_1=B_2$ or (ii) $A_1=B_2$, $B_1=A_2$.

Remark 2.4. Every connected 2-variegated graph on $2n$ vertices can be obtained from a 2-variegated tree on $2n$ vertices.

3. A characterization of 3-variegated graphs

From the definition of a k-variegated graph, $k\geq 3$, it immediately follows that a graph G of order kn is k-variegated if and only if there exists k mutually disjoint sets $P_1,\ldots,P_k$ of vertices of G, $|P_i|=n$ $\forall$ i, $1\leq i\leq k$, such that each of the subgraphs of G on $P_i\sqcup P_j$, $i\neq j$, $1\leq i,j\leq k$, is 2-variegated. We give below a nontrivial characterization of 3-variegated graphs.

Theorem 3.1. *A graph G of order $3n$ is 3-variegated if and only if there exist three mutually disjoint sets S_1, S_2, S_3 of independent edges, called special edges,*

$|S_1| = |S_2| = |S_3| = n$, such that the following holds:

(i) Through every vertex there are precisely two special edges.

(ii) If $e = uv$ is an edge of G that is not a special edge and if the two special edges through u belong to S_i, S_j then the special edges through v also belong to S_i, S_j.

(iii) If $e = uv$ is a special edge of G, and if $e \in S_i$ then the other special edges through u and v belong to S_j and S_k, $j \neq k$, $i \neq j$, $i \neq k$.

Proof. Suppose that G is a 3-variegated graph of order $3n$ with $V(G) = X \sqcup Y \sqcup Z$ as a 3-variegation. Let $S_1 = S_{xy}$ be the set of n independent edges joining a vertex of X to a vertex of Y. Similarly let $S_2 = S_{yz}$, $S_3 = S_{zx}$. Then S_1, S_2, S_3 satisfy (i), (ii) and (iii) follows immediately.

Conversely, suppose that G is a graph of order $3n$ such that there exist mutually disjoint sets S_1, S_2, S_3 of n independent edges each, called special edges, satisfying (i), (ii) and (iii). We then show that G is 3-variegated.

Let $S_1 = \{a_1b_1, a_2b_2, \ldots, a_nb_n\}$. Without loss of generality assume that through a_i the other special edge is from S_2 and through b_i the other special edge is from S_3, $1 \leq i \leq n$ (see Fig. 1). Note that the other end vertices of all the edges from S_2 and S_3 together account for only the n remaining vertices. Call them $\{c_1, \ldots, c_n\}$ and suppose, without loss of generality, that $a_ic_i \in S_2$, $1 \leq i \leq n$. Now, a_ib_j, $i \neq j$, $1 \leq i, j \leq n$, cannot be an edge of G since otherwise we will get a contradiction to (ii). From hypothesis it follows immediately that a_ic_j, $i \neq j$, $1 \leq i, j \leq n$ cannot be an edge of G. If $b_ic_{\alpha_i} \in S_3$, $1 \leq i \leq n$, then it similarly follows from the hypothesis that $b_ic_{\alpha_j}$, $i \neq j$, $1 \leq i, j \leq n$, cannot be an edge of G. Let $X = \{a_1, \ldots, a_n\}$, $Y = \{b_1, \ldots, b_n\}$, $Z = \{c_1, \ldots, c_n\}$. Then $V(G) = X \sqcup Y \sqcup Z$ is a 3-variegation of G.

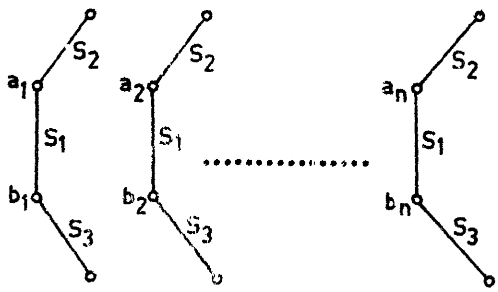

Fig. 1.

We next give a necessary condition for 3-variegation similar to that for 2-variegation given in Theorem 2.1. However, we illustrate that this condition is not sufficient for 3-variegation.

Proposition 3.2. *If a graph G of order $3n$ is 3-variegated then there exist three mutually disjoint sets S_1, S_2, S_3 of independent edges, called special edges $|S_1| = |S_2| = |S_3| = n$, such that* (A) *holds where*

(A): *If C is a cycle in G that contains $n_1 + n_2 + n_3$ special edges, n_i from S_i, $i = 1, 2, 3$ then either all n_i's are even or all n_i's are odd.*

Proof. Let $V(G)=X\sqcup Y\sqcup Z$ be a 3-variegation of G, $S_1=S_{xy}$, $S_2=S_{yz}$, $S_3=S_{zx}$. Let C be a cycle in G and let C have $n_1+n_2+n_3$ special edges, n_i from S_i, $i=1,2,3$. If $n_1+n_2+n_3=0$ then trivially (A) holds. Suppose $n_1+n_2+n_3\geqslant 1$. If $n_1=n_{xy}$, $n_2=n_{yz}$, $n_3=n_{zx}$, we say that C is of type (n_{xy}, n_{yz}, n_{zx}). Without loss of generality, let $x_i\in X$ be a vertex on C. Suppose that $x_{i+1}\in X$ is the next vertex on C from X when we travel C, say clockwise. Of course, $x_i=x_{i+1}$ is possible. We have the following three cases for distribution of vertices of G on the path P_i on C between x_i and x_{i+1}, taken clockwise.

Case 1. x_i and x_{i+1} are adjacent vertices on C. Hence P_i is of type $(0,0,0)$.

Case 2. On P_i the vertices adjacent to x_i and x_{i+1} are, one from Y and the other from Z (see Fig. 2).

$$P_i: \underset{x_i}{0}\text{———}\underset{y}{0}\cdots\underset{z}{0}\text{———}\underset{x_{i+1}}{0} \quad\text{or}\quad P_i: \underset{x}{0}\text{———}\underset{z}{0}\cdots\underset{y}{0}\text{———}\underset{x_{i+1}}{0}.$$

Fig. 2.

Case 3. On P_i the vertices adjacent to x_i and x_{i+1} are both from either Y or Z (see Fig. 3).

$$P_i: \underset{x_i}{0}\text{———}\underset{y}{0}\cdots\underset{y}{0}\text{———}\underset{x_{i+1}}{0} \quad\text{or}\quad P_i: \underset{x}{0}\text{———}\underset{z}{0}\cdots\underset{z}{0}\text{———}\underset{x_{i+1}}{0}$$

Fig. 3.

In case 2 (x_i, y) (or (y, x_{i+1})) is the only special edge from S_{xy}, (z, x_{i+1}) (or (x_i, z)) is the only special edge from S_{zx}, and all other special edges on P_i are from S_{yz} and obviously they have to be odd in number. Hence P_i is of type $(1, n_0, 1)$ where n_0 denotes an odd integer. In case 3 P_i is of the type $(2, n_e, 0)$ where n_e denotes an even integer. Summing over P_i's to get C we see that (A) holds.

Remarks 3. Consider the graph H given in Fig. 4. The graph H is not 3-variegated. For, since degree of the vertex 2 is two, the vertices 1, 2, 3 have to go in different parts in any suitable partition. Since 4 is adjacent to 1, 4 has to be in the same part as that of 1. Now, in whichever of the three parts, 5 is placed we

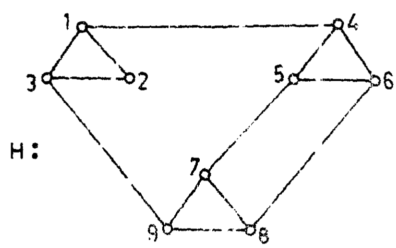

Fig. 4

arrive at a contradiction. However, with $S_1 = \{(1,3),(5,6),(7,8)\}$, $S_2 = \{(1,2),(4,5),(8,9)\}$ and $S_3 = \{(2,3),(4,6),(7,9)\}$ condition (A) of Proposition 3.2 is satisfied. It follows that condition (A), by itself, is not sufficient for a graph to be 3-variegated.

References

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