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Close-to-convexity properties of Gaussian hypergeometric functions

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Abstract

Let $\mathscr{A}$ be the class of normalized analytic functions in the unit disk Δ. Let $\phi(z)$ be either $zF(a,b;c;z)$ or $(c/ab)[F(a,b;c;z)-1]$, where $F(a,b;c;z)$ denotes the classical hypergeometric function. The purpose of this paper is to study close-to-convexity (and hence univalency) of $\phi(z)$ in the unit disc. More generally, we find conditions on a,b,c and β such that ϕ satisfies $\mathrm{Re}\, e^{i\eta}((1-z)\phi'(z)-\beta)>0$ for all $z\in\Delta$ and for some real $\eta\in(-\frac{1}{2}\pi,\frac{1}{2}\pi)$. © 1997 Elsevier Science B.V. All rights reserved.

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1. Introduction and main results

The theory of Gauss hypergeometric function

$$ {}_2F_1(a,b;c;z):=F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}z^n \quad (|z|<1), $$

which is the solution of the homogeneous hypergeometric differential equation

$$ z(1-z)w''(z)+[c-(a+b+1)z]w'(z)-abw(z)=0 $$

is fully set out in [4]. Euler, Gauss, Kummer, Riemann and Ramanujan all contributed to the theory of hypergeometric equation which appears in many situations and is connected with conformal mappings [13], quasiconformal theory [11], differential equations [9], continued fraction

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and so on. Here a,b,c are complex numbers such that $c \neq -m$, $m = 0,1,2,3,\ldots$, $(a,0)=1$ for $a \neq 0$ and, for each positive integer n, $(a,n) := a(a+1)\cdots(a+n-1)$, see [4]. In the exceptional case $c=-m$, $m=0,1,2,3,\ldots$, $F(a,b;c;z)$ is defined if $a=-j$ or $b=-j$, where $j=0,1,2,\ldots$ and $j \leqslant m$. It is clear that if $a=-m$, a negative integer, then $F(a,b;c;z)$ becomes a polynomial of degree m in z. We are concerned with the normalized hypergeometric function $f(z)=zF(a,b;c;z)$ or $(c/ab)[F(a,b;c;z)-1]$. The hypergeometric function satisfies numerous identities [1, 3, 4] and we observe that the behaviour of the hypergeometric function $F(a,b;c;z)$ near $z=1$ is classified into three cases according to $c>a+b$, $c=a+b$ and $c<a+b$, respectively:

(i) For $c>a+b$ (see [20, p. 49, 4, 24])

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty. \tag{1.1}$$

(ii) $F(a,b;a+b;z) \sim -\log(1-z)/B(a,b)$ as $z \to 1$; $B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$, see [6, 4].

(iii) $F(a,b;c;z) \sim (B(c,a+b-c)/B(a,b))(1-z)^{c-a-b}$, as $z \to 1$, $c<a+b$, see [24, p. 299, 4].

The case $c=a+b$ is called the *zero-balanced.* When $z=x$, $x \in (0,1)$, Cases (ii) and (iii) above have been extended and improved in [2, 18], see also [3, 4]. In this paper, we focus our attention to study the geometric nature of the hypergeometric function, in particular the univalency part. In [19, 15], examples have been constructed to demonstrate that in each of the above three cases there exist functions of the form $zF(a,b;c;z)$ or $F(a,b;c;z)$, containing univalent as well as nonunivalent functions. However, the exact range of the parameters (a,b,c) for which $zF(a,b;c;z)$ or $F(a,b;c;z)$ is univalent remains unknown [15, 19, 21, 23, 12, 22]. Our theorems are basically related with certain subfamilies of the family $\mathscr{A}$ of all normalized analytic functions f $(f(0)=0=f'(0)-1)$ in the unit disc Δ, and so we include here some basic definitions and notations: denote by $\mathscr{S}$, $\mathscr{K}(\beta)$, $\mathscr{S}^*(\beta)$ the subclasses of $\mathscr{A}$ that consist of functions that are univalent, convex of order $\beta<1$, and starlike of order $\beta<1$, respectively. We write $\mathscr{K}=\mathscr{K}(0)$, $\mathscr{S}^*=\mathscr{S}^*(0)$ and it is a well-known fact that $f \in \mathscr{K}(\beta)$ if and only if $zf' \in \mathscr{S}^*(\beta)$. We also introduce the class of close-to-convex functions. According to a standard analytic definition, a function $f \in \mathscr{A}$ is said to be *close-to-convex of order $\beta<1$ with respect to a fixed starlike function g* if and only if

$$\operatorname{Re}\left[e^{i\eta}\left(\frac{zf'(z)}{g(z)}-\beta\right)\right]>0, \quad z \in \Delta, \tag{1.2}$$

for some real $\eta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$. The family of close-to-convex functions of order β relative to $g \in \mathscr{S}^*$ is denoted by $\mathscr{C}(\beta;g)$. If $\eta=0$, we simply denote it by $\mathscr{C}_0(\beta;g)$. Thus, we remark that the usual class of all close-to-convex functions of order β, denoted by $\mathscr{C}(\beta)$, is the set $\{\mathscr{C}(\beta;g)\colon g \in \mathscr{S}^*\}$. Set $\mathscr{C}(0)=\mathscr{C}$. It is important to note that the chain of proper inclusions for $0 \leqslant \beta < 1$: $\mathscr{S}^*(\beta) \subset \mathscr{C}(\beta) \subset \mathscr{S}$. For general properties of these classes of functions, we refer to the book by Duren [5]. For $0 \leqslant \beta < 1$, we also introduce the class

$$\mathscr{P}(\beta) = \{p(z)\colon \exists \eta \in \mathbb{R},\ zp \in \mathscr{A},\ \text{such that } p(0)=1,\ \operatorname{Re}[e^{i\eta}(p(z)-\beta)]>0,\ z \in \Delta\}$$

and define

$$\mathscr{R}(\beta) = \{f \in \mathscr{A}\colon f'(z) \in \mathscr{P}(\beta)\}.$$

When $\eta=0$, we denote $\mathscr{P}(\beta)$ and $\mathscr{R}(\beta)$ simply by $\mathscr{P}_0(\beta)$ and $\mathscr{R}_0(\beta)$, respectively. Clearly $\mathscr{C}(\beta;z) \equiv \mathscr{R}(\beta)$ for $\beta<1$, and therefore, if $0 \leqslant \beta < 1$, we have that $\mathscr{R}(\beta)$ is included in $\mathscr{C}$, but not in $\mathscr{S}^*$, and neither is the smaller class $\mathscr{R}_0(\beta)$. The question about the inclusion of $\mathscr{R}_0(\beta)\subset\mathscr{S}^*$ was raised in [25], and settled in the negative in [10]. There has been considerable interest to study the properties of the transformations of the type $f\mapsto V_{a,b;c}(f):=zF(a,b;c;z)*f(z)$. For $\operatorname{Re}c>\operatorname{Re}b>0$, we have the representation

$$V_{a,b;c}(f):=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left(\frac{z}{(1-tz)^a}*f(z)\right)\mathrm{d}t \tag{1.3}$$

and this operator has been studied in [7, 16, 17]. Our attempt in this paper is to study the operator $V_{a,b;c}(f)$ but only for the special choice $f(z)=z/(1-z)$ which is the extremal functions together with its rotations for the convex family $\mathscr{K}$. Then in this case, it is clear that the transform becomes the normalized hypergeometric function: $V_{a,b;c}(z/(1-z))=zF(a,b;c;z)$. We note that the function $z/(1-z)$ is not included in $\mathscr{R}_0(0)$. However, the problems of finding the exact range of the parameters (a,b,c) for the function $F(a,b;c;z)$ or $(c/ab)[F(a,b;c;z)-1]$ to be univalent, starlike, close-to-convex, or convex remain open. For partial answers and the latest improvements to these questions, I refer to [15, 19, 23, 12, 22]. In this connection, the authors in [19] found sufficient conditions for the function $zF(a,b;c;z)$ to belong to $\mathscr{C}_0(0,z/(1-z))$ and hence univalent in Δ. Now we state one of our main results which by a slightly different method of proof improves the result of [19].

Theorem 1. *Suppose that a,b and $\beta<1$ are associated by any one of the following conditions*:
(1) $a\in(0,\infty)$, $b\in(0,1/a]$ *and* $\beta\leqslant 1-(1/\cos\eta)(1-\Gamma(a+b)/\Gamma(a)\Gamma(b))$.
(2) $a\in(\frac{1}{2},\infty)$, $b\in[a/(2a-1),\infty)$ *and* $\beta\leqslant 1-(1/\cos\eta)(\Gamma(a+b)/\Gamma(a)\Gamma(b)-1)$.
(3) $\operatorname{Re}a>0$, $|a|\leqslant\min\{1,\sqrt{\operatorname{Re}a}\}$, $b=\bar a$ *and* $\beta\leqslant 1-(1/\cos\eta)(1-\Gamma(2\operatorname{Re}a)/\Gamma(a)\Gamma(\bar a))$.
(4) $\operatorname{Re}a>0$, $|a|\geqslant\max\{1,\sqrt{\operatorname{Re}a}\}$, $b=\bar a$ *and* $\beta\leqslant 1-(1/\cos\eta)(\Gamma(2\operatorname{Re}a)/\Gamma(a)\Gamma(\bar a)-1)$.
(5) $c\geqslant\max\{a+b,a+b+(ab-1)/4,(3(a+b+ab)-1)/4\}$ *and* $\beta\leqslant 1-(|c-2ab|+2ab)/(c\cos\eta)$, *where a,b satisfy either $a,b>0$, or $a\in\mathbb{C}\setminus\{0\}$, $b=\bar a$.*
Then the function $zF(a,b;a+b;z)$ belongs to $\mathscr{C}(\beta;g)$ with $g(z)=z/(1-z)$.

The special case $\eta=\beta=0$ in Parts (1) and (2) of Theorem 1 for $a,b,c>0$ has been obtained by the author in [19] and the proof of this theorem will be given in Section 2

Example 2. Let $f(z)=zF(a,b;a+b;z)$ and $g(z)=zF(a,\bar a;2\operatorname{Re}a;z)$. Then taking $\eta=0$ in Theorem 1 we have the following results which improve on Theorem 2.1 in [19]:
(i) $f(z)\in\mathscr{C}_0(1/B(a,b);z/(1-z))$ if $a>0$ and $b\in(0,1/a]$.
(ii) $f(z)\in\mathscr{C}_0(2-1/B(a,b);z/(1-z))$ if $a>\frac{1}{2}$ and $b\geqslant a/(2a-1)$.
(iii) $g(z)\in\mathscr{C}_0(1/B(a,\bar a);z/(1-z))$ if $\operatorname{Re}a>0$ and $|a|\leqslant\min\{1,\sqrt{\operatorname{Re}a}\}$.
(iv) $g(z)\in\mathscr{C}_0(2-1/B(a,\bar a);z/(1-z))$ if $\operatorname{Re}a>0$ and $|a|\geqslant\max\{1,\sqrt{\operatorname{Re}a}\}$.

It is easy to give sufficient coefficient conditions for f to belong the class $\mathscr{C}(\beta;g)$, at least when $g(z)\in\mathscr{S}^*$ takes one of the following forms:

$$z,\quad \frac{z}{1\pm z},\quad \frac{z}{1\pm z^2},\quad \frac{z}{(1\pm z)^2}\quad\text{or}\quad \frac{z}{1\pm z+z^2},$$

so that $z/g(z)$ takes one of the equivalent forms

$$1, \quad 1\pm z, \quad 1\pm z^2, \quad (1\pm z)^2 \quad \text{or} \quad 1\pm z+z^2,$$

respectively. According to Frideman [8], these are the only nine functions of the class $\mathcal{S}$ whose coefficients are rational integers. Using these we first state and prove the following simple results which in particular give sufficient conditions for the univalency of the normalized analytic functions.

Proposition 3. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$. Then we have the following*:

(i) $\sum_{n\geqslant 1} |na_n-(n+1)a_{n+1}|\leqslant(1-\beta)\cos\eta$ *implies that* $f\in\mathscr{C}(\beta;g)$ *with* $g(z)=z/(1-z)$.

(ii) $\sum_{n\geqslant 1} |(n-1)a_{n-1}-2na_n+(n+1)a_{n+1}|\leqslant(1-\beta)\cos\eta$ *implies that* $f\in\mathscr{C}(\beta;g)$ *with* $g(z)=z/(1-z)^2$.

(iii) $\sum_{n\geqslant 1} |(n-1)a_{n-1}-(n+1)a_{n+1}|\leqslant(1-\beta)\cos\eta$ *implies that* $f\in\mathscr{C}(\beta;g)$ *with* $g(z)=z/(1-z^2)$.

(iv) $\sum_{n=1}^{\infty} |(n-1)a_{n-1}-na_n+(n+1)a_{n+1}|\leqslant(1-\beta)\cos\eta$ *implies that* $f\in\mathscr{C}(\beta;g)$ *with* $g(z)=z/(1-z+z^2)$.

Proof. (i) Suppose that $g(z)=z/(1-z)$ and f satisfies the condition

$$\sum_{n\geqslant 1} |na_n-(n+1)a_{n+1}|\leqslant(1-\beta)\cos\eta.$$

Then for $|z|<1$, we can write

$$\begin{aligned}\operatorname{Re} e^{i\eta}\left(\frac{zf'(z)}{g(z)}-\beta\right) &= \operatorname{Re} e^{i\eta}[(1-z)f'(z)-\beta] \\ &= (1-\beta)\cos\eta-\operatorname{Re} e^{i\eta}\left(\sum_{n\geqslant 1}(na_n-(n+1)a_{n+1})z^n\right) \\ &> (1-\beta)\cos\eta-\sum_{n\geqslant 1}|na_n-(n+1)a_{n+1}|\geqslant 0.\end{aligned}$$

Therefore, $f\in\mathscr{C}(\beta;g)$ with $g(z)=z/(1-z)$.

The remaining parts follow on the similar lines of the proof of part (i). □

Parts (i)–(iii) of the special case $\eta=0$ and $\beta=0$ of Proposition 3 are due to [14]. Also, we point out that it would not be difficult to state sufficient conditions, such as in Proposition 3, for f to belong to $\mathscr{C}(\beta;g)$ at least when the choice of $g\in\mathcal{S}^*$ satisfies a property that $z/g(z)$ is a polynomial function and $\lim_{z\to 1}(z/g(z))=1$. Using Proposition 3, we can easily draw the following corollaries, and these are in some sense important in special circumstances, see [15, 16, 19].

Corollary 4. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$. Suppose that*

$$1\geqslant 2a_2\geqslant\cdots\geqslant na_n\geqslant\cdots\geqslant 1-(1-\beta)\cos\eta \tag{1.4}$$

or

$$1 \leqslant 2a_2 \leqslant \cdots \leqslant na_n \leqslant \cdots \leqslant 1 + (1-\beta)\cos\eta. \tag{1.5}$$

Then $f \in \mathscr{C}(\beta; g)$ with $g(z) = z/(1-z)$.

Corollary 5. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Suppose that*

$$1 \geqslant 2a_2 - 1 \geqslant 3a_3 - 2a_2 \geqslant \cdots \geqslant (n+1)a_{n+1} - na_n \geqslant \cdots \geqslant 1 - (1-\beta)\cos\eta \tag{1.6}$$

or

$$1 \leqslant 2a_2 - 1 \leqslant 3a_3 - 2a_2 \leqslant \cdots \leqslant (n+1)a_{n+1} - na_n \leqslant \cdots \leqslant 1 + (1-\beta)\cos\eta. \tag{1.7}$$

Then $f \in \mathscr{C}(\beta; g)$ with $g(z) = z/(1-z)^2$.

Corollary 6. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If one of the following four conditions,*

$$1 \geqslant 3a_3 \geqslant \cdots \geqslant (2n+1)a_{2n+1} \geqslant \cdots \geqslant 2a_2 \geqslant \cdots \geqslant 2na_{2n} \geqslant \cdots \geqslant 1 - (1-\beta)\cos\eta,$$

$$1 \leqslant 3a_3 \leqslant \cdots \leqslant (2n+1)a_{2n+1} \leqslant \cdots \leqslant 2a_2 \leqslant \cdots \leqslant 2na_{2n} \leqslant \cdots \leqslant 1 + (1-\beta)\cos\eta,$$

$$1 \geqslant 3a_3 \geqslant \cdots \geqslant (2n+1)a_{2n+1} \geqslant \cdots \geqslant 2na_{2n} \geqslant \cdots \geqslant 2a_2 \geqslant 1 - (1-\beta)\cos\eta,$$

$$1 \leqslant 3a_3 \leqslant \cdots \leqslant (2n+1)a_{2n+1} \leqslant \cdots \leqslant 2na_{2n} \leqslant \cdots \leqslant 2a_2 \geqslant 1 + (1-\beta)\cos\eta$$

is satisfied, then $f \in \mathscr{C}(\beta; g)$ with $g(z) = z/(1-z^2)$.

Now we are in a position to state our next result and the proof of the following theorem will be given in Section 2.

Theorem 7. *Suppose that a, b and c are related by any one of the following conditions:*

(1) $a, b \in [1, \infty)]$, $c = a+b-1$ *and* $\beta \leqslant 1 - (1/\cos\eta)\left(\dfrac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)} - 1\right)$,

(2) $a \in (0,1)$, $b \in (1-a, 1)$, $c = a+b-1$ *and* $\beta \leqslant 1 - (1/\cos\eta)\left(\dfrac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)} - 1\right)$,

(3) $a \in (0,1)$ $b \in (1,\infty)$, $c = a+b-1$ *and* $\beta \leqslant 1 - (1/\cos\eta)\left(1 - \dfrac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}\right)$,

(4) $a, b \in (1,\infty)$, *or* $a, b \in (0,1)$, $c \geqslant ab$ *and* $\beta \leqslant 1 - 1/\cos\eta$,

(5) $a \in (1,\infty)$ *and* $b \in (0,1]$, $c > a+b-1$ *and* $\beta \leqslant 1 - 1/\cos\eta$,

(6) $\mathrm{Re}\, a > \frac{1}{2}$, $b = \bar{a}$, $c = 2\mathrm{Re}\, a - 1$ *and* $\beta \leqslant 1 - (1/\cos\eta)\left(\dfrac{\Gamma(2\mathrm{Re}\, a - 1)}{\Gamma(a)\Gamma(\bar{a})} - 1\right)$,

(7) $a \in \mathbb{C}\backslash\{0,1\}$, $b = \bar{a}$, $0 \neq c \geqslant \{0, |a^2|, 2\mathrm{Re}\, a - 1\}$ *and* $\beta \leqslant 1 - 1/\cos\eta$.

Then for $f(z)=zF(a,b;c;z)$ *the Alexander transform* Λ_f *of the function* f *defined by*

$$\Lambda_f(z)=\int_0^z \frac{f(t)}{t}\,\mathrm{d}t=\sum_{n=1}^{\infty}\frac{A_n}{n}z^n$$

is in $\mathscr{C}(\beta;g)$ *with* $g(z)=z/(1-z)$.

Corollary 8. *Suppose that a and b are related by any one of the following conditions*:

(1) $a\in(-1,0]$, $b\in[0,\infty)]$ *and* $\beta\leqslant 1-(1/\cos\eta)\left(1-\dfrac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}\right)$,

(2) $a\in(-1,0]$, $b\in(-1-a,0]$ *and* $\beta\leqslant 1-(1/\cos\eta)\left(\dfrac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}-1\right)$,

(3) $a\in[0,\infty)$, $b\in(0,\infty)$ *and* $\beta\leqslant 1-(1/\cos\eta)\left(\dfrac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}-1\right)$,

(4) $\mathrm{Re}\,a>-\frac{1}{2}$, $b=\bar{a}$ *and* $\beta\leqslant 1-(1/\cos\eta)\left(\dfrac{\Gamma(2\mathrm{Re}\,a+1)}{\Gamma(a+1)\Gamma(\bar{a}+1)}-1\right)$.

Then the function $((a+b)/ab)[F(a,b;a+b;z)-1]$ *is in* $\mathscr{C}(\beta;g)$ *with* $g(z)=z/(1-z)$.

Proof. We note that for $g(z)=zF(a+1,b+1;c+1;z)$ we have

$$\Lambda_g(z)=\int_0^z \frac{g(t)}{t}\,\mathrm{d}t=(c/ab)[F(a,b;c;z)-1]$$

and therefore the required conclusion follows from Theorem 7. □

Example 9. Let $f(z)=((a+b)/ab)[zF(a,b;a+b;z)-1]$, $g(z)=(2\mathrm{Re}\,a/|a|^2)[F(a,\bar{a};2\mathrm{Re}\,a;z)-1]$, and $\beta(a,b)=\Gamma(a+b+1)/\Gamma(a+1)\Gamma(b+1)$. Then taking $\eta=0$ in Corollary 8 we have the following results which extend and improve the theorems in [15].

(i) $f(z)\in\mathscr{C}_0(\beta(a,b);z/(1-z))$ if $a\in(-1,0)$ and $b>0$.
(ii) $f(z)\in\mathscr{C}_0(2-\beta(a,b);z/(1-z))$ if $a\in(-1,0]$ and $b\in[-1-a,0)$.
(iii) $f(z)\in\mathscr{C}_0(2-\beta(a,b);z/(1-z))$ if $a,b>0$.
(iv) $g(z)\in\mathscr{C}_0(2-\beta(a,\bar{a});z/(1-z))$ if $\mathrm{Re}\,a>-\frac{1}{2}$.

To cover the situation where $c>a+b$, we state the following theorem without proof as it follows in the same lines of proof of Theorem 7.

Theorem 10. *Suppose that any one of the following conditions is satisfied*:
(1) $a\in(-1,0)$, $b\in(-1,0)$ *and* $c\geqslant a+b+ab$.
(2) $a\in(0,\infty)$, $b\in(0,\infty)$ *and* $c\geqslant a+b+ab$.
(3) $a\in\mathbb{C}\backslash\{-1\}$, $b=\bar{a}$ *and* $0\neq c\geqslant \max\{-1,2\mathrm{Re}\,a+|a|^2\}$ *with* $\beta=0$.

Then the function $(c/ab)[F(a,b;c;z)-1]$ *belongs to* $\mathscr{C}(\beta;z/(1-z))$ *with* $\beta\leqslant 1-1/\cos\eta$.

To emphasize the importance of dealing with complex values for a and b, we give below two examples. We remark that the examples of this type are not available in the literature.

Examples 11. (1) Choose $\beta=0=\eta$ and $a=-m$, where $m\geqslant 2$ is a positive integer. Then from Theorem 10(3) we deduce that if $0\neq c\geqslant m(m-2)$, the polynomial

$$\frac{c}{m^2}[F(-m,-m;c;z)-1]=z+\sum_{n=2}^{m}\frac{|(-m+1,n-1)|^2}{(c+1,n-1)(1,n)}z^n$$

is close-to-convex with respect to $z/(1-z)$ and hence univalent in Δ.

(2) Choose $\beta=0=\eta$ and $a=\mathrm{i}d$, where d is a nonzero real number. Then from Theorem 10(3) we deduce that if $c\geqslant d^2$, the function

$$\frac{c}{d^2}[F(\mathrm{i}d,-\mathrm{i}d;c;z)-1]=z+\sum_{n=2}^{\infty}\frac{|(1+\mathrm{i}d,n-1)|^2}{(c+1,n-1)(1,n)}z^n$$

is close-to-convex with respect to $z/(1-z)$ and hence univalent in Δ.

Finally, we state the following theorem without proof as it follows if we use Corollary 6 and adopt the method of proof of Theorem 7.

Theorem 12. *Suppose that a,b and $\beta<1$ are associated by any one of the following conditions*:
(1) $\operatorname{Re}a>0$, $|a|\leqslant\min\{1/\sqrt{2},\sqrt{(2\operatorname{Re}a/3)}\}$, $b=\bar a$ *and* $\beta\leqslant 1-(1/\cos\eta)(1-(2\Gamma(2\operatorname{Re}a)/\Gamma(a)\Gamma(\bar a)))$.
(2) $\operatorname{Re}a>0$, $|a|\geqslant\max\{1/\sqrt{2},\sqrt{(2\operatorname{Re}a/3)}\}$, $b=\bar a$, *and* $\beta\leqslant 1-(1/\cos\eta)(2\Gamma(2\operatorname{Re}a)/\Gamma(a)\Gamma(\bar a)-1)$.
(3) $a\in(0,\frac{1}{3}]$ *and* $b\in(0,1/2a]$ (*or* $a\in(\frac{1}{3},\infty)$ *and* $b\leqslant\min\{1/2a,a/(3a-1)\}$), *and β is given by*

$$\beta\leqslant 1-\frac{1}{\cos\eta}\left(1-\frac{2\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right).$$

(4) $a\in(\frac{1}{3},\infty)$ *and* $b\geqslant\max\{1/2a,a/(3a-1)\}$, *and* $\beta\leqslant 1-(1/\cos\eta)((2\Gamma(a+b)/\Gamma(a)\Gamma(b))-1)$.
Then the odd hypergeometric function $zF(a,b;a+b;z^2)$ belongs to $\mathscr{C}(\beta;z/(1-z^2))$.

Theorem 12 shows that the function $zF(\frac{1}{2},\frac{1}{2};1;z^2)=(2/\pi)zK(z)$, where $K(z)=\int_0^{\pi/2}(1-z^2\sin^2 t)^{-1/2}\,\mathrm{d}t$, is in $\mathscr{C}(\beta;z/(1-z^2))$ with $\beta=1-(\pi-2)/(\pi\cos\eta)$.

2. Proofs of main theorems

Proof of Theorem 1. Consider $zF(a,b;c;z)=z+\sum_{n=2}^{\infty}A_nz^n$ where $A_1=1$ and for $n\geqslant 2$,

$$A_n=\frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}. \tag{2.1}$$

Then

$$nA_n-(n+1)A_{n+1}=\frac{(a,n-1)(b,n-1)}{(c,n1)(1,n)}X(n),$$

where

$$X(n) = n^2(c-a-b) + n(1-ab) - (a-1)(b-1).$$

Now we let $T := \sum_{n\geqslant 1} |nA_n - (n+1)A_{n+1}|$ and therefore, by Proposition 3(i), it sufficies to show that $T \leqslant (1-\beta)\cos\eta$. We first deal with the case $c = a+b$ and divide the proof into two parts.

(1) If a, b are related by (1) then for all $n \geqslant 1$, $X(n) = n(1-ab) - (a-1)(b-1) \geqslant 0$ so that

$$\begin{aligned} T &= \sum_{n=1}^{\infty} \frac{(a,n-1)(b,n-1)}{(a+b,n)(1,n)}[n(1-ab)-(a-1)(b-1)] \\ &= \frac{1-ab}{a+b}\sum_{n=1}^{\infty}\frac{(a,n-1)(b,n-1)}{(a+b+1,n-1)(1,n-1)} - \sum_{n=1}^{\infty}\frac{(a-1,n)(b-1,n)}{(a+b,n)(1,n)} \\ &= \frac{1-ab}{a+b}\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} - \left(\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b+1)} - 1\right), \quad \text{by (1.1)}, \end{aligned}$$

which gives

$$T = 1 - \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

(2) If a, b are related by (2) then for all $n \geqslant 1$, $X(n) = n(1-ab) - (a-1)(b-1) \leqslant 0$ so that

$$T = -\sum_{n=1}^{\infty} \frac{(a,n-1)(b,n-1)}{(a+b,n)(1,n)}[n(1-ab)-(a-1)(b-1)]$$

which gives

$$T = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} - 1.$$

Therefore the conclusions for (1) and (2) of Theorem 1 follow from the above observations. Cases (3) and (4) can be obtained from the above two cases.

(5) Now we shall deal with the case $c > a+b$. Assume the hypothesis that $c \geqslant \max\{a+b, a+b+(ab-1)/4, (3(a+b+ab)-1)/4\}$. Using this, it can be easily seen that the function $X(n)$ is increasing for $n \geqslant 2$, and that $X(n) \geqslant X(2) \geqslant 0$ for all $n \geqslant 2$. Now, we rewrite the expression for $X(n)$ and T as

$$X(n) = (c-a-b)n(n-1) + (c-a-b-ab+1)n - (a-1)(b-1)$$

and

$$T = \frac{|c-2ab|}{c} + T_1 + T_2 + T_3,$$

where, using (1.1), we compute

$$T_1 = \frac{(c-a-b)ab}{c(c+1)} \sum_{n=2}^{\infty} \frac{(a+1,n-2)(b+1,n-2)}{(c+2,n-2)(1,n-2)}$$
$$= \frac{ab\Gamma(c)\Gamma(c+1-a-b)}{\Gamma(c+1-a)\Gamma(c+1-b)},$$
$$T_2 = \frac{(c-a-b-ab+1)}{c} \sum_{n=2}^{\infty} \frac{(a,n-1)(b,n-1)}{(c+1,n-1)(1,n-1)}$$
$$= (c-a-b-ab+1)\left(\frac{\Gamma(c)\Gamma(c+1-a-b)}{\Gamma(c+1-a)\Gamma(c+1-b)} - \frac{1}{c}\right),$$
$$T_3 = -\sum_{n=2}^{\infty} \frac{(a-1,n)(b-1,n)}{(c,n)(1,n)}$$
$$= -\left(\frac{(c+1-a-b)\Gamma(c)\Gamma(c+1-a-b)}{\Gamma(c+1-a)\Gamma(c+1-b)} - 1 - \frac{(a-1)(b-1)}{c}\right).$$

Simplifying the value of the sum $T_1 + T_2 + T_3$, we find that

$$T = \frac{|c-2ab| + 2ab}{c}$$

and the conclusion is immediate from Proposition 3(i). □

Proof of Theorem 7. Consider $\Lambda_f(z) = \sum_{n=1}^{\infty}(A_n/n)z^n$, where $A_1 = 1$ and for $n \geqslant 2$, A_n is defined by (2.1). For convenience, we let $S := \sum_{n\geqslant 1} |A_n - A_{n+1}|$ and note that it sufficies to show that $S \leqslant (1-\beta)\cos\eta$. Again we divide the proof into several parts. First we assume that $c = a + b - 1$. Then we have two possibilities.

Case 1: Let $a, b \in [1, \infty)$, or $a \in (0, 1)$ and $b \in (1-a, 1)$. First we observe that

$$|A_n - A_{n+1}| = \left|\frac{(a,n-1)(b,n-1)}{(c,n)(1,n)}[n(c+1-a-b) - (1-a)(1-b)]\right|,$$

so that

$$S := \sum_{n\geqslant 1} |A_n - A_{n+1}| = \sum_{n=1}^{\infty} \frac{(a-1,n)(b-1,n)}{(a+b-1,n)(1,n)}.$$

Using the formula (1.1) we find that

$$S = \frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)} - 1.$$

Case 2: Let $a\in(0,1)$ and $b\in[1,\infty)$. Then, in this case we see that

$$S=-\sum_{n=1}^{\infty}\frac{(a-1,n)(b-1,n)}{(a+b-1,n)(1,n)}=-\left[\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}-1\right].$$

Secondly, for the next case, we assume that $c\geqslant ab$.

Case 3: Let $a,b\in(1,\infty)$, or $a,b\in(0,1)$. Then, in this case we see that $ab>a+b-1$ which means that we are actually considering the situation where $c>a+b-1$. Thus for all $n\geqslant 1$ we have

$$n(c+1-a-b)-(1-a)(1-b)\geqslant c+1-a-b-(1-a)(1-b)=c-ab\geqslant 0.$$

Therefore, writing

$$S=\frac{c+1-a-b}{c}\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c+1,n)(1,n)}-\left\{-1+\sum_{n=0}^{\infty}\frac{(a-1,n)(b-1,n)}{(c,n)(1,n)}\right\}$$

we obtain that

$$S=\frac{c+1-a-b}{c}\left(\frac{\Gamma(c+1)\Gamma(c+1-a-b)}{\Gamma(c+1-a)\Gamma(c+1-b)}\right)-\left\{\frac{\Gamma(c)\Gamma(c+2-a-b)}{\Gamma(c+1-a)\Gamma(c+1-b)}-1\right\}=1.$$

Finally, for the last case, we assume that $c>a+b-1$.

Case 4: Let $a\in(1,\infty)$ and $b\in(0,1]$. Then in this case we see that $a+b-1\geqslant ab$ which means that, for all $n\geqslant 1$, the inequality

$$n(c+1-a-b)-(1-a)(1-b)\geqslant c+1-a-b-(1-a)(1-b)=c-ab>0$$

holds and hence, we have $S=1$. Now, the conclusion for all the other cases follow from these observations, and therefore, the proof is complete. □

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