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Acyclicity of Symmetric and Exterior Powers of Complexes

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INTRODUCTION

Given a module M over a commutative ring R, it is of considerable interest to obtain information on the homological properties of its symmetric powers $\mathrm{S}^i M$ and and exterior powers $\bigwedge^i M$. One possible approach to this problem is to start from a free resolution $\mathbf{F}$ of M and produce "approximate resolutions" of $\mathrm{S}^i M$ and $\bigwedge^i M$. These are complexes with the correct homology in degree zero, which are minimal if R is local and $\mathbf{F}$ is minimal, and which are acyclic under certain conditions on $\mathbf{F}$. In case $\mathbf{F}$ has length ≤ 1 such constructions have been proposed, and necessary and sufficient conditions have been given for the exterior powers by Lebelt [9] and for the symmetric powers by Avramov [2].

In Section 4 we consider the case when $\mathbf{F}$ has length at most 2. By using combinations of divided, exterior, and symmetric powers of the free modules in $\mathbf{F}$, we give approximate resolutions of $\mathrm{S}^i M$ and provide a criterion for their acyclicity.

The situation is more complicated for longer complexes. When R is a $\mathbb{Q}$-algebra, Lebelt [11] gives approximate resolutions of the exterior powers and proves that they are acyclic if M has sufficiently high torsion-freeness. In [14] Weyman proposed a variation of Lebelt's construction for both symmetric and exterior powers over arbitrary rings and formulated necessary and sufficient conditions for its acyclicity. A close examination of the boundary maps of [14] shows that in most cases they do not produce a complex; cf. Example 6.2. The reason is that their definition uses some noncanonical families of maps.

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For a finite complex of free R-modules $\mathbf{F}$ of arbitrary length, with $\mathrm{H}_0(\mathbf{F}) = M$, we construct canonical complexes of free modules $\mathscr{S}^i\mathbf{F}$ and $\mathscr{L}^i\mathbf{F}$, whose zeroth homology is $\mathrm{S}^i M$ and $\bigwedge^i M$, respectively. These complexes are built from appropriate combinations of symmetric and exterior powers of the free modules in $\mathbf{F}$, with naturally induced maps between them. Our main results, Theorems 2.1 and 2.2, provide necessary and sufficient conditions for the acyclicity of $\mathscr{S}^i\mathbf{F}$ and $\mathscr{L}^i\mathbf{F}$; Theorems 3.11 and 3.12 do the same for the complexes constructed by Lebelt [11] and for their variants for symmetric powers.

Each acyclicity criterion involves two types of conditions. On the one hand, as in [2], [9], and [14], there are hypotheses on the grades of appropriate ideals of minors for the differentials of the complex $\mathbf{F}$, which are analogous to the conditions in the Buchsbaum−Eisenbud criterion [3] for acyclicity of $\mathbf{F}$. On the other hand, there is a hypothesis on the additive torsion of R. This condition indicates (except when $\mathbf{F}$ has length at most 2) that the constructions considered in [11] and in the present paper give a strongly characteristic-dependent approach to the approximate resolutions of the symmetric or the exterior powers of M.

As an example, consider the case where R is Noetherian and the complex

$$\mathbf{F}\colon 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to 0$$

is a finite free resolution of M with $\varphi_j \neq 0$ for $j = 1, \ldots, n$. Denote by r_j the rank of φ_j, and by $I_s(\varphi_j)$ its ideal of minors of order s. In this special situation some of our results can be formulated as follows:

Let $k \geq 2$ be an integer. If $n \geq 2$, then the following three conditions are equivalent:

(i) $\mathscr{S}^i\mathbf{F}$ is a free resolution of $\mathrm{S}^i M$ for $i = 1, \ldots, k$.

(ii) $\mathscr{S}^k\mathbf{F}$ is acyclic; $k!$ is invertible in R.

(iii) grade $I_{r_j}(\varphi_j) \geq kj$ for j even; grade $I_{r_j - t}(\varphi_j) \geq k(j-1) + 1 + t$ for j odd and $t = 0, \ldots, k-1$; $k!$ is invertible in R.

As an application of our acyclicity criteria, in Section 5 we generalize a result of Avramov [2] on the q-torsion-freeness of the symmetric powers of a finite module of projective dimension 1 over a Noetherian ring.

PRELIMINARIES

Throughout this paper R denotes a commutative ring with unity, unadorned tensor products are over R, and all considered graded objects are positive, i.e., their homogeneous parts indexed by negative integers are zero.

A graded R-algebra A is called *strictly commutative* if $ab = (-1)^{|a||b|}ba$ for all homogeneous $a, b \in A$, and $a^2 = 0$ for all $a \in A$ of odd degree. The tensor product $A \otimes B$ of graded R-algebras A and B has the multiplication

$$(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|}(aa') \otimes (bb').$$

It is a strictly commutative algebra provided A and B are. A *derivation* of A is an R-linear endomorphism ∂ of degree -1 of the underlying graded module of A, satisfying the Leibnitz formula $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$ for all homogeneous $a, b \in A$. It is a *differential* of A if in addition $\partial^2 = 0$.

A complex $\mathbf{M} = (M, \mu)$ is *acyclic* if $\mathrm{H}_i(\mathbf{M}) = 0$ for each $i > 0$ and *exact* if in addition $\mathrm{H}_0(\mathbf{M}) = 0$. For an integer c, we write $\mathbf{M}[c]$ for the complex $(M[c], \mu[c])$, where $M[c]$ is the graded R-module with $M[c]_i = M_{i-c}$, and the differential is given by $\mu[c]_i = (-1)^c\mu_{i-c}$. We also consider the canonical degree c bijective map of graded modules $\Sigma^c\colon M \to M[c]$, given for each $u \in M_i$ by $\Sigma^c(u) = u \in M[c]_{i+c}$, and write Σ for Σ^1.

The tensor product of the complexes of R-modules $\mathbf{M} = (M, \mu)$ and $\mathbf{N} = (N, \nu)$ is the complex of R-modules $\mathbf{M} \otimes \mathbf{M} = (M \otimes N, \mu \otimes 1 + 1 \otimes \nu)$.

1. SYMMETRIC AND EXTERIOR POWERS

Let M be a graded R-module. For an integer m let $M\{m\}$ be the graded submodule of M with $M\{m\}_m = M_m$ and $M\{m\}_i = 0$ for $i \neq m$. Let $\mathrm{C}(M\{m\})$ denote the symmetric algebra $\mathrm{S}(M\{m\})$ when m is even and the exterior algebra $\Lambda(M\{m\})$ when m is odd. In general, note that $M = \bigoplus_{m \geq 0} M\{m\}$ and set

$$\mathrm{C}(M) = \bigoplus_{m \geq 0} C(M\{m\}).$$

We endow $\mathrm{C}(M)$ with the canonical grading (for which it is strictly commutative)

$$\mathrm{C}(M)_t = \bigoplus_{a_1 + 2a_2 + \cdots + ta_t = t} \mathrm{C}^{a_0}M_0 \otimes \cdots \otimes \mathrm{C}^{a_t}M_t,$$

where

$$\mathrm{C}^a M_j = \begin{cases} \mathrm{S}^a M_j & \text{for even } j, \\ \wedge^a M_j & \text{for odd } j, \end{cases}$$

and consider M as a graded submodule of $\mathrm{C}(M)$.

The properties of symmetric and exterior algebras easily yield:

(1.1) PROPOSITION. *If M is a graded R-module, A is strictly commutative graded R-algebra, and $\tau\colon M \to A$ is a degree zero homomorphism of graded R-modules, then there exists a unique canonical extension of τ to a homomorphism of graded R-algebras $\theta\colon \mathrm{C}(M) \to A$, such that $\tau = \theta|_M$.*

A canonical bigraded R-module structure on $\mathrm{C}(M)$ is given by

$$(1.2)\qquad \mathrm{C}(M)_{t,i} = \bigoplus_{\substack{a_1+2a_2+\cdots+ta_t=t\\ a_0+a_1+\cdots+a_t=i}} \mathrm{C}^{a_0}M_0 \otimes \cdots \otimes \mathrm{C}^{a_t}M_t.$$

In this bigrading $M_t = \mathrm{C}(M)_{t,1}$ for each t.

(1.3) PROPOSITION. *Let $\mathbf{M} = (M, \mu)$ be a complex with differential μ. Then μ extends uniquely to a different ∂_μ of the algebra $\mathrm{C}(M)$.*

Proof. On the graded R-module $A = \mathrm{C}(M) \oplus \mathrm{C}(M)[1]$ consider the product

$$(1.4)\qquad (a,x)(b,y) = \big(ab, xb + (-1)^{|a|}ay\big)$$

for $a, b \in \mathrm{C}(M)$ and $x, y \in \mathrm{C}(M)[1]$.

It is easy to check that A becomes a strictly commutative graded R-algebra. Since the homomorphism of graded R-modules $\tau\colon M \to A$ given by $u \mapsto \tau(u) = (u, \Sigma\mu(u))$ has degree zero, by the universal property (1.1) of $\mathrm{C}(M)$ we obtain a map of graded R-algebras $\theta\colon \mathrm{C}(M) \to A$. The desired derivation ∂_μ is then given by the composition of θ with the canonical projection $A \to \mathrm{C}(M)[1]$, followed by Σ^{-1}.

Note that $\partial_\mu^2(uv) = \partial_\mu^2(u)v + u\partial_\mu^2(v)$ for all $u, v \in \mathrm{C}(M)$. As $\partial_\mu^2(u) = \mu^2(u) = 0$ for the algebra generators of $\mathrm{C}(M)$, it follows that $\partial_\mu^2 = 0$. This shows existence. Uniqueness is clear. ∎

We call $\mathrm{C}(\mathbf{M}) = (\mathrm{C}(M), \partial_\mu)$ the *free strictly commutative DG algebra* of the complex $\mathbf{M}$. Note that ∂_μ is a map of bigraded R-modules of bidegree $(-1, 0)$.

(1.5) BASE CHANGE. *For a complex of R-modules $\mathbf{M} = (M, \mu)$ and a homomorphism of commutative rings $\rho\colon R \to Q$ the canonical extension*

$$\theta\colon \mathrm{C}_Q(\mathbf{M} \otimes_R Q) \to \mathrm{C}_R(\mathbf{M}) \otimes_R Q$$

of the canonical inclusion $M \otimes_R Q \to \mathrm{C}_R(M) \otimes_R Q$ is an isomorphism of DG algebras over Q which is compatible with the bigrading. In particular, for a multiplicatively closed set U in R there is an isomorphism $U^{-1}\mathrm{C}_R(\mathbf{M}) \cong \mathrm{C}_{U^{-1}R}(U^{-1}\mathbf{M})$.

Proof. By the universal property of tensor products, the homomorphism of R-algebras $\mathrm{C}_R(M) \to \mathrm{C}_Q(M \otimes_R Q)$ extending $M \to M \otimes 1$ induces a Q-algebra homomorphism θ': $\mathrm{C}_R(M) \otimes_R Q \to \mathrm{C}_Q(M \otimes_R Q)$. Clearly θ and θ' are inverse isomorphisms. Since $\theta' \circ (\partial_\mu \otimes Q) \circ \theta$ is a Q-differential on $\mathrm{C}_Q(\mathbf{M} \otimes Q)$ which extends $\partial_\mu \otimes Q$, by (1.3) we get $\theta^{-1} \circ (\partial_\mu \otimes Q) \circ \theta = \partial_{(\mu \otimes Q)}$. ∎

(1.6) PROPOSITION. *For complexes of R-modules $\mathbf{M} = (M, \mu)$ and $\mathbf{N} = (N, \nu)$, the canonical extension*

$$\vartheta : \mathrm{C}(\mathbf{M} \otimes \mathbf{N}) \to \mathrm{C}(\mathbf{M}) \otimes \mathrm{C}(\mathbf{N})$$

of the inclusion of graded R-modules τ: $M \oplus N \to \mathrm{C}(M) \otimes \mathrm{C}(N)$ given by $\tau(u, v) = u \otimes 1 + 1 \otimes v$, is an isomorphism of DG algebras over R, which is compatible with the bigrading.

Proof. The inverse to ϑ is given by the homomorphism of graded R-algebras $\mathrm{C}(M) \otimes \mathrm{C}(N) \to \mathrm{C}(M \oplus N)$ derived from the canonical inclusions $M \to M \oplus N \leftarrow N$ by the universal properties of $\mathrm{C}(M), \mathrm{C}(N)$ and of the tensor product. Since $\vartheta^{-1} \circ (\partial_\mu \otimes 1 + 1 \otimes \partial_\nu) \circ \vartheta$ is a differential of $\mathrm{C}(M \oplus N)$ and extends $\mu \oplus \nu$, by (1.3) we obtain $\vartheta^{-1} \circ (\partial_\mu \otimes 1 + 1 \otimes \partial_\nu) \circ \vartheta = \partial_{\mu \oplus \nu}$. ∎

Let $\mathbf{M} = (M, \mu)$ be a complex of R-modules. As the differential ∂_μ on $\mathrm{C}(\mathbf{M})$ is a map of bidegree $(-1, 0)$, the complex $\mathrm{C}(\mathbf{M})$ splits into a direct sum of subcomplexes

$$\mathrm{C}(\mathbf{M}) = \bigoplus_{i \geq 0} \mathrm{C}(\mathbf{M})_{*,i}. \tag{1.7}$$

We call $\mathscr{S}^i\mathbf{M} = \mathrm{C}(\mathbf{M})_{*,i}$ the *ith symmetric power* of $\mathbf{M}$ and call the complex $\mathscr{L}^i\mathbf{M} = \mathscr{S}^i(\mathbf{M}[1])[-i]$ the *ith exterior power* of $\mathbf{M}$. By abuse of notation, the differential in both cases is written as ∂.

For $i > 0$ the differential ∂: $(\mathscr{S}^i\mathbf{M})_1 \to (\mathscr{S}^i\mathbf{M})_0$ is the map $(\mathrm{S}^{i-1}M_0) \otimes M_1 \to \mathrm{S}^iM_0$ given by $(f \otimes u) \mapsto \mu_1(u)f$. Thus we obtain the first of the isomorphisms

$$\mathrm{H}_0(\mathscr{S}^i\mathbf{M}) \cong \mathrm{S}^i\mathrm{H}_0(\mathbf{M}) \quad \text{and} \quad \mathrm{H}_0(\mathscr{L}^i\mathbf{M}) \cong \textstyle\bigwedge^i \mathrm{H}_0(\mathbf{M}) \quad \text{for } i \geq 0; \tag{1.8}$$

the second one follows in a similar manner.

For a complex of free modules $\mathbf{F} = (F, \varphi)$ set $\lambda(\mathbf{F}) = \sup\{i | F_i \neq 0\}$. We say that $\mathbf{F}$ has *no gaps* if $F_i \neq 0$ for $0 \leq i < \lambda(\mathbf{F})$.

Assume that $\mathbf{F}$ has no gaps and that $\lambda(\mathbf{F}) = m < \infty$. Let r_m be the rank of F_m. Then by (1.2) the complexes $\mathscr{S}^k\mathbf{F}$ and $\mathscr{L}^k\mathbf{F}$ are finite free for each $k \geq 0$ and

$$\lambda(\mathscr{S}^k\mathbf{F}) = \begin{cases} km, & \text{for even } m, \\ k(m-1) + \min(r_m, k), & \text{for odd } m, \end{cases}$$

(1.9)

$$\lambda(\mathscr{L}^k\mathbf{F}) = \begin{cases} km, & \text{for odd } m, \\ k(m-1) + \min(r_m, k), & \text{for even } m. \end{cases}$$

If $f_{s,1}, \ldots, f_{s,b_s}$ is a basis of F_s for $s = 0, \ldots, m$, then the R-module $(\mathscr{S}^i\mathbf{F})_t$ has a basis given by all products in $\mathrm{C}(\mathbf{F})$ of the form

(1.10) $$\left(f_{0,1}^{c_{0,1}} \cdots f_{0,b_0}^{c_{0,b_0}}\right) \cdots \left(f_{t,1}^{c_{t,1}} \cdots f_{t,b_t}^{c_{t,b_t}}\right)$$

$$\text{with } \sum c_{u,v} = i, \ \sum u c_{u,v} = t,$$

such that when u is odd the exponents $c_{u,v}$ are either zero or one. Similarly, $(\mathscr{L}^i\mathbf{F})_t$ has a basis given by all products in $\mathrm{C}(\mathbf{F}[1])$ of the form (1.10), such that when u is even the exponents $c_{u,v}$ are either zero or one.

For an integer $n \geq 1$ let $\mathbf{E}(n) = (E, \epsilon)$ be the complex

$$\mathbf{E}(n)\colon 0 \to E_n \xrightarrow{\epsilon} E_{n-1} \to 0,$$

where $E_n = Rf$ and $E_{n-1} = Rg$ are free R-modules on generators f and g of degrees n and $n-1$, respectively, and ϵ is the isomorphism defined by $\epsilon(f) = g$.

For an R-module L and an integer $c \in \mathbb{Z}$ set $L/(c) = L/cL$ and $(c) \setminus L = (0{:}c)_L$.

(1.11) Proposition. *Let $\mathbf{M}$ be a complex over R and let n, c, t, a be integers such that $n, c \geq 1$ and $t, a \geq 0$.*

If n is even, then there is a canonical exact sequence

$$\begin{aligned} 0 \to \mathrm{H}_{t+1-cn}(\mathscr{S}^a\mathbf{M})/(c) &\to \mathrm{H}_t\left(\mathscr{S}^c(\mathbf{E}(n)) \otimes \mathscr{S}^a\mathbf{M}\right) \\ &\to (c) \setminus \mathrm{H}_{t-cn}(\mathscr{S}^a\mathbf{M}) \to 0 \end{aligned}$$

and the complex $\mathscr{L}^c(\mathbf{E}(n)) \otimes \mathscr{L}^a\mathbf{M}$ is exact.

If n is odd, then there is a canonical exact sequence

$$\begin{aligned} 0 \to \mathrm{H}_{t+1-cn}(\mathscr{L}^a\mathbf{M})/(c) &\to \mathrm{H}_t\left(\mathscr{L}^c(\mathbf{E}(n)) \otimes \mathscr{L}^a\mathbf{M}\right) \\ &\to (c) \setminus \mathrm{H}_{t-cn}(\mathscr{L}^a\mathbf{M}) \to 0 \end{aligned}$$

and the complex $\mathscr{S}^c(\mathbf{E}(n)) \otimes \mathscr{S}^a\mathbf{M}$ is exact.

Proof. When $c \geq 1$ we have

$$\mathscr{S}^c(\mathbf{E}(n)) \colon \begin{cases} 0 \to Rf^c \xrightarrow{\partial} Rf^{c-1}g \to 0, & \text{for even } n, \text{ with } \partial(f^c) = cf^{c-1}g, \\ 0 \to Rfg^{c-1} \xrightarrow{\partial} Rg^c \to 0, & \text{for odd } n, \text{ with } \partial(fg^{c-1}) = g^c, \end{cases}$$

Therefore we obtain canonical exact sequence of complexes

$$0 \to (\mathscr{S}^a\mathbf{M})[cn-1] \to \mathscr{S}^c(\mathbf{E}(n)) \otimes \mathscr{S}^a\mathbf{M} \to (\mathscr{S}^a\mathbf{M})[cn] \to 0, \qquad \text{for even } n;$$

$$0 \to (\mathscr{S}^a\mathbf{M})[cn-c] \to \mathscr{S}^c(\mathbf{E}(n)) \otimes \mathscr{S}^a\mathbf{M} \to (\mathscr{S}^a\mathbf{M})[cn+1-c] \to 0, \qquad \text{for odd } n.$$

Their homology long exact sequences imply that $\mathscr{S}^c(\mathbf{E}(n)) \otimes \mathscr{S}^a\mathbf{M}$ is exact when n is odd. When n is even they induce for each $t \geq 0$ the desired canonical exact sequence in the symmetric case.

The corresponding results for the exterior case are obtained analogously. ∎

(1.12) COROLLARY. *Let $n, k \geq 1$ be integers.*

If either $n = 1$ or $k!$ is invertible in R, then the acyclicity of $\mathscr{S}^k\mathbf{M}$ is equivalent to that of $\mathscr{S}^k(\mathbf{E}(n) \oplus \mathbf{M})$.

If $k!$ is invertible in R, then the complex $\mathscr{L}^k\mathbf{M}$ is acyclic if and only if $\mathscr{L}^k(\mathbf{E}(n) \oplus \mathbf{M})$ is acyclic.

Proof. The canonical isomorphism $\mathrm{C}(\mathbf{E}(n) \oplus \mathbf{M}) \cong \mathrm{C}(\mathbf{E}(n)) \otimes \mathrm{C}(\mathbf{M})$ and the canonical decomposition (1.7) induce for each $k \geq 0$ a canonical isomorphism of complexes

$$\mathscr{S}^k(\mathbf{E}(n) \oplus \mathbf{M}) \cong \bigoplus_{a+c=k} \mathscr{S}^c(\mathbf{E}(n)) \otimes \mathscr{S}^a\mathbf{M}. \tag{1.13}$$

Now apply (1.11).

The proof of the assertion for $\mathscr{L}^i\mathbf{M}$ is analogous. ∎

2. ACYCLICITY OF SYMMETRIC AND EXTERIOR POWERS

First we recall the notion of grade of an ideal $I \subseteq R$. If I is a proper ideal, then set $\mathrm{gr}_R\, I = \sup\{s \mid \text{there is an } R\text{-regular sequence in } I \text{ of length } s\}$; else set $\mathrm{gr}_R\, I = \infty$. Define

$$\text{grade } I = \lim_{s \to \infty} \mathrm{gr}_{R[X_1, \dots, X_s]}\, IR[X_1, \dots, X_s],$$

where $R[X_1, \dots, X_s]$ is the polynomial ring over R in the indeterminates $X_1, \dots, X_s$. We refer to [12, Chaps. 5 and 6] for the properties of this notion of grade (denoted there by $\mathrm{Gr}_R\{I\}$ and termed *true grade* or *polynomial grade*).

When R is local with maximal ideal $\mathfrak{m}$ we set depth $R = \operatorname{grade}\mathfrak{m}$. It follows from [12, Chap. 6, Theorem 5] by a standard argument, that $\operatorname{grade} I = \inf\{\operatorname{depth} R_{\mathfrak{p}} \mid I \subseteq \mathfrak{p} \in \operatorname{Spec}(R)\}$ when I is finitely generated.

Let $\mathbf{F} = (F, \varphi)$ be a finite free complex

$$\mathbf{F}\colon 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_1} F_0 \to 0$$

with $F_i \neq 0$ for $i = 0, \ldots, n$ and let $M = \mathrm{H}_0(\mathbf{F})$. Set $b_i = \operatorname{rank} F_i$, call the number $r_i = \sum_{k=i}^{n}(-1)^{k-i}b_k$ the *expected rank* of φ_i, and write $I_s(\varphi_i)$ for the ideal of $s \times s$ minors of φ_i, where $I_s(\varphi_i) = 0$ for $s > \min(b_i, b_{i-1})$ and $I_s(\varphi_i) = R$ for $s \leq 0$.

(2.0) *Grade conditions*. For integers $j, k \geq 1$ we consider the *grade condition*

$\mathbf{GC}^k(j)$: $\operatorname{grade} I_{r_j}(\varphi_j) \geq kj$

and the *sliding grade condition*

$\mathbf{SGC}^k(j)$: $\operatorname{grade} I_{r_j - t}(\varphi_j) \geq k(j-1) + 1 + t$ for $t = 0, \ldots, k-1$.

Our main results give acyclicity criteria for the symmetric and exterior powers of $\mathbf{F}$, in terms similar to those of the Buchsbaum–Eisenbud criterion [3].

(2.1) THEOREM. *Let $k \geq 2$ be an integer. Conditions* (i) *and* (ii) *below are equivalent*:

(i) $\mathscr{S}^k\mathbf{F}$ *is acyclic*; $\operatorname{grade} I_{r_j}(\varphi_j) \geq 1$ *for each odd* $j \geq 1$; $k!$ *is invertible in* R.

(ii) *the grade condition* $\mathbf{GC}^k(j)$ *holds when j is even*; *the sliding grade condition* $\mathbf{SGC}^k(j)$ *holds when j is odd*; $k!$ *is invertible in* R.

They imply

(iii) $\mathscr{S}^i\mathbf{F}$ *is a free resolution of* S^iM *for* $i = 1, \ldots, k$.

If $\varphi_m \neq 0$ *for some even* m, *then all three conditions are equivalent*.

Analogously, for the exterior powers we have

(2.2) THEOREM. *Let $k \geq 2$ be an integer. Conditions* (i) *and* (ii) *below are equivalent*:

(i) $\mathscr{L}^k\mathbf{F}$ *is acyclic*; $\operatorname{grade} I_{r_j}(\varphi_j) \geq 1$ *for each even* $j \geq 2$; $k!$ *is invertible in* R.

(ii) *the grade condition* $\mathbf{GC}^k(j)$ *holds when j is odd*; *the sliding grade condition* $\mathbf{SGC}^k(j)$ *holds when j is even*; $k!$ *is invertible in* R.

They imply

(iii) $\mathscr{L}^i\mathbf{F}$ *is a free resolution of* $\bigwedge^i M$ *for* $i = 1, \ldots, k$.

If $\varphi_m \neq 0$ *for some odd* m, *then all three conditions are equivalent*.

The proofs depend on the acyclicity criteria of Peskine and Szpiro [13] and Buchsbaum and Eisenbud [3] in the form given by Northcott [12], which applies to arbitrary commutative rings:

(2.3) The complex $\mathbf{F}$ is acyclic if and only if $\mathbf{F}_{\mathfrak{p}}$ is acyclic for all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that depth $R_{\mathfrak{p}} < \lambda(\mathbf{F})$ [12, Chap. 5, Theorem 21].

(2.4) The complex $\mathbf{F}$ is acyclic if and only if grade $I_{r_j}(\varphi_j) \geq j$ for $j = 1, \dots, \lambda(\mathbf{F})$ [12, Chap. 6, Theorem 15].

We start with an easy observation:

(2.5) LEMMA. *Let* $\lambda(\mathbf{F}) = n \geq 1$ *and* $\mathbf{F} \cong \mathbf{E}(n) \oplus \mathbf{F}'$ *for some complex of free modules* $\mathbf{F}' = (F', \varphi')$. *If* $\mathscr{P}$ *denotes one of the grade conditions* (2.0), *then* $\mathscr{P}$ *holds for* $\mathbf{F}$ *if and only if it holds for* $\mathbf{F}'$.

Proof. Note that

$$r'_n = r_n - 1 \qquad \text{and} \qquad r'_i = r_i \qquad \text{for } i < n$$

and that for each integer $s \in \mathbb{Z}$ we have

$$I_s(\varphi'_n) = I_{s+1}(\varphi_n) \qquad \text{and} \qquad I_s(\varphi'_i) = I_s(\varphi_i) \qquad \text{for } i < n.$$

The lemma now follows by an elementary application of the equalities above. ∎

We give the proof of (2.1). The proof of (2.2) is analogous.

Proof of Theorem 2.1. (i) $\Rightarrow$ (ii) We recall that $\lambda(\mathbf{F}) = n$ and proceed by induction on the lexicographically ordered set of pairs (n, r_n). As the statement is trivially true when $n = 0$ for any value of r_n, we assume that $n \geq 1$ and that the assertion holds for every pair $(s, r_s) < (n, r_n)$.

Case (a). n is odd and $1 \leq r_n < k$. By (1.9) we have $\lambda(\mathscr{S}^k\mathbf{F}) = kn - k + r_n$. Set $G' = (\bigwedge^{r_n - 1} F_n) \otimes (\mathrm{S}^{k - r_n + 1} F_{n-1})$ and $G'' = (\bigwedge^{r_n} F_n) \otimes (\mathrm{S}^{k - r_n - 1} F_{n-1}) \otimes F_{n-2}$. Then by (1.2), the tail of $\mathscr{S}^k\mathbf{F}$ has the form

$$0 \to \left(\textstyle\bigwedge^{r_n} F_n\right) \otimes \left(\mathrm{S}^{k-r_n} F_{n-1}\right) \xrightarrow{\partial} G' \oplus G'' \to \cdots.$$

Set $\partial' = \pi' \circ \partial$ and $\partial'' = \pi'' \circ \partial$, where $\pi' \colon G' \oplus G'' \to G'$ and $\pi'' \colon G' \oplus G'' \to G''$ are the canonical projections. Set $q = \operatorname{rank}((\bigwedge^{r_n} F_n) \otimes (\mathrm{S}^{k-r_n} F_{n-1}))$. Then $I_q(\partial) \subseteq \sum_{i=0}^{q} I_{q-i}(\partial') I_i(\partial'')$; hence

$$\operatorname{grade}\left(\sum_{i=0}^{q} I_{q-i}(\partial') I_i(\partial'')\right) \geq \operatorname{grade} I_q(\partial) \geq kn - k + r_n,$$

where the second inequality follows from the acyclicity of $\mathscr{S}^k\mathbf{F}$ and the Buchsbaum–Eisenbud criterion (2.4). Also, we have

$$(*) \qquad \sum_{i=0}^{q} I_{q-i}(\partial') I_i(\partial'') \subseteq I_1(\varphi_n).$$

Indeed, if $q - i \geq 1$, then $I_{q-i}(\partial') \subseteq I_q(\varphi_n)$ by the construction of ∂'. Thus, it is enough to show that $I_q(\partial'') = 0$.

This is clear for $n = 1$. To see it for $n > 1$, consider the multiplicative set U of all nonzero divisors of R. As grade $I_{r_n}(\varphi_n) \geq 1$, by adjoining an indeterminate to R we may assume that $I_{r_n}(\varphi_n) \cap U \neq \varnothing$. Hence, in the localized sequence

$$0 \to U^{-1}F_n \xrightarrow{U^{-1}\varphi_n} U^{-1}F_{n-1} \to U^{-1}F_{n-2} \to \cdots$$

the image of $U^{-1}\varphi_n$ splits off as a nonzero free direct summand of $U^{-1}F_{n-1}$ over the total ring of fractions $U^{-1}R$ of R. Therefore in the induced sequence

$$0 \to U^{-1}\big(\mathrm{S}^{k-r_n}F_n\big) \xrightarrow{U^{-1}(\mathrm{S}^{k-r_n}\varphi_n)} U^{-1}\big(\mathrm{S}^{k-r_n}F_{n-1}\big)$$

the image $\operatorname{Im} U^{-1}(\mathrm{S}^{k-r_n}\varphi_n)$ also splits off as a nonzero free summand of $U^1(\mathrm{S}^{k-r_n}F_{n-1})$. Furthermore, for arbitrary $y \in \bigwedge^{r_n} F_n$ and $x_1 \cdots x_{k-r_n} \in \mathrm{S}^{k-r_n}F_n$ we obtain an equality

$$\begin{aligned} &\partial''\big(y \otimes \varphi_n(x_1) \cdots \varphi_n(x_{k-r_n})\big) \\ &\qquad = \sum_{i=1}^{k-r_n} y \otimes \varphi_n(x_1) \cdots \widehat{\varphi_n(x_i)} \cdots \varphi_n(x_{k-r_n}) \otimes (\varphi_{n-1}\varphi_n)(x_i) = 0, \end{aligned}$$

where $\widehat{\varphi_n(x_i)}$ means that $\varphi_n(x_i)$ is omitted. It follows that there is an inclusion of $U^{-1}R$-modules

$$U^{-1}(\wedge^{r_n}F_n) \otimes \operatorname{Im} U^{-1}\big(\mathrm{S}^{k-r_n}\varphi_n\big) \subseteq \operatorname{Ker}(U^{-1}\partial'');$$

hence $I_q(U^{-1}\partial'') = 0$. Therefore $I_q(\partial'') = 0$, which concludes the proof of $(*)$.

From $(*)$ we obtain grade $I_1(\varphi_n) \geq kn - k + r_n$, which is the last nontrivial one among the inequalities to be proved. It is enough to establish the remaining ones after localization at each prime $\mathfrak{p} \in \operatorname{Spec}(R)$ with depth $R_\mathfrak{p} < kn - k + r_n$. As grade $I_1(\varphi_n) \geq kn - k + r_n$, for such a prime the complex $\mathbf{F}_\mathfrak{p}$ splits over $R_\mathfrak{p}$ into a direct sum $\mathbf{E}(n) \oplus \mathbf{F}'$ of free complexes. Since grade $I_{r_n}(\varphi_n) \geq 1$, we have $b_n \leq b_{n-1}$; therefore $\mathbf{F}'$ has no gaps. Thus by (1.12) and by the inductive hypothesis, the desired inequalities hold for $\mathbf{F}'$. Applying (2.5), we conclude the proof of Case (a).

Case (b). n is odd and $2 \leq k \leq r_n$. Here $\lambda(\mathscr{S}^k\mathbf{F}) = kn$ and the end of $\mathscr{S}^k\mathbf{F}$ has the form

$$0 \to \textstyle\bigwedge^k F_n \xrightarrow{\partial} \big(\bigwedge^{k-1}F_n\big) \otimes F_{n-1} \to \cdots .$$

Its exactness implies by the Buchsbaum–Eisenbud criterion that grade $I_q(\partial) \geq kn$, where $q = \operatorname{rank} \bigwedge^k F_n$. As $I_1(\partial) = I_1(\varphi_n)$ by the construction of ∂, this yields $I_q(\partial) \subseteq I_1(\varphi_n)$; hence grade $I_1(\varphi_n) \geq kn$. It is enough to show the desired inequalities after localization at each prime $\mathfrak{p} \in \operatorname{Spec}(R)$ with depth $R_\mathfrak{p} < kn$. However, for such a prime the complex $\mathbf{F}_\mathfrak{p}$ splits as a direct sum just as in the proof of Case (a), and the argument given there applies.

(2.6) *Remark.* If $n = 1$, then the proof of the implication (i) ⇒ (ii) is complete and has not used the assumption that $k!$ is invertible in R.

Case (c). n is even. In this case $\lambda(\mathscr{S}^k \mathbf{F}) = kn$ and the tail of $\mathscr{S}^k \mathbf{F}$ has the form

$$0 \to \mathrm{S}^k F_n \xrightarrow{\partial} \left(\mathrm{S}^{k-1} F_n\right) \otimes F_{n-1} \to \cdots .$$

Since it is acyclic, the Buchsbaum–Eisenbud criterion gives grade $I_q(\partial) \geq kn$, where $q = \operatorname{rank} \mathrm{S}^k F_n$. As $I_q(\partial) \subseteq I_1(\partial) = I_1(\varphi_n)$, we obtain grade $I_1(\varphi_n) \geq kn$. It is enough to show the desired inequalities after localization at each prime $\mathfrak{p} \in \operatorname{Spec}(R)$ with depth $R_\mathfrak{p} < kn$. Note that $\mathbf{F}'$ has gaps only if $b_n \geq 2$ and $b_{n-1} = 1$. Since $\operatorname{rank} \mathrm{S}^k F_n = \binom{b_n + k - 1}{b_n - 1}$ and $\operatorname{rank}((\mathrm{S}^{k-1} F_n) \otimes F_{n-1}) = \binom{b_n + k - 2}{b_n - 1} b_{n-1}$, the acyclicity of $\mathscr{S}^k \mathbf{F}$ yields

$$\binom{b_n + k - 1}{b_n - 1} \leq \binom{b_n + k - 2}{b_n - 1} b_{n-1},$$

which reduces to $b_n + k - 1 \leq k b_{n-1}$, that is, to $b_n \leq k(b_{n-1} - 1) + 1$. Thus $b_{n-1} = 1$ implies $b_n = 1$; hence $\mathbf{F}'$ has no gaps. Therefore by (1.12) and by the induction hypothesis the desired inequalities hold for $\mathbf{F}'$. Applying (2.5), we conclude the proof of the implication (i) ⇒ (ii).

(ii) ⇒ (i) We induct again on the lexicographically ordered set of pairs (n, r_n). When $n = 0$ our statement is trivially true for any value of r_n. Let $n \geq 1$ and assume the assertion is true for any pair $(s, r_s) < (n, r_n)$.

Case (a). n is odd and $1 \leq r_n < k$. Then by (1.9) the complex $\mathscr{S}^k \mathbf{F}$ has length $k(n-1) + r_n$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be such that depth $R_\mathfrak{p} < kn - k + r_n$. We have

$$\operatorname{grade} I_1(\varphi_n) = \operatorname{grade} I_{r_n - (r_n - 1)}(\varphi_n) \geq kn - k + r_n,$$

where the inequality holds by assumption. Therefore the localized complex $\mathbf{F}_\mathfrak{p}$ splits into a direct sum of free complexes over $R_\mathfrak{p}$ as $\mathbf{F}_\mathfrak{p} \cong \mathbf{E}(n) \oplus \mathbf{F}'$. Since grade $I_{r_n}(\varphi_n) \geq 1$, we have $b_n \leq b_{n-1}$; therefore $\mathbf{F}'$ has no gaps. As grade does not decrease under localization, by (2.5) the inductive hypothesis holds for $\mathbf{F}'$. Thus by (1.12) we obtain that $(\mathscr{S}^k\mathbf{F})_{\mathfrak{p}}$ is acyclic for every prime $\mathfrak{p}$ with depth $R_{\mathfrak{p}} < \lambda(\mathscr{S}^k\mathbf{F})$. Therefore the complex $\mathscr{S}^k\mathbf{F}$ is acyclic by (2.3) and we are done in this case.

Case (b). n is odd and $k \le r_n$. Then $\lambda(\mathscr{S}^k\mathbf{F}) = kn$ and, as in Case (a), take $\mathfrak{p} \in \operatorname{Spec}(R)$ with depth $R_{\mathfrak{p}} < nk$. Since $I_1(\varphi_n) \supseteq I_{r_n-k+1}(\varphi_n)$, we get

$$\operatorname{grade} I_1(\varphi_n) \ge \operatorname{grade} I_{r_n-k+1}(\varphi_n) \ge kn,$$

where the second inequality is one of our assumptions. The rest of the argument is the same as in Case (a) from this implication.

(2.7) *Remark.* If $n = 1$, then the proof of the implication (ii) $\Rightarrow$ (i) is complete and has not used the assumption that $k!$ is invertible in R.

Case (c). n is even. Then $\lambda(\mathscr{S}^k\mathbf{F}) = nk$. As above, take $\mathfrak{p} \in \operatorname{Spec}(R)$ with depth $R_{\mathfrak{p}} < nk$. Since $r_n \ge 1$, we have grade $I_1(\varphi_n) \ge$ grade $I_{r_n}(\varphi_n) \ge kn$, and the argument from Case (a) completes the proof of the implication (ii) $\Rightarrow$ (i) and, hence, of the equivalence of (i) and (ii).

(ii) $\Rightarrow$ (iii) This implication follows directly from the implication (ii) $\Rightarrow$ (i), once we note that if $2 \le i \le k$ and (ii) holds for k, then it holds for i.

Assume next that $\varphi_m \ne 0$ for some even m.

(iii) $\Rightarrow$ (i) Since $\mathbf{F}$ is acyclic, by (2.4) we have for every $j \ge 1$ that grade $I_{r_j}(\varphi_j) \ge j \ge 1$; in particular this holds for every odd $j \ge 3$. Therefore the proof of the theorem will be complete once we show that (iii) implies $k!$ is invertible in R.

Assume that $p \le k$ is a prime number, which is not a unit in R. Take a prime ideal $\mathfrak{p} \supseteq pR$ with depth $R_{\mathfrak{p}} =$ grade $pR \le 1$. As $m \ge 2$, the acyclicity of $\mathbf{F} = \mathscr{S}^1\mathbf{F}$ and (2.4) yield $r_m \ge 0$ and grade $I_{r_{m-1}}(\varphi_{m-1}) \ge 1$. If $r_m = 0$, then $r_{m-1} = \operatorname{rank} F_{m-1}$ and φ_{m-1} must be injective, contradicting the fact that $\operatorname{Ker} \varphi_{m-1} = \operatorname{Im} \varphi_m \ne 0$. Thus $r_m \ge 1$, therefore grade $I_1(\varphi_m) \ge 2$, yielding a decomposition of $\mathbf{F}_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ in the form $\mathbf{E}(m) \oplus \mathbf{F}'$. As (iii) holds for $\mathbf{F}_{\mathfrak{p}}$, the complex $\mathscr{S}^p(\mathbf{E}(m))$ is acyclic by (1.13), hence exact. Thus (1.11) implies that p is a unit in $R_{\mathfrak{p}}$, yielding the desired contradiction.

The proof of the theorem is now complete. ∎

We conclude this section with variations on the preceding theorems.

(2.8) Theorem. *Let $k \ge 2$ be an integer. Assume that $k = 2$ or that* $\operatorname{Supp}(M) = \operatorname{Spec}(R)$. *If* grade $I_1(\varphi_m) \ge 2$ *for some even m, then*

(i-) *$\mathscr{S}^k\mathbf{F}$ is acyclic and* grade $I_{r_j}(\varphi_j) \ge 1$ *for each odd integer $j \ge 1$*
is equivalent to each one of the conditions (ii) *and* (iii) *from* (2.1).

Proof. In view of (2.1), it suffices to show that (i-) implies $k!$ is invertible in R.

Assume that $p \le k$ is a prime number, which is not invertible in R. Take a prime ideal $\mathfrak{p} \supseteq pR$ with depth $R_{\mathfrak{p}}$ = grade $pR \le 1$. As grade $I_1(\varphi_m) \ge 2$, it follows that $\mathbf{F}_{\mathfrak{p}} \cong \mathbf{E}(m) \oplus \mathbf{F}'$ over $R_{\mathfrak{p}}$. By (1.13) we have the decomposition $\mathscr{S}^k \mathbf{F}_{\mathfrak{p}} \cong \bigoplus_{i=0}^{k} \mathscr{S}^i(\mathbf{E}(m)) \otimes \mathscr{S}^{k-i}\mathbf{F}'$; thus the complexes $\mathscr{S}^k(\mathbf{E}(m))$ and $\mathscr{S}^p(\mathbf{E}(m)) \otimes \mathscr{S}^{k-p}\mathbf{F}'$ are acyclic. Now (1.11) yields that k is invertible in $R_{\mathfrak{p}}$ and the multiplication by $p \in \mathfrak{p}R_{\mathfrak{p}}$ is an isomorphism on $\mathrm{H}_0(\mathscr{S}^{k-p}\mathbf{F}') = \mathrm{S}^{k-p}M_{\mathfrak{p}}$. If $k = 2$, then $2 = p = k$ is invertible in R, contradicting our assumption on p and concluding the proof in this case. If $\mathrm{Supp}(M) = \mathrm{Spec}(R)$, then $\mathrm{S}^{k-p}M_{\mathfrak{p}}$ is nonzero and finitely generated over $R_{\mathfrak{p}}$ (because $M_{\mathfrak{p}}$ is nonzero and finitely generated), and Nakayama's lemma gives the desired contradiction. ▮

By a similar argument we obtain

(2.9) THEOREM. *Let $k \ge 2$ be an integer. Assume that $k = 2$ or that* $\mathrm{Supp}(M) = \mathrm{Spec}(R)$. *If* grade $I_1(\varphi_m) \ge 2$ *for some odd m, then*

(i-) *$\mathscr{L}^k\mathbf{F}$ is acyclic and* grade $I_{r_j}(\varphi_j) \ge 1$ *for each even integer $j \ge 2$*

is equivalent to each one of the conditions (ii) *and* (iii) *from* (2.2).

3. DIVIDED POWERS

Recall that A is a *divided powers algebra* if it is a strictly commutative graded R-algebra such that to every element $x \in A_n$ of positive even degree n and to every integer $k \ge 0$ there is associated an element $x^{(k)} \in A_{kn}$, called the *kth divided power* of x, whose list of defining properties can be found in [5, Sect. 7] or [6, Chap. 1, Sect. 7]. In addition, for such an algebra we set $x^{(k)} = x^k$ when x is homogeneous of zero or odd degree.

A differential ∂ of a divided powers R-algebra A is said to be *compatible with the divided powers structure* if for every $k \ge 0$ and every homogeneous element $x \in A$ of positive even degree we have $\partial(x^{(k)}) = x^{(k-1)}\partial(x)$.

In the first part of this section we consider the divided powers DG algebra $\mathrm{D}(\mathbf{M})$ of a complex $\mathbf{M} = (M, \mu)$ and some of its relevant properties. Their proofs are analogous to the proofs of the corresponding statements for $\mathrm{C}(\mathbf{M})$ and are omitted.

Let $m \ge 2$ be an even integer and let M be a graded R-module such that $M = M\{m\}$. Fix a free graded presentation

$$F \xrightarrow{\varphi} G \xrightarrow{\pi} M \to 0$$

of M, such that $G = G\{m\}$ and $F = F\{m\}$. If M is free, set $\Gamma(M)$ to be the free divided powers algebra of M as defined in [5, Sect. 8.4] (where it is called $S(M)$). In the general case let I be the ideal in $\Gamma(G)$ generated by the elements $\{\varphi(f)^{(k)} | f \in F \text{ and } k \geq 1\}$ and set $\Gamma(M) = \Gamma(G)/I$.

We observe that $\Gamma(M)_k = 0$ when m does not divide k and write $\Gamma^a M_m$ for the R-module $\Gamma(M)_{am}$.

Let now M be an arbitrary graded R-module. Set $\mathrm{D}(M\{0\}) = \mathrm{S}(M\{0\})$; for odd m set $\mathrm{D}(M\{m\}) = \Lambda(M\{m\})$; for even $m \geq 2$ set $\mathrm{D}(M\{m\}) = \Gamma(M\{m\})$; in general set

$$\mathrm{D}(M) = \bigotimes_{m \geq 0} \mathrm{D}(M\{m\}).$$

We endow $\mathrm{D}(M)$ with the canonical grading

$$\mathrm{D}(M)_t = \bigoplus_{a_1 + 2a_2 + \cdots + ta_t = t} \mathrm{D}^{a_0} M_0 \otimes \cdots \otimes \mathrm{D}^{a_t} M_t,$$

where

$$\mathrm{D}^a M_j = \begin{cases} \mathrm{S}^a M_j, & \text{for } j = 0; \\ \wedge^a M_j, & \text{for odd } j; \\ \Gamma^a M_j, & \text{for even } j \geq 2; \end{cases}$$

and consider M as a graded submodule of $\mathrm{D}(M)$.

An argument analogous to that of [5, Sect. 11, Theorem 3] yields:

(3.1) PROPOSITION. *The R-algebra $\mathrm{D}(M)$ has a canonical structure of a divided powers algebra. If τ: $M \to A$ is a homomorphism of degree zero of the graded R-module M into a divided powers R-algebra A, then there exists a unique canonical extension of τ to a homomorphism of divided powers R-algera θ: $\mathrm{D}(M) \to A$, such that $\tau = \theta|_M$.*

A canonical bigraded R-module structure on $\mathrm{D}(M)$ is given by

$$\mathrm{D}(M)_{t,i} = \bigoplus_{\substack{a_1 + 2a_2 + \cdots + ta_t = t \\ a_0 + a_1 + \cdots + a_t = i}} \mathrm{D}^{a_0} M_0 \otimes \cdots \otimes \mathrm{D}^{a_i} M_i. \tag{3.2}$$

In this bigrading $M_i = \mathrm{D}(M)_{i,1}$ for each i.

(3.3) PROPOSITION. *Let $\mathbf{M} = (M, \mu)$ be a complex with differential μ. Then μ extends uniquely to a compatible with the divided powers structure differential ∂_μ of the algebra $\mathrm{D}(M)$.*

Proof. The proof is *mutatis mutandis* that of (1.3). The only new point is that the algebra $\mathrm{D}(M) \oplus \mathrm{D}(M)[1]$ with multiplication defined as in (1.4) has a canonical system of divided powers, given by $(a, x)^{(k)} = (a^{(k)}, a^{(k-1)}x)$ for $k \geq 1$ and (a, x) of even positive degree; cf. the proof of [11, (2.2)]. ▮

We call $\mathrm{D}(\mathbf{M}) = (\mathrm{D}(M), \partial_\mu)$ the *divided powers DG algebra* of the complex $\mathbf{M}$. Note that ∂_μ is a map of bigraded R-modules of bidegree $(-1, 0)$.

(3.4) BASE CHANGE. *For a complex of R-modules $\mathbf{M} = (M, \mu)$ and a homomorphism of commutative rings ρ: $R \to Q$ the canonical extension*

$$\theta\colon \mathrm{D}_Q(\mathbf{M} \otimes_R Q) \to \mathrm{D}_R(\mathbf{M}) \otimes_R Q$$

of the canonical inclusion $M \otimes_R Q \to \mathrm{D}_R(M) \otimes_R Q$ is an isomorphism of divided powers DG algebras over Q which is compatible with the bigrading. In particular, for a multiplicatively closed set U in R we have $U^{-1}\mathrm{D}_R(\mathbf{M}) \cong \mathrm{D}_{U^{-1}R}(U^{-1}\mathbf{M})$ canonically.

(3.5) PROPOSITION. *For complexes of R-modules $\mathbf{M} = (M, \mu)$ and $\mathbf{N} = (N, \nu)$, the canonical extension*

$$\vartheta\colon \mathrm{D}(\mathbf{M} \oplus \mathbf{N}) \to \mathrm{D}(\mathbf{M}) \otimes \mathrm{D}(\mathbf{N})$$

of the inclusion of graded R-modules τ: $M \oplus N \to \mathrm{D}(M) \otimes \mathrm{D}(N)$, given by $\tau(u, v) = u \otimes 1 + 1 \otimes v$, is an isomorphism of divided powers DG algebras over R, which is compatible with the bigrading.

Let $\mathbf{M} = (M, \mu)$ be a complex of R-modules. As the differential ∂_μ on $\mathrm{D}(\mathbf{M})$ is a map of bidegree $(-1, 0)$, the complex $\mathrm{D}(\mathbf{M})$ splits into a direct sum of subcomplexes

$$\mathrm{D}(\mathbf{M}) = \bigoplus_{i \geq 0} \mathrm{D}(\mathbf{M})_{*,i}.$$

We write $\mathscr{G}^i\mathbf{M}$ for the subcomplex $\mathrm{D}(\mathbf{M})_{*,i}$ and $\mathscr{D}^i\mathbf{M}$ for the complex $\mathscr{G}^i(\mathbf{M}[1])[-i]$. By abuse of notation, the differential in both cases is denoted by ∂.

(3.6) *Remarks*. (a) If $M_t = 0$ for $t \geq 2$, then $\mathscr{G}^i\mathbf{M} = \mathscr{S}^i\mathbf{M}$ for each $i \geq 0$.

(b) When $\mathbf{F}$ is a finite complex of free modules, the complexes $\mathscr{D}^i\mathbf{F}$ coincide (up to the sign of the differentials) with the complexes $C^i_*\mathbf{F}$ constructed by Lebelt [11, Sect. 1].

For $i \geq 1$ the differentials ∂_1: $(\mathrm{S}^{i-1}M_0) \otimes M_1 \to \mathrm{S}^iM_0$ of $\mathscr{G}^i\mathbf{M}$ and ∂_1: $(\wedge^{i-1}M_0) \otimes M_1 \to \wedge^iM_0$ of $\mathscr{D}^i\mathbf{M}$ are given by $(f \otimes u) \mapsto \mu_1(u)f$. Thus we obtain isomorphisms

$$\mathrm{H}_0(\mathscr{G}^i\mathbf{M}) \cong \mathrm{S}^i\mathrm{H}_0(\mathbf{M}) \qquad \text{and} \qquad \mathrm{H}_0(\mathscr{D}^i\mathbf{M}) \cong \textstyle\bigwedge^i\mathrm{H}_0(\mathbf{M}) \qquad \text{for } i \geq 0.$$

Let $\mathbf{F} = (F, \varphi)$ be a finite free complex with no gaps and with $\lambda(\mathbf{F}) = m$. By (3.2) the complexes $\mathscr{G}^k\mathbf{F}$ and $\mathscr{D}^k\mathbf{F}$ are finite free for each $k \geq 0$ and

$$(3.7)\qquad \begin{aligned} \lambda(\mathscr{G}^k\mathbf{F}) &= \begin{cases} km, & \text{for even } m, \\ k(m-1) + \min(r_m, k), & \text{for odd } m, \end{cases} \\ \lambda(\mathscr{D}^k\mathbf{F}) &= \begin{cases} km, & \text{for odd } m, \\ k(m-1) + \min(r_m, k), & \text{for even } m. \end{cases} \end{aligned}$$

If $f_{s,1}, \dots, f_{s,b_s}$ is a basis of F_s for $s = 0, \dots, m$, then the R-module $(\mathscr{G}^i\mathbf{F})_t$ has a basis given by the set of products in $\mathrm{D}(\mathbf{F})$ of the form

$$(3.8)\qquad \left(f_{0,1}^{(c_{0,1})} \cdots f_{0,b_0}^{(c_{0,b_0})}\right) \cdots \left(f_{t,1}^{(c_{t,1})} \cdots f_{t,b_t}^{(c_{t,b_t})}\right) \quad \text{with } \sum c_{u,v} = i,\ \sum u c_{u,v} = t,$$

such that when u is odd, the exponents $c_{u,v}$ are either zero or one.

Similarly, $(\mathscr{D}^i\mathbf{F})_t$ has a basis given by the set of products in $\mathrm{D}(\mathbf{F}[1])$ of the form (3.8), such that when u is even, the exponents $c_{u,v}$ are either zero or one.

(3.9) PROPOSITION. *Let $\mathbf{M}$ be a complex over R, and let n, c, t, a be integers such that $n, c \geq 1$ and $t, a \geq 0$.*

The complexes $\mathscr{D}^c(\mathbf{E}(1)) \otimes \mathscr{D}^a\mathbf{M}$ and $\mathscr{G}^c(\mathbf{E}(1)) \otimes \mathscr{G}^a\mathbf{M}$ are exact.

If $n \geq 2$ is even, then there is a canonical exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}_{t-cn+1}(\mathscr{D}^a\mathbf{M})/(c) \to \mathrm{H}_t\left(\mathscr{D}^c(\mathbf{E}(n)) \otimes \mathscr{D}^a\mathbf{M}\right) \\ &\to (c) \setminus \mathrm{H}_{t-cn}(\mathscr{D}^a\mathbf{M}) \to 0 \end{aligned}$$

and the complex $\mathscr{G}^c(\mathbf{E}(n)) \otimes \mathscr{G}^a\mathbf{M}$ is exact.

If $n \geq 3$ is odd, then there is a canonical exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}_{t-cn+1}(\mathscr{G}^a\mathbf{M})/(c) \to \mathrm{H}_t\left(\mathscr{G}^c(\mathbf{E}(n)) \otimes \mathscr{G}^a\mathbf{M}\right) \\ &\to (c) \setminus \mathrm{H}_{t-cn}(\mathscr{G}^a\mathbf{M}) \to 0 \end{aligned}$$

and the complex $\mathscr{D}^c(\mathbf{E}(n)) \otimes \mathscr{D}^a\mathbf{M}$ is exact.

Proof. Use that when $c \geq 1$ we have

$$\mathscr{G}^c(\mathbf{E}(n)) : \begin{cases} 0 \to Rfg^{c-1} \xrightarrow{\partial} Rg^c \to 0, & \text{for } n=1, \text{ with } \partial\left(fg^{c-1}\right) = g^c, \\ 0 \to Rf^{(c)} \xrightarrow{\partial} Rf^{(c-1)}g \to 0, & \text{for even } n \geq 2, \text{ with} \\ & \quad \partial\left(f^{(c)}\right) = f^{(c-1)}g, \\ 0 \to Rfg^{(c-1)} \xrightarrow{\partial} Rg^{(c)} \to 0, & \text{for odd } n \geq 3, \text{ with} \\ & \quad \partial\left(fg^{(c-1)}\right) = cg^{(c)}, \end{cases}$$

and argue as in (1.11). ▮

(3.10) COROLLARY. *Let $n, k \geq 1$ be integers.*
If either $n \leq 2$ of $k!$ is invertible in R, then the acyclicity of $\mathscr{G}^k\mathbf{M}$ is equivalent to that of $\mathscr{G}^k(\mathbf{E}(n) \oplus \mathbf{N})$.
If either $n = 1$ or $k!$ is invertible in R, then the acyclicity of $\mathscr{D}^k\mathbf{M}$ is equivalent to that of $\mathscr{D}^k(\mathbf{E}(n) \oplus \mathbf{M})$.

The following theorems, in which $\mathbf{F} = (F, \varphi)$ is a finite free complex with no gaps and $M = \mathrm{H}_0(\mathbf{F})$, are the main results of this section. The initial proofs are replaced by more direct ones, suggested by the referee.

(3.11) THEOREM. *Let $k \geq 2$ be an integer. Conditions* (i) *and* (ii) *below are equivalent*:

(i) $\mathscr{G}^k\mathbf{F}$ *is acyclic*; grade $I_{r_j}(\varphi_j) \geq 1$ *for each odd $j \geq 1$*; $k!$ *is invertible in R.*

(ii) *the grade condition $\mathbf{GC}^k(j)$ holds when j is even*; *the sliding grade condition $\mathbf{SGC}^k(j)$ holds when j is odd*; $k!$ *is invertible in R.*
They imply

(iii) $\mathscr{G}^i\mathbf{F}$ *is a free resolution of* S^iM *for $i = 1, \ldots, k$.*
If $\varphi_m \neq 0$ for some odd $m \geq 3$, then all three conditions are equivalent.

Proof. Consider the canonical extension $\mathrm{C}(\mathbf{F}) \to \mathrm{D}(\mathbf{F})$ of the inclusion $\mathbf{F} \to \mathrm{D}(\mathbf{F})$. When $k!$ is invertible in R and $1 \leq i \leq k$, the induced map of complexes $\mathscr{S}^i\mathbf{F} \to \mathscr{G}^i\mathbf{F}$ is an isomorphism, with inverse given in the notation of (1.10) and (3.8) by

$$\prod_{j=0}^{t}\left(f_{j,1}^{(c_{j,1})} \cdots f_{j,b_j}^{(c_{j,b_j})}\right) \mapsto \frac{1}{\prod_{j=1}^{t}\left(c_{j,1}! \cdots c_{j,b_j}!\right)} \prod_{j=0}^{t}\left(f_{j,1}^{c_{j,1}} \cdots f_{j,b_j}^{c_{j,b_j}}\right).$$

Thus the implications (i) ⇔ (ii) ⇒ (iii) follow from (2.1). For the implication (iii) ⇒ (i) one argues as in the corresponding implication of (2.1). ▮

Remark. Another way to obtain Theorem 3.11 is by going through the proof of (2.1) with (3.10) substituting (1.12).

An analogous argument yields

(3.12) THEOREM. *Let $k \geq 2$ be an integer. Conditions* (i) *and* (ii) *below are equivalent*:

(i) $\mathscr{D}^k\mathbf{F}$ *is acyclic*; grade $I_{r_j}(\varphi_j) \geq 1$ *for each even* $j \geq 2$; $k!$ *is invertible in* R.

(ii) *the grade condition* $\mathbf{GC}^k(j)$ *holds when j is odd*; *the sliding grade condition* $\mathbf{SGC}^k(j)$ *holds when j is even*; $k!$ *is invertible in* R.

They imply

(iii) $\mathscr{D}^i\mathbf{F}$ *is a free resolution of* $\bigwedge^i M$ *for* $i = 1, \ldots, k$.

If $\varphi_m \neq 0$ *for some even* m, *then all three conditions are equivalent*.

In view of Remark 3.6(b), the preceding theorem generalizes [11, (3.1a)].

Corresponding to (2.8) and (2.9) we have

(3.13) THEOREM. *Let $k \geq 2$ be an integer. Assume that $k = 2$ or that* $\operatorname{Supp}(M) = \operatorname{Spec}(R)$. *If* grade $I_1(\varphi_m) \geq 2$ *for some odd* $m \geq 3$, *then*

(i-) $\mathscr{G}^k\mathbf{F}$ *is acyclic and* grade $I_{r_j}(\varphi_j) \geq 1$ *for each odd integer* $j \geq 1$

is equivalent to each one of the conditions (ii) *and* (iii) *from* (3.11).

(3.14) THEOREM. *Let $k \geq 2$ be an integer. Assume that $k = 2$ or that* $\operatorname{Supp}(M) = \operatorname{Spec}(R)$. *If* grade $I_1(\varphi_m) \geq 2$ *for some even* m, *then*

(i-) $\mathscr{D}^k\mathbf{F}$ *is acyclic and* grade $I_{r_j}(\varphi_j) \geq 1$ *for each even integer* $j \geq 2$

is equivalent to each one of the conditions (ii) *and* (iii) *from* (3.12).

4. COMPLEXES OF LENGTH AT MOST 2

For complexes of length at most 2 we have acyclicity criteria, which do not involve conditions on the additive torsion of R.

The proof of Theorem 4.1 is obtained by using (3.10) in the same way as (1.12) was used in the proof of (2.1); cf. Remarks 2.6 and 2.7.

(4.1) THEOREM. *Let* $\mathbf{F}$ *be a finite free complex with no gaps and with* $\lambda(\mathbf{F}) = 1$. *Set* $M = \mathrm{H}_0(\mathbf{F})$ *and let* $k \geq 1$ *be an integer. The following are equivalent*:

(i) $\mathscr{D}^i\mathbf{F}$ *is a free resolution of* $\bigwedge^i M$ *for* $i = 1, \ldots, k$.

(ii) $\mathscr{D}^k\mathbf{F}$ *is acyclic*.

(iii) grade $I_{r_1}(\varphi_1) \geq k$.

In the case when R is Noetherian, the equivalence of (i) and (iii) in the preceding result is due to Lebelt [9, Corollary 1 to Theorem 5 and Corollary to Theorem 13].

An analogous argument yields

(4.2) THEOREM. *Let* $\mathbf{F}$ *be a finite free complex with no gaps and with* $\lambda(\mathbf{F}) \leq 2$. *Set* $M = \mathrm{H}_0(\mathbf{F})$ *and let* $k \geq 1$ *be an integer. The following are equivalent*:

(i) $\mathscr{G}^i\mathbf{F}$ *is a free resoltuion of* S^iM *for* $i = 1, \ldots, k$.

(ii) $\mathscr{G}^k\mathbf{F}$ *is acyclic and* grade $I_{r_1}(\varphi_1) \geq 1$.

(iii) grade $I_{r_2}(\varphi_2) \geq 2k$; grade $I_{r_1 - t}(\varphi_1) \geq 1 + t$ *for* $t = 0, \ldots, k-1$.

In the case when $\lambda(\mathbf{F}) \leq 1$, Theorem 4.2 is due to Avramov; cf. the proof of [2, Proposition 1].

Remark. As noted by the referee, for a finite free complex $\mathbf{F}$ of length at most 2 the DG algebra $\mathrm{D}(\mathbf{F})$ and the complexes $\mathscr{G}^i\mathbf{F}$ are defined (for purposes different from the above) in [7], where they are called $S\mathbf{F}$ and $S_i\mathbf{F}$, respectively.

5. PROJECTIVE DIMENSION 1

Throughout this section the ring R is assumed to be Noetherian.

Let $q \geq 1$ be an integer. An R-module M is said to be *q-torsion-free* if every R-regular sequence of length $\leq q$ is also M-regular. If M is finite and of finite projective dimension, then M is q-torsion-free precisely when the inequality depth $M_\mathfrak{p} \geq \min(q, \text{depth } R_\mathfrak{p})$ holds for each $\mathfrak{p} \in \mathrm{Spec}(R)$ [1, (4.25)]. In this case the property of being q-torsion-free localizes.

Let U be the set of all nonzero divisors of R. We say that the finite R-module M has rank r if $U^{-1}M$ is a free $U^{-1}R$-module of rank r. If $R^s \xrightarrow{\varphi} R^m \to M \to 0$ is a free presentation of M, the kth Fitting invariant of M is the ideal $\mathrm{F}_k(M) = I_{m-k+1}(\varphi)$ of minors of order $m - k + 1$ of φ. When the ring R is local and $\mathbf{F} = (F, \varphi)$ is a minimal free resolution of M, we write $b_1(M) = \text{rank } F_1$ for the first Betti number of M.

The next theorem is the main result of this section.

(5.1) THEOREM. *For a finite R-module M set* $b(M) = \sup\{b_1(M_\mathfrak{p}) | \mathfrak{p} \in \mathrm{Spec}(R)\}$ *and let* $q \geq 1$ *be an integer. If* $\mathrm{pd}_R\, M = 1$, *then the following two conditions are equivalent and imply* $\mathrm{pd}_R\, \mathrm{S}^tM \leq \min(b(M), t)$ *for each* $t \geq 1$:

(i) S^tM *is q-torsion-free for each* $t \geq 1$.

(ii) S^tM *is q-torsion-free for* $t = 1, \ldots, b(M)$.

If in addition M has rank r, then they are also equivalent to

(iii) grade $\mathrm{F}_{r+t}(M) \geq t + q$ *for* $t = 1, \ldots, b(M)$.

Remark. When R is a Cohen–Macaulay domain and $q = 1$, the equivalence of conditions (i) and (iii) above is due to Huneke [8, Theorem 1.1]. Under the assumptions of (5.1), this equivalence is due to Avramov [2, Proposition 4].

For the proof of the theorem we need an elementary characterization of q-torsion-freeness:

(5.2) LEMMA. *Let* $\mathbf{F} = (F, \varphi)$ *be a free complex. Set* $M = \mathrm{H}_0(\mathbf{F})$ *and let* $q \geq 1$ *be an integer. The following two conditions are equivalent*:

(i) *The module* M *is* q*-torsion-free and* $\mathbf{F}$ *is a resolution of* M.

(ii) *For every* R*-regular sequence* $\mathbf{x} = (x_1, \ldots, x_s)$ *of length* $s \leq q$ *the complex* $\mathbf{F}/(\mathbf{x})\mathbf{F}$ *is an* $R/(\mathbf{x})$*-free resolution of the module* $M/(\mathbf{x})M$.

Proof. By induction on q, it suffices to treat the case $\mathbf{x} = (x_1)$. The assertion is trivial if R contains no regular elements. If $x_1 = x \in R$ is an R-regular element, consider the exact sequence of complexes

$$0 \to \mathbf{F} \xrightarrow{x} \mathbf{F} \to \mathbf{F}/x\mathbf{F} \to 0.$$

We have $\mathrm{H}_0(\mathbf{F}/x\mathbf{F}) = M/xM$ and, if $\mathbf{F}$ is a resolution of M, then the homology exact sequence yields $\mathrm{H}_1(\mathbf{F}/x\mathbf{F}) = \ker(M \xrightarrow{x} M)$ and $\mathrm{H}_i(\mathbf{F}/x\mathbf{F}) = 0$ for $i \geq 2$. Thus $\mathbf{F}$ is a resolution of M and x is M-regular if and only if $\mathbf{F}$ is a resolution of M and $\mathbf{F}/x\mathbf{F}$ is a resolution of M/xM. ∎

The next two lemmas contain the main ingredients of the proof of (5.1).

(5.3) LEMMA. *Let* M *be a finite* R*-module, let* P *be a finite projective* R*-module, and let* $q, b \geq 1$ *be integers*.

The R*-modules* $\mathrm{S}^t M$ *are* q*-torsion-free* (*resp. of finite projective dimension*) *for* $t = 1, \ldots, b$ *if and only if the* R*-modules* $\mathrm{S}^t(M \oplus P)$ *are* q*-torsion-free* (*resp. of finite projective dimension*) *for* $t = 1, \ldots, b$.

Proof. As P is a direct summand of a finite free R-module F, we have theh split inclusions $M \hookrightarrow M \oplus P \hookrightarrow M \oplus F$. They in turn induce for $t = 1, \ldots, b$ the split inclusions

$$\mathrm{S}^t M \hookrightarrow \mathrm{S}^t(M \oplus P) \hookrightarrow \mathrm{S}^t(M \oplus F) \cong \bigoplus_{i=0}^{t} (\mathrm{S}^i M \otimes \mathrm{S}^{t-i} F).$$

Since each $\mathrm{S}^k F$ is a finite free R-module, the assertion of the lemma is immediate from the fact that a finite direct sum of R-modules is q-torsion-free (resp. of finite projective dimension) if and only if each direct summand is q-torsion-free (resp. of finite projective dimension). ∎

Consider a complex of length 1,

$$(**) \qquad \mathbf{F}: 0 \to R^k \xrightarrow{\varphi} R^m \to 0,$$

where $k \geq 1$ and $m \geq 1$, and set $M = \mathrm{H}_0(\mathbf{F})$.

(5.4) LEMMA. *Let $q, b \geq 1$ be integers. The following are equivalent*:

(i) $\mathbf{F}$ *is acyclic, and* $\mathrm{S}^t M$ *is q-torsion-free for* $t = 1, \dots, b$.

(ii) $\mathscr{G}^t \mathbf{F}$ *is acyclic and* $\mathrm{S}^t M$ *is q-torsion-free for* $t = 1, \dots, b$.

(iii) $\mathscr{G}^b \mathbf{F}$ *is acyclic and* $\mathrm{S}^b M$ *is q-torsion-free*.

(iv) *For each R-regular sequence* $\mathbf{x} = (x_1, \dots, x_s)$ *of length $s \leq q$ there are inequalities* $\operatorname{grade}_{R/(\mathbf{x})} I_{k-t}(\varphi \otimes R/(\mathbf{x})) \geq 1 + t$ *for* $t = 0, \dots, b-1$.

(v) $\operatorname{grade}_R I_{k-t}(\varphi) \geq 1 + t + q$ *for* $t = 0, \dots, b-1$.

Proof. (v) $\Leftrightarrow$ (iv) is immediate from the basic properties of grade; cf., e.g., [12, Chap. 5, Theorems 15 and 19].

(iv) $\Rightarrow$ (ii)By (4.2) the complexes $(\mathscr{G}^t \mathbf{F}) \otimes R/(\mathbf{x})$ are acyclic for $t = 1, \dots, b$ and for every R-regular sequence $\mathbf{x}$ of length $s \leq q$. This implies (ii) by (5.2).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv) By (5.2) the complexes $(\mathscr{G}^b \mathbf{F}) \otimes R/(\mathbf{x})$ are acyclic for every R-regular sequence $\mathbf{x}$ of length $s \leq q$. This implies (iv) by (4.2).

(ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (ii) We induct on b. The case $b = 1$ is trivial. Assume that $b \geq 2$ and that the implication holds for $b - 1$. The induction hypothesis shows that $\mathscr{G}^t \mathbf{F}$ is acyclic for $t = 1, \dots, b-1$. In view of the already established implication (ii) $\Rightarrow$ (v) this gives grade $I_{k-t}(\varphi) \geq 1 + t + q \geq 1 + t$ for $t = 0, \dots, b-2$. Thus

$$\operatorname{grade} I_{k-b+1}(\varphi) \geq \operatorname{grade} I_{k-b+2}(\varphi) \geq 1 + (b-2) + q \geq 1 + (b-1);$$

hence $\mathscr{G}^b \mathbf{F}$ is acyclic by (4.2). ∎

Proof of Theorem 5.1. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) If R is local, then $k = b(M)$ in the minimal free resolution $(**)$ of M, and by (5.4) condition (5.4(v)) holds with $b = b(M)$. Therefore condition (5.4(v)) holds for each $b \geq 1$. Since it implies condition (5.4(ii)), we obtain the desired conclusion in the local case.

By the argument above, (ii) implies also the acyclicity of $\mathscr{G}^t \mathbf{F}$ for each $t \geq 1$; in particular, (3.7) gives $\operatorname{pd}_R \mathrm{S}^t M \leq \min(b(M), t)$ for each $t \geq 1$, which establishes the assertion on the projective dimensions in the local case.

Now consider the general case. As $\operatorname{pd}_R M = 1$, there exists a projective module P such that $M \oplus P$ has a finite free resolution of minimal length of the form $(**)$. By (5.3) condition (ii) holds for $M \oplus P$. Thus (5.4) implies for $t = 1, \dots, b(M)$ that the complex $\mathscr{G}^t\mathbf{F}$ is a free resolution of $\mathrm{S}^t(M \oplus P)$. In particular $\mathrm{S}^t(M \oplus P)$ has finite projective dimension over R for $t = 1, \dots, b(M)$ and (5.3) yields finite projective dimension of S^tM for $t = 1, \dots, b(M)$.

As pointed out in the beginning of this section, for finite modules of finite projective dimension the property of being q-torsion-free is local. Thus the already considered local case yields for each $t \geq 1$ and each $\mathfrak{p} \in \operatorname{Spec}(R)$ that $(\mathrm{S}^tM)_\mathfrak{p}$ is q-torsion-free over $R_\mathfrak{p}$ and that $\operatorname{pd}_{R_\mathfrak{p}}(\mathrm{S}^tM)_\mathfrak{p} \leq \min(b(M), t)$.

Therefore (ii) implies $\operatorname{pd}_R \mathrm{S}^tM \leq \min(b(M), t)$ for each $t \geq 1$; in particular, the torsion-freeness of the symmetric powers of M can be calculated locally. With this, the proofs of (i) $\Leftrightarrow$ (ii) and of the assertion on the projective dimensions are complete.

Assume next that M has rank r.

(iii) $\Leftrightarrow$ (ii) By the argument above, condition (ii) localizes. As the same holds for condition (iii), we may assume that R is local and then $k = b(M)$ in the minimal resolution $(**)$ of M. Rewriting (iii) gives grade $I_{k-t}(\varphi) \geq 1 + t + q$ for $t = 0, \dots, b(M) - 1$, which is equivalent to (ii) by (5.4). ∎

6. EXAMPLES

The first example shows that the assumption "grade $I_{r_j}(\varphi_j) \geq 1$ for each odd $j \geq 1$" in condition (2.1(i)) does not follow from the other two assumptions made there:

(6.1) Example. The complex of free R-modules

$$\mathbf{F}\colon 0 \to Rg_1 \oplus Rg_1 \xrightarrow{(0 \quad 1)} Rh \to 0$$

is not acyclic and has grade $I_{r_1}(\varphi_1) = 0$, yet $\mathscr{S}^2\mathbf{F}$ is the exact complex

$$0 \to R(g_1g_2) \xrightarrow{\begin{pmatrix} -1 \\ 0 \end{pmatrix}} R(g_1h) \oplus R(g_2h) \xrightarrow{(0 \quad 1)} Rh^2 \to 0.$$

In [14, Sect. 2] Weyman associates to an integer i and a finite complex of free modules $\mathbf{F} = (F, \varphi)$ a graded R-module $L_i\mathbf{F}$, together with an endomorphism d of $L_i\mathbf{F}$ of degree -1. However, this construction does not in general produce a complex:

(6.2) Example. Consider the exact complex $\mathbf{F}$ of length 2 of free R-modules:

$$0 \to Re \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} Rf_1 \oplus Rf_2 \xrightarrow{(1 \quad -1)} Rg \to 0.$$

The construction in [14, Sect .2] gives $L_2\mathbf{F}$ in the form

$$0 \to R(e\otimes f_1) \oplus R(e\otimes f_2) \xrightarrow{d_3=\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}} R(e\otimes g) \oplus Rf_1^{(2)} \oplus R(f_1f_2) \oplus Rf_2^{(2)}$$

$$\xrightarrow{d_2=\begin{pmatrix} -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 \end{pmatrix}} R(f_1\otimes g) \oplus R(f_2\otimes g) \to 0$$

in which

$$d_2 \circ d_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0.$$

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REFERENCES

1. M. Auslander and M. Bridger, Stable module theory, *Mem. Amer. Math. Soc.* **94** (1969).
2. L. L. Avramov, Complete intersections and symmetric algebras, *J. Algebra* **73** (1981), 248–263.
3. D. A. Buchsbaum and D. Eisenbud, What makes a complex exact?, *J. Algebra* **25** (1973), 259–268.
4. D. A. Buchsbaum and D. Eisenbud, Some structure thoerems for finite free resolutions, *Adv. in Math.* **12** (1974), 84–139.
5. H. Cartan, Algèbres d'Eilenberg–MacLane, *Seminaire H. Cartan* 1954–55, *Exposés* 2 *à* 11, *École Normale Supérieure*, 2nd ed., Vol. III, Springer-Verlag, Berlin/New York, 1979, p. 1309.
6. T. H. Gulliksen and G. Levin, Homology of local rings, *Queen's Papers in Pure and Appl. Math.* **20** (1969).
7. M. Hashimoto and K. Kurano, Resolutions of determinantal ideals: n-minors of $(n+2)$-square matrices, *Adv. in Math.* **94** (1992), 1–66.
8. C. Huneke, On the symmetric algebra of a module, *J. Algebra* **69** (1981), 113–119.
9. K. Lebelt, Über Torsion äußerer Potenzen von Moduln der homologischen Dimension 1, Thesis, Technical Univ. Clausthal, 1973.
10. K. Lebelt, Zur homologischen Dimension äußerer Potenzen, *Arch. Math. (Basel)* **26** (1975), 595–601.
11. K. Lebelt, Freie Auflösungen äußerer Potenzen, *Manuscripta Math.* **21** (1977), 341–355.
12. D. G. Northcott, Finite free resolutions, *Cambridge Tracts in Math.* **71** (1976).
13. C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, *Inst. Hautes Études Sci. Publ. Math.* **42** (1973), 47–119.
14. J. Weyman, Resolutions of exterior and symmetric powers, *J. Algebra* **58** (1979), 333–341.