Inverses of Quasi-tridiagonal Matrices

S. A. H. Rizvi
Department of Mathematics and Astronomy
University of Lucknow
Lucknow-226007, India

Submitted by Hans Schneider

ABSTRACT

Discretizations in various types of problems lead to quasi-tridiagonal matrices. In this paper, the inverse of a (nonsingular) quasi-tridiagonal matrix is obtained. In addition, a necessary and sufficient condition for a block matrix to have a quasi-tridiagonal inverse is derived.

1. INTRODUCTION

Inverses of nonsingular tridiagonal matrices have been obtained under different sets of conditions by various authors (see Haley [1] for a recent survey of methods for inverting tridiagonal matrices; see also Valvi [2] and Yamomoto and Ikebe [3]). Haley [1] has used the method of projection recurrences for obtaining inverses of three different types of nonsingular tridiagonal matrices, viz. those having (i) no vanishing element in the secondary diagonals, (ii) vanishing elements in both secondary diagonals, and (iii) a constant diagonal and arbitrary secondary diagonals. His results have been obtained as special cases of results on band matrices. For the case of block matrices, Schechter [4] obtained an *LU*-decomposition of quasi-tridiagonal matrices in connection with a direct method for solving a certain set of linear equations, and used his results for solving a certain boundary problem for equations of mixed type. Ikebe [5] has obtained the inverse of tridiagonal matrices of type (i) above as a special case of Hessenberg matrices. He has also generalized the results to quasi-tridiagonal matrices having nonsingular super- and subdiagonal blocks.

Recently, Barrett [6] has given a characterization of matrices having tridiagonal inverses, and has obtained an explicit tridiagonal form for the inverses of such matrices. Barrett has defined the "triangle property" for

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matrices, and proved that the possession of this property is a necessary and sufficient condition for a matrix to have a tridiagonal inverse under certain conditions on the elements of the leading diagonal. The results of Valvi [2] are special cases of Barrett's results.

The aim of this paper is twofold: to obtain an explicit expression for the inverse of a nonsingular quasi-tridiagonal matrix, and to obtain a necessary and sufficient condition for a block matrix to have a quasi-tridiagonal inverse. In Section 2, the inverse of an upper quasi-bidiagonal matrix is given, and the inverse of a quasi-tridiagonal matrix is obtained with its help. In Section 3, analogous to the triangle property defined by Barrett [6], a "quasi-triangle property" is defined for block matrices. A formula for the determinant of a block matrix possessing this quasi-triangle property is obtained, and an explicit expression for the block inverse of a nonsingular block matrix of this type is given. The paper concludes by establishing in Section 4 a necessary and sufficient condition for a nonsingular block matrix to have a quasi-tridiagonal inverse.

NOTATION. $i = j(k)m$ stands for $i = j, j+k, j+2k, \ldots, m$. In the sequel, unless otherwise stated, both i and j will take all integral values from 1 to n. Uppercase Latin letters denote block matrices with square diagonal blocks or blocks of square matrices. I and 0 denote the identity and the null matrix, respectively.

2. QUASI-TRIDIAGONAL MATRICES

In this section, we shall obtain the block inverse of a nonsingular quasi-tridiagonal matrix T, i.e. a matrix $T = [T_{ij}]$ such that $T_{ij} = 0$ whenever $|i - j| > 1$. For this, we shall need certain preliminary lemmas.

LEMMA 1 (Singh [7]). *The block inverse $H = [H_{ij}]$ of the nonsingular lower quasi-bidiagonal matrix L, i.e. a matrix $L = [L_{ij}]$ such that $L_{ij} = 0$ whenever $i < j$ or $i > j+1$, is given by the following relations*:

$$H_{ij} = 0, \quad i < j; \qquad H_{ii} = L_{ii}^{-1};$$

$$H_{ij} = (-1)^{i+j}\left\{\prod_{k=0}^{i-j-1}\left(L_{i-k,i-k}^{-1}L_{i-k,i-k-1}\right)\right\}L_{jj}^{-1}, \qquad i > j.$$

For upper quasi-bidiagonal matrices, using induction, one can easily prove

LEMMA 2. *The block inverse $H = [H_{ij}]$ of the nonsingular upper quasi-bidiagonal matrix U, i.e. a matrix $U = [U_{ij}]$ such that $U_{ij} = 0$ whenever $i > j$ or $i < j - 1$, is given by the following relations*:

$$H_{ij} = 0, \quad i > j; \qquad H_{ii} = U_{ii}^{-1};$$

$$H_{ij} = (-1)^{i+j} U_{ii}^{-1} \left\{ \prod_{k=i}^{j-1} \left(U_{k,k+1} U_{k+1,k+1}^{-1} \right) \right\}, \qquad i < j.$$

It may be noted here that Lemma 2 provides an extension, to the case of block matrices, of the form of the inverse of a bidiagonal matrix obtained by Chatterjee [8].

LEMMA 3 (Schechter [4]). *The quasi-tridiagonal matrix T can be expressed as $T = LU$, where L and U are lower and upper quasi-bidiagonal matrices, respectively, and are given by the following relations*:

$$\begin{aligned} &U_{11} = T_{11}, \qquad U_{jj} = T_{jj} - T_{j,j-1} U_{j-1,j-1}^{-1} T_{j-1,j}, \\ &U_{j-1,j} = T_{j-1,j}, \qquad L_{jj} = I, \qquad L_{j+1,j} = T_{j+1,j} U_{jj}^{-1}. \end{aligned} \tag{1}$$

A sufficient condition for the validity of this decomposition is that the leading principal block submatrices $T_k = [T_{ij}]$, $i, j = 1(1)k$, $k = 1(1)(n-1)$, are nonsingular. This follows from the fact that the decomposition requires only the nonsingularity of the matrices U_{jj}, $j = 1(1)(n-1)$, which, in turn, is equivalent to the nonsingularity of T_k, $k = 1(1)(n-1)$, because $\det T_k = \prod_{s=1}^{k} \det U_{ss}$, as one can easily see with the help of induction, using [9, p. 46, Equation I].

We now give the inverse of a nonsingular quasi-tridiagonal matrix T.

THEOREM 1. *If $T = [T_{ij}]$ is a nonsingular quasi-tridiagonal matrix all of whose leading principal block submatrices $T_k = [T_{ij}]$, $i, j = 1(1)k$, $k = 1(1)(n-1)$ are nonsingular, then the block inverse $H = [H_{ij}]$ of T is given by the following relations*:

$$H_{ij} = \begin{cases} (-1)^{i+j} U_{ii}^{-1} P(i,j) \left\{ I + \sum\limits_{s=j+1}^{n} P(j,s) Q(s,j) \right\}, & i < j, \\ U_{ii}^{-1} \left\{ I + \sum\limits_{s=i+1}^{n} P(i,s) Q(s,i) \right\}, & i = j, \\ (-1)^{i+j} U_{ii}^{-1} \left\{ I + \sum\limits_{s=i+1}^{n} P(i,s) Q(s,i) \right\} Q(i,j), & i > j, \end{cases}$$

where $P(i,\ j) = \prod_{k=i}^{j-1}(T_{k,\ k+1}U_{k+1,\ k+1}^{-1})$, $Q(i,\ j) = \prod_{k=0}^{i-j-1}(T_{i-k,\ i-k-1}U_{i-k-1,\ i-k-1}^{-1})$, *the* U_{jj} *being given by* (1) *and an empty sum being taken to be* 0.

Proof. Since, by Lemma 3, $T = LU$, we have $H = U^{-1}L^{-1}$. Substituting the values of U^{-1} and L^{-1} given by Lemmas 2 and 1, respectively, we easily obtain the values of H_{ij} given in the theorem. ∎

It may be noted here that Theorem 1 extends both Lemmas 1 and 2 given above.

3. THE QUASI-TRIANGLE PROPERTY

In this section, we define the "quasi-triangle property" for block matrices, and study matrices having this property. We first give a definition and certain preliminary lemmas.

DEFINITION. The block matrix $R = [R_{ij}]$ is said to have the quasi-triangle property if R_{ii}, $i = 2(1)(n-1)$, are nonsingular, and

$$R_{ij} = R_{ik}R_{kk}^{-1}R_{kj} \qquad \text{for all} \quad i<k<j \quad \text{and} \quad \text{for all} \quad i>k>j. \tag{2}$$

Throughout this section, $R = [R_{ij}]$ will denote a block matrix having the quasi-triangle property, and $A = [A_{ij}]$ will denote the block inverse of R.

From the matrix identity [10, Equation (13)] $(D - VW^{-1}U)^{-1}VW^{-1} = D^{-1}V(W - UD^{-1}V)^{-1}$, we have

LEMMA 4. *If* R_{pp}, R_{qq},

$$B_{pq} = R_{pp} - R_{pq}R_{qq}^{-1}R_{qp}, \tag{3}$$

and B_{qp} *are all nonsingular, then*

$$R_{pp}^{-1}R_{pq}B_{qp}^{-1} = B_{pq}^{-1}R_{pq}R_{qq}^{-1}.$$

LEMMA 5.

$$\det R = \det B_{12} \det R_{22} \prod_{k=3}^{n} \det B_{k,k-1}. \tag{4}$$

Proof. Subtracting $[R_{i2}]R_{22}^{-1}R_{21}$ from the block column $[R_{i1}]$, we get [9, p. 43]

$$\det R = \det B_{12}\det\big[R_{ij}\big], \qquad i,j=2(1)n.$$

Further, subtracting $[R_{i,s-1}]R_{s-1,s-1}^{-1}R_{s-1,s}$ from the block column $[R_{is}]$, $s=n(-1)4$, we similarly get

$$\begin{aligned}\det R &= \det B_{12}\det\begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33}\end{bmatrix}\prod_{k=4}^{n}\det\big[R_{kk}-R_{k,k-1}R_{k-1,k-1}^{-1}R_{k-1,k}\big] \\ &= \det B_{12}\det\begin{bmatrix} R_{22} & R_{23} \\ R_{32} & R_{33}\end{bmatrix}\prod_{k=4}^{n}\det B_{k,k-1}. \end{aligned}\tag{5}$$

Hence [9, p. 46, Equation I] the lemma follows. ■

It may be noted here that the formula (4) reduces to the formula obtained by Barrett [6] for matrices having the triangle property if we take the matrices R_{ij} to be of order unity and identify them with the corresponding scalars. Further, it should be mentioned here that, if either R_{11} or R_{nn} is nonsingular, we get the simple formulae

$$\det R = \begin{cases} \det R_{11}\left(\prod_{k=2}^{n}\det B_{k,k-1}\right), & \det R_{11}\neq 0, \\ \left(\prod_{k=1}^{n-1}\det B_{k,k+1}\right)\det R_{nn}, & \det R_{nn}\neq 0. \end{cases}$$

The formula of Lemma 5 has been given for the sake of generality.

We now give an explicit expression for the block inverse A of R. In fact, we prove

THEOREM 2. *Let A denote the block inverse of the nonsingular block matrix R (which has the quasi-triangle property). For n = 3, A is quasi-tridiagonal and is given by the following relations:*

$$A_{ii}=B_{i2}^{-1},\quad A_{i2}=-B_{i2}^{-1}R_{i2}R_{22}^{-1},\quad A_{2i}=-R_{22}^{-1}R_{2i}B_{i2}^{-1},\qquad i=1,3;$$

$$A_{22}=R_{22}^{-1}+R_{22}^{-1}R_{21}B_{12}^{-1}R_{12}R_{22}^{-1}+R_{22}^{-1}R_{23}B_{32}^{-1}R_{32}R_{22}^{-1},\qquad A_{13}=A_{31}=0.$$

For $n \geqslant 4$, A is again quasi-tridiagonal and is given by the following relations:

$$A_{11} = B_{12}^{-1}, \qquad A_{12} = -B_{12}^{-1}R_{12}R_{22}^{-1},$$

$$A_{21} = -R_{22}^{-1}R_{21}B_{12}^{-1}, \qquad A_{22} = B_{23}^{-1} + R_{22}^{-1}R_{21}B_{12}^{-1}R_{12}R_{22}^{-1};$$

$$A_{ij} = 0, \qquad |i-j| > 1;$$

$$A_{ii} = B_{i,i-1}^{-1} + R_{ii}^{-1}R_{i,i+1}B_{i+1,i}^{-1}R_{i+1,i}R_{ii}^{-1}, \qquad i = 3(1)(n-1),$$

$$A_{i,i+1} = -R_{ii}^{-1}R_{i,i+1}B_{i+1,i}^{-1}, \quad A_{i+1,i} = -B_{i+1,i}^{-1}R_{i+1,i}R_{ii}^{-1},$$

$$i = 2(1)(n-1); \quad A_{nn} = B_{n,n-1}^{-1}.$$

Proof. Except for B_{23}, the inverses involved in the theorem all exist either because of the quasi-triangle property or because of Lemma 5. B_{23} can also be easily seen to be nonsingular from Equation (5) using Equation II of [10, p. 46]. The above forms for the inverses are easily verified to be correct by direct multiplication of R and A making use of Lemma 4 and the relations (2) and (3). As an illustration, we show that $(RA)_{ij} = 0$ for $i > j$, $3 \leqslant j \leqslant n-1$.

Since $A_{ij} = 0$ for $|i-j| > 1$, $(RA)_{ij} = \sum_{k=j-1}^{j+1} R_{ik}A_{kj} = (R_{ij} - R_{i,j-1}R_{j-1,j-1}^{-1}R_{j-1,j})B_{j,j-1}^{-1} - (R_{i,j+1} - R_{ij}R_{jj}^{-1}R_{j,j+1})B_{j+1,j}^{-1}R_{j+1,j}R_{jj}^{-1}$. Since R has the quasi-triangle property (2), we have

$$R_{i,j-1} = R_{ij}R_{jj}^{-1}R_{j,j-1}, \qquad R_{ij} = R_{i,j+1}R_{j+1,j+1}^{-1}R_{j+1,j}. \tag{6}$$

Substituting these expressions for $R_{i,j-1}$ and R_{ij} in the first and second terms, respectively, in the latter expression for $(RA)_{ij}$, and using relation (3), we get

$$(RA)_{ij} = R_{ij}R_{jj}^{-1}B_{j,j-1}B_{j,j-1}^{-1} - R_{i,j+1}R_{j+1,j+1}^{-1}B_{j+1,j}B_{j+1,j}^{-1}R_{j+1,j}R_{jj}^{-1} = 0,$$

because of (6). ∎

Theorem 2 reduces to the result (3.2) of Barrett [6] on the inverse of a nonsingular matrix with the triangle property if we take the matrices R_{ij} to be square of order unity and identify them with the corresponding scalars. It may be noted here that if, in the case $n = 3$, R_{33} is nonsingular, the exceptional value of A_{22} reduces to the general form of A_{22} for the case $n \geqslant 4$ by virtue of Lemma 4. The greatest difficulty in generalizing the result (3.2) of Barrett [6] is the noncommutativity of the matrix blocks. The formulae obtained in Theorem 2 become analogous to those of Barrett in the case where the R_{ij} commute. In the general case, if either R_{11} or R_{nn} is nonsingular, the number of equations required to define A is reduced. The latter number is equal to that of Barrett [6] only when both R_{11} and R_{nn} are nonsingular. It is the generality of the formulae in Theorem 2 which makes the forms involved asymmetrical and cumbersome.

4. A NECESSARY AND SUFFICIENT CONDITION

In this section we obtain the necessary and sufficient condition for a block matrix to have a quasi-tridiagonal inverse. For this, we need

LEMMA 6. *If $H=[H_{ij}]$ denotes the inverse of a quasi-tridiagonal matrix $T=[T_{ij}]$, then, for $r=2(1)(n-1)$, H_{rr} is nonsingular only if the leading principal block submatrix $T_{r-1}=[T_{ij}]$, $i,j=1(1)(r-1)$, is nonsingular.*

Proof. Regarding the matrix T as an ordinary (and not a block) matrix and using the formula for minors of the inverse matrix [9, p. 21], we have

$$\det H_{rr}=(\det T)^{-1}\det T_{r-1}\det T^{+}_{r+1},$$

where $T^{+}_{r+1}=[T_{ij}]$, $i,j=(r+1)(1)n$. Hence $\det H_{rr}$ is nonzero if $\det T_{r-1}$ is nonzero. This proves the lemma. ■

We now prove a theorem which generalizes Theorem 1 of Barrett [6] to the case of block matrices.

THEOREM 3. *Let $K=[K_{ij}]$ be a nonsingular block matrix whose diagonal blocks K_{ii}, $i=2(1)(n-1)$, are nonsingular. Then the block matrix K has the quasi-triangle property if and only if its inverse is quasi-tridiagonal.*

Proof. The necessity part follows from Theorem 2 above. For the sufficiency part, we observe that, because of Lemma 6, if K_{ii}, $i=2(1)(n-1)$, are nonsingular, then the leading principal block submatrices of K^{-1} are also nonsingular. The quasi-triangle property is now easily verified using the form for K^{-1} given by Theorem 1 above. The theorem is thus proved. ■

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