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On the linear weak topology and dual pairings over rings

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Abstract

In this note we study the weak topology on paired modules over a (not necessarily commutative) ground ring. Over QF rings we are able to recover most of the well-known properties of this topology in the case of commutative base fields. The properties of the linear weak topology and the dense pairings are then used to characterize pairings satisfying the so-called α -condition.

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Introduction

Let R be a commutative field, V, W be vector spaces over R with a non-degenerating R -bilinear form $\beta: V \times W \rightarrow R$, $P := (V, W)$ be the induced R -pairing and consider $V \xrightarrow{K_P} W^*$ and $W \xrightarrow{X_P} V^*$ as vector subspaces. For every subset $K \subseteq W$ (respectively $X \subseteq W^*$) set

$$\text{An}(K) = \{f \in W^* \mid f(K) = 0\} \quad \left(\text{respectively } \text{Ke}(X) = \bigcap \{\text{Ker}(f) \mid f \in X\} \right).$$

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Considering R with the discrete topology, R^W with the product topology, the induced relative topology on $V \subseteq W^* \subset R^W$ is called the *linear weak topology* $V[\mathfrak{T}_{ls}(W)]$ and has basis of neighbourhoods of 0_V :

$$\{F^\perp: V \cap \text{An}(F) \mid F = \{w_1, \dots, w_k\} \subset W \text{ a finite subset}\}.$$

The linear weak topology on $W \subseteq V^* \subset R^V$ is defined analogously. The closure of any vector subspace $X \subset V$ is given by $\overline{X} := X^{\perp\perp}$. A closed (open) vector subspace $X \subset V$ has the form $X = K^\perp$, where $K \subset W$ is any (finite dimensional) vector subspace. The embeddings $W \hookrightarrow V^*$ respectively $V \hookrightarrow W^*$ imply that $V \subseteq W^*$ respectively $W \subseteq V^*$ are dense. The properties of this topology are well-known and were studied by several authors (e.g., [8,9,12,10]).

For the case of arbitrary base rings most of the properties of this topology (including the characterizations of closure, closed, open and dense submodules) are not valid anymore. The aim of this note is to study the properties of this topology induced on paired modules over arbitrary ground rings. In particular we extend results obtained by the author [2] on this topology from the case of commutative base rings to the arbitrary case. In contrast to the proofs in the case of base fields, which depend heavily on the existence of bases, our proofs are in module theoretic terms.

Throughout this note R denotes a (*not necessarily commutative*) associative ring with $1_R \neq 0_R$. We consider R as a right (and a left) linear topological ring with the discrete topology. The category of unitary left (right) R -modules will be denoted with ${}_R\mathcal{M}$ (\mathcal{M}_R). The category of unitary R -bimodules is denoted with ${}_R\mathcal{M}_R$. For a right (a left) R -module L we denote with L^* (*L) the set of all R -linear mappings from L to R . If V is an R -module, then an R -submodule $X \subset V$ is called R -cofinite, if V/X is finitely generated as an R -module.

Let L be a right (a left) R -module and $K \subset L$ be an R -submodule. We call $K \subset L$ N -pure for some left (right) R -module N , if the canonical mapping $\iota_K \otimes \text{id}_N: K \otimes_R N \rightarrow L \otimes_R N$ ($\text{id}_N \otimes \iota_L: N \otimes_R K \rightarrow N \otimes_R L$) is an embedding. We call $K \subset L$ pure (in the sense of Cohn), if $K \subset L$ is N -pure for every left (right) R -module N .

1. The linear weak topology

1.1. R -pairings. A *left R -pairing* $P = (V, W)$ consists of a left R -module W and a right R -module V with an R -linear mapping $\kappa_P: V \rightarrow {}^*W$ (equivalently $\chi_P: W \rightarrow V^*$). For left R -pairings $(V, W), (V', W')$ a morphism $(\xi, \theta): (V', W') \rightarrow (V, W)$ consists of R -linear mappings $\xi: V \rightarrow V'$ and $\theta: W' \rightarrow W$, such that

$$\langle \xi(v), w' \rangle = \langle v, \theta(w') \rangle \quad \text{for all } v \in V \text{ and } w' \in W'. \quad (1)$$

Let $P = (V, W)$ be a left R -pairing, $V' \subset V$ be a right R -submodule, $W' \subset W$ be a (pure) left R -submodule with $\langle V', W' \rangle = 0$. Then $Q := (V/V', W')$ is a left R -pairing, $(\pi, \iota): (V/V', W') \rightarrow (V, W)$ is a morphism of left R -pairings and we call $Q \subset P$ a (*pure*) *left R -subpairing*. The left R -pairings with the morphisms defined above build a category, which we denote by \mathcal{P}_l . The category of right R -pairings \mathcal{P}_r is defined analogously.

1.2. The finite topology. Consider R with the discrete topology. For every set A we consider R^A with the *product topology* and identify it with the set of all mappings from A to R . If W is a left R -module, then the induced relative topology on the right R -submodule ${}^*W \subset R^W$ is called the *finite topology* and makes *W a linear topological right R -module with basis of neighbourhoods of $0_{{}^*W}$:

$$\mathcal{B}_f(0_{{}^*W}) := \{ \text{An}(F) \mid F = \{w_1, \dots, w_k\} \subset W \text{ is a finite subset} \}.$$

1.3. Let $P = (V, W)$ be a left R -pairing and consider the right R -submodule ${}^*W \subset R^W$ with the finite topology. Then there is a unique topology on V , the *linear weak topology* $V[\mathfrak{T}_{ls}^r(W)]$, such that $\kappa_P : V \rightarrow {}^*W$ is continuous. A basis of neighbourhoods of 0_V is given by the *neighbourhoods*

$$\mathcal{B}_f(0_V) := \{ F^\perp := \kappa_P^{-1}(\text{An}(F)) \mid F = \{w_1, \dots, w_k\} \subset W \text{ is a finite subset} \}.$$

The closure \bar{X} of any subset $X \subseteq V$ is then given by

$$\bar{X} = \bigcap \{ X + F^\perp \mid F \subset W \text{ is a finite subset} \}.$$

Analogously one can consider W as a left linear topological R -module with the linear weak topology $W[\mathfrak{T}_{ls}^l(V)]$, which is the finest topology on W that makes $\chi_W : W \rightarrow V^*$ continuous (we consider $V^* \subset R^V$ with the finite topology).

1.4. Lemma. Let $P = (V, W)$ be a left R -pairing and consider V with the linear weak topology $V[\mathfrak{T}_{ls}^r(W)]$.

- (1) $V[\mathfrak{T}_{ls}^r(W)]$ is Hausdorff if and only if $V \xrightarrow{\kappa_P} {}^*W$.
- (2) If $\kappa_P(V) \subset {}^*W$ is dense and ${}_R R$ is W -injective, then $\widehat{V} \simeq {}^*W$ (where \widehat{V} is the completion of V w.r.t. $V[\mathfrak{T}_{ls}^r(W)]$).
- (3) The finite topology on *W is Hausdorff. If ${}_R R$ is W -injective, then *W is complete.

Proof. Denote with \mathcal{W}^f the class of all finitely generated R -submodules of W .

(1) This is evident, while

$$\overline{0_V} := \bigcap \{ K^\perp \mid K \in \mathcal{W}^f \} = \left(\sum \{ K \in \mathcal{W}^f \} \right)^\perp = W^\perp = \text{Ker}(\kappa_P).$$

(2) Consider for every left R -submodule $K \xrightarrow{\iota_K} W$ the R -linear mapping

$$\varphi_K : V \rightarrow {}^*K, v \mapsto [k \mapsto \langle v, k \rangle].$$

Since ${}_R R$ is W -injective, $\iota_K^* : {}^*W \rightarrow {}^*K$ is surjective. By assumption $\kappa_P(V) \subset {}^*W$ is dense and consequently for every finitely generated left R -submodule $K \subset W$, the R -linear mapping φ_K is surjective, hence $V/K^\perp \simeq {}^*K$. If we write $W = \varinjlim K_\lambda$ as a direct system $\{K_\lambda\}_\Lambda$ of its finitely generated R -submodules, then

$$\widehat{V} := \varprojlim V/K_\lambda^\perp \simeq \varprojlim {}^*K_\lambda \simeq \text{Hom}_{R-}(\varinjlim K_\lambda, R) = {}^*W.$$

(3) The result follows from (1) and (2). \square

1.5. Definition. An R -module U is called *FP-injective*, if every diagram of R -modules

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & R^{(\mathbb{N})} \\ & & \downarrow f & \nearrow g & \\ & & U & & \end{array}$$

with exact row and K finitely generated can be completed commutatively with some R -linear mapping $g: R^{(\mathbb{N})} \rightarrow U$.

An important role in studying the linear weak topology is played by the so-called

1.6. Annihilator conditions [13, 28.1]. Let N be an R -module.

(1) For every R -submodule $L \subset N$ we have

$$\text{Ke An}(L) = L \iff N/L \text{ is } R\text{-cogenerated.}$$

(2) If R is N -injective, then

$$\text{An}(L_1 \cap L_2) = \text{An}(L_1) + \text{An}(L_2) \quad \text{for all } R\text{-submodules } L_1, L_2 \subset N.$$

(3) If R is injective, or if N is finitely generated and R is FP-injective, then for every finitely generated R -submodule $X \subset \text{Hom}(N, R)$ we have $\text{An Ke}(X) = X$.

We call the ring R a *QF ring*, if R_R (equivalently ${}_R R$) is Noetherian and a cogenerator (e.g., [13, 48.15]).

1.7. Lemma. Let $P = (V, W)$ be a left R -pairing and consider V with the linear weak topology $V[\mathfrak{T}_{l_s}^r(W)]$.

- (1) $\bar{X} \subseteq X^{\perp\perp}$ for any subset $X \subset V$. Consequently every orthogonally closed right R -submodule of V is closed.
- (2) If R_R is Noetherian, then all open right R -submodules of V are R -cofinite.
- (3) Let $X \subset V$ be a right R -submodule, so that V/X is R -cogenerated. If $\text{An}(X) = \chi_P(X^\perp)$, then X is closed. If moreover ${}_R R$ is Noetherian, $X \subset V$ is R -cofinite and $W \xrightarrow{\chi_P} V^*$, then X is open.
- (4) Let R_R be Artinian.
 - (a) A right R -submodule $X \subset V$ is open if and only if it is closed and R -cofinite.
 - (b) Let $X \subset Y \subset V$ be right R -submodules. If $X \subset V$ is closed and R -cofinite, then $Y \subset V$ is also closed and R -cofinite.
- (5) Assume $V \subseteq {}^*W$.
 - (a) If ${}_R R$ is injective, or if ${}_R W$ is finitely generated and ${}_R R$ is FP-injective, then every finitely generated right R -submodule $X \subset V$ is closed.
 - (b) Let V_R be finitely generated. If ${}_R R$ is injective and R_R is Noetherian (e.g., R is a QF ring), then all right R -submodules of V are closed.

Proof. (1) Let $\tilde{x} \in \bar{X}$ be arbitrary. For every $w \in X^\perp$ there exist $x_w \in X$ and $v_w \in \{w\}^\perp$ with $\tilde{x} = x_w + v_w$ and so $\langle \tilde{x}, w \rangle = 0$. Consequently $\bar{X} \subseteq X^{\perp\perp}$. If X is orthogonally closed, then $\bar{X} \subseteq X^{\perp\perp} = X$, i.e., X is closed.

(2) Let $X \subset V$ be an open right R -submodule. By definition there exists a finitely generated left R -submodule $K \subset W$, such that $K^\perp \subset X$. If R_R is Noetherian, then *K_R is finitely generated, hence $K^\perp \subset V$ is R -cofinite. Consequently $X \subset V$ is R -cofinite.

(3) Let $X \subset V$ be a right R -submodule, so that V/X is R -cogenerated. If $\text{An}(X) = \chi_P(X^\perp)$, then it follows by 1.6(1) that

$$X = \text{Ke An}(X) = \text{Ke}(\chi_P(X^\perp)) = X^{\perp\perp}.$$

By (1) X is closed. Assume now that ${}_R R$ is Noetherian, $X \subset V$ is R -cofinite and $W \xrightarrow{\chi_P} V^*$. Then by assumption $X^\perp = \text{An}(X) \simeq (V/X)^*$ is finitely generated in ${}_R \mathcal{M}$ and so $X = (X^\perp)^\perp$ is open.

(4) Assume R_R to be Artinian and let $X \subset V$ be a right R -submodule.

- (a) Every open R -submodule $X \subset V$ is closed without any assumptions on R and is R -cofinite by (2). On the other hand, let $X \subset V$ be R -cofinite and closed. Since R_R is Artinian V/X is finitely cogenerated (e.g., [13, 31.4]), hence open by [4, 1.8].
- (b) Let $X \subset V$ be R -cofinite and closed. Then X is by (a) open and so $Y \supset X$ is open, hence closed. Obviously $Y \subset V$ is R -cofinite.

(5) Let $V \xrightarrow{\kappa_P} {}^*W$ be an embedding.

- (a) If $X \subset V$ is a finitely generated right R -submodule, then we have under our assumptions and applying 1.6(3): $X^{\perp\perp} = V \cap \text{An Ke}(X) = X$, hence X is closed by (1).
- (b) Since V_R is finitely generated and R_R is Noetherian, all right R -submodules of V are finitely generated. Since, by assumption, ${}_R R$ is injective, the result follows by (a). \square

Closed and open submodules

For a left R -pairing (V, W) we characterize in what follows the closed (the open) R -submodules of V w.r.t. $V[\mathfrak{I}'_S(W)]$ in case R_R is an injective cogenerator (R a QF ring).

1.8. Theorem. *Let $P = (V, W)$ be a left R -pairing and consider V with the linear weak topology $V[\mathfrak{I}'_S(W)]$. Assume R_R to be an injective cogenerator.*

- (1) *The closure of a right R -submodule $X \subseteq V$ is given by $\bar{X} = X^{\perp\perp}$.*
- (2) *Let $X \subset Y \subseteq V$ be right R -submodules. Then X is dense in Y if and only if $X^\perp = Y^\perp$. If $W \xrightarrow{\chi_P} V^*$, then $X \subset V$ is dense if and only if $X^\perp = 0$.*
- (3) *Let R be a QF ring and $X \subset V$ be an R -cofinite right R -submodule. Then X is closed if and only if $\text{An}(X) = \chi_P(X^\perp)$.*
- (4) *The class of closed R -submodules of V is given by*

$$\{K^\perp \mid K \subset W \text{ is an arbitrary left } R\text{-submodule}\}.$$

(5) If R is a QF-ring and $W \xrightarrow{\chi_P} V^*$ is an embedding, then the class of open R -submodules of V is given by

$$\{K^\perp \mid K \subset W \text{ is a finitely generated left } R\text{-submodule}\}.$$

Proof. (1) By Lemma 1.7(1) $\overline{X} \subseteq X^{\perp\perp}$. On the other hand, let $\tilde{v} \in X^{\perp\perp} \setminus \overline{X}$ be arbitrary. Then there exists by 1.6(1) a finitely generated left R -submodule $K \subset W$, such that $\tilde{v} \notin X + K^\perp = \text{Ke An}(X + K^\perp)$. Consequently there exists $\delta \in V^*$, such that $\delta(X + K^\perp) = 0$ and $\delta(\tilde{v}) \neq 0$. By assumption R_R is injective and it follows from 1.6(3) that $\delta \in \text{An}(K^\perp) = \text{An Ke}(\chi_P(K)) = \chi_P(K)$, i.e., $\delta = \chi_P(w)$ for some $w \in K$. So

$$0 = \langle \tilde{v}, w \rangle = \chi_P(w)(\tilde{v}) = \delta(\tilde{v}) \neq 0,$$

a contradiction. It follows then that $\overline{X} = X^{\perp\perp}$.

(2) $X \subset Y$ is dense if and only if $\overline{X} = \overline{Y}$ and the result follows from (1).

(3) Let R be a QF ring and $X \subseteq V$ be an R -cofinite right R -submodule. Let X be closed, i.e., $X = X^{\perp\perp}$ by (1). Since R_R is Noetherian, $\chi_P(X^\perp) \subseteq \text{An}(X) \simeq (V/X)^*$ is finitely generated in ${}_R\mathcal{M}$. Since R_R is injective, we have by 1.6(3):

$$\begin{aligned} \text{An}(X) &= \text{An Ke An}(X) = \text{An Ke An}(X^{\perp\perp}) \\ &= \text{An Ke An}(\text{Ke}(\chi_P(X^\perp))) = \chi_P(X^\perp). \end{aligned}$$

On the other hand, if $\text{An}(X) = \chi_P(X^\perp)$, then it follows by Lemma 1.7(3) that X is closed and we are done.

(4) Follows from (1) and Lemma 1.7(1).

(5) Let R be a QF ring and $W \xrightarrow{\chi_P} V^*$. If $K \subset W$ is a finitely generated left R -submodule, then $K^\perp \subset V$ is open by definition. On the other hand, if $X \subset V$ is an open right R -submodule, then X is closed, i.e., $X = X^{\perp\perp}$. By Lemma 1.7(2) $X \subset V$ is R -cofinite and so $X^\perp \xrightarrow{\chi_P} \text{An}(X) \simeq (V/X)^*$ is finitely generated in ${}_R\mathcal{M}$. \square

1.9. Corollary. Let $(V, W), (V', W')$ be left R -pairings and consider V and V' with the linear weak topology $V[\mathfrak{I}'_s(W)], V'[\mathfrak{I}'_s(W')]$ respectively. Let $(\xi, \theta) : (V', W') \rightarrow (V, W)$ be a morphism of left R -pairings.

- (1) If $K' \subset W'$ is a left R -submodule, then $\xi^{-1}(K'^\perp) = (\theta(K'))^\perp$. In particular, $\xi : V \rightarrow V'$ is continuous. In particular, $\xi^{-1}(Y') \subset V$ is closed for every closed right R -submodule $Y' \subset V'$.
- (2) If R_R is an injective cogenerator, then $\xi^{-1}(Y') \subset V$ is orthogonally closed for every closed right R -submodule $Y' \subset V'$.

Proof. (1) Trivial.

(2) If $Y' \subset V'$ is closed, then it follows by Theorem 1.8(3) that $Y' = K'^\perp$ for some R -submodule $K' \subset W'$. It follows then by (1) that $\xi^{-1}(Y') = \xi^{-1}(K'^\perp) = (\theta(K'))^\perp$, i.e., $\xi^{-1}(Y') \subset V$ is orthogonally closed. \square

1.10. Proposition. Let W, W' be left R -modules and consider ${}^*W, {}^*W'$ with the finite topology. Let $\theta \in \text{Hom}_{R-}(W', W)$ and consider the morphism of left R -pairings $(\theta^*, \theta): ({}^*W', W') \rightarrow ({}^*W, W)$.

- (1) $\theta^{*-1}(\text{An}(K')) = \text{An}(\theta(K'))$ for every left R -submodule $K' \subset W'$. In particular $\theta^*: {}^*W \rightarrow {}^*W'$ is continuous.
- (2) If ${}_R R$ is W -injective, then $\theta^*(\text{An}(K)) = \text{An}(\theta^{-1}(K))$ for every left R -submodule $K \subseteq W$.
- (3) If R_R is an injective cogenerator and ${}_R R$ is W -injective (e.g., R is a QF-ring), then
 - (a) $\theta^*: {}^*W \rightarrow {}^*W'$ is linearly closed (i.e., $\theta^*(X) \subset {}^*W'$ is closed for every closed right R -submodule $X \subset {}^*W$).
 - (b) $\overline{\theta^*(X)} = \theta^*(\overline{X})$ for every right R -submodule $X \subset {}^*W$.
 - (c) $\text{Ke}(\theta^*(X)) = \theta^{-1}(\text{Ke}(X))$ for every right R -submodule $X \subset {}^*W$.
 - (d) For R -submodules $X_1, \dots, X_k \subset {}^*W$ we have $\overline{X_1 + \dots + X_k} = \overline{X_1} + \dots + \overline{X_k}$. Hence every finite sum of closed right R -submodules of *W is closed.

Proof. (1) Trivial.

(2) Let $K \subseteq W$ be a left R -submodule. Clearly $\theta^*(\text{An}(K)) \subseteq \text{An}(\theta^{-1}(K))$. On the other hand, consider the R -linear mapping

$$0 \rightarrow W'/\theta^{-1}(K) \xrightarrow{\iota} W/K. \tag{2}$$

By assumption ${}_R R$ is W -injective and so it is W/K -injective (e.g., [13, 16.2]). Hence (2) induces the epimorphism

$${}^*(W/K) \xrightarrow{\iota^*} {}^*(W'/\theta^{-1}(K)) \rightarrow 0,$$

or equivalently the epimorphism

$$\text{An}(K) \xrightarrow{\theta^*} \text{An}(\theta^{-1}(K)) \rightarrow 0.$$

(3) Let R_R be an injective cogenerator and ${}_R R$ be W -injective.

- (a) The result follows from Theorem 1.8(1), Lemma 1.7(1) and (2).
- (b) Let $X \subset {}^*W$ be a right R -submodule. By (a) θ^* is linearly closed, so $\overline{\theta^*(X)} \subseteq \theta^*(\overline{X})$. By (1) $\theta^{*-1}(\overline{\theta^*(X)})$ is closed and it follows that $\overline{X} \subseteq \theta^{*-1}(\overline{\theta^*(X)})$, i.e., $\theta^*(\overline{X}) \subseteq \overline{\theta^*(X)}$ and the result follows.
- (c) For every right R -submodule $X \subset {}^*W$ we get by the results above:

$$\begin{aligned} \text{Ke}(\theta^*(X)) &= \text{Ke An Ke}(\theta^*(X)) = \text{Ke}(\overline{\theta^*(X)}) \\ &= \text{Ke}(\theta^*(\overline{X})) = \text{Ke}(\theta^*(\text{An Ke}(X))) \\ &= \theta^{-1}(\text{Ke An Ke}(X)) = \theta^{-1}(\text{Ke}(X)). \end{aligned}$$

- (d) Let $X_1, \dots, X_k \subset {}^*W$ be right R -submodules. By Theorem 1.8(1) and induction on k in 1.6(2) we have

$$\begin{aligned} \overline{\sum_{i=1}^k X_i} &= \text{An Ke} \left(\sum_{i=1}^k X_i \right) = \text{An} \left(\bigcap_{i=1}^k \text{Ke}(X_i) \right) \\ &= \sum_{i=1}^k \text{An Ke}(X_i) = \sum_{i=1}^k \overline{X_i}. \quad \square \end{aligned}$$

2. The α -condition

In a joint work with Gómez-Torrecillas and Lobillo [3] we presented the so-called α -condition for pairings of modules over commutative rings, which has shown to be a natural assumption in the author's study of *duality theorems* for Hopf algebras [2]. Recently that condition has shown to be a natural assumption in the study of the category of right (left) comodules of a *coring* \mathcal{C} as a full subcategory of the category right (left) modules of its *dual ring* ${}^*\mathcal{C}$ (\mathcal{C}^*) (e.g., [1]). In this section we consider this condition for pairings over arbitrary (not necessarily commutative) rings and give examples of pairings satisfying it. In particular we extend our observations in [2] on such pairings from the commutative case to the arbitrary one.

2.1. The category \mathcal{P}_l^α . We say a left R -pairing $P = (V, W)$ satisfies the α -condition (or P is an α -pairing) iff for every right R -module M the following mapping is injective

$$\begin{aligned} \alpha_M^P : M \otimes_R W &\rightarrow \text{Hom}_{-R}(V, M), \\ \sum m_i \otimes w_i &\mapsto \left[v \mapsto \sum m_i \langle v, w_i \rangle \right]. \end{aligned} \quad (3)$$

With $\mathcal{P}_l^\alpha \subset \mathcal{P}_l$ we denote the *full* subcategory of left R -pairings satisfying the α -condition (we call these *left α -pairings*). We call a left R -pairing $P = (V, W)$ *dense*, if $\kappa_P(V) \subseteq {}^*W$ is dense w.r.t. the finite topology. The subcategory of *right α -pairings* $\mathcal{P}_r^\alpha \subset \mathcal{P}_r$ is defined analogously.

2.2. Remark. Let $P = (V, W) \in \mathcal{P}_l^\alpha$. Then $W \xrightarrow{\chi_P} V^*$, hence ${}_R W$ is in particular R -co-generated. If M is an arbitrary right R -module, then we have for every right R -submodule $N \subset M$ the commutative diagram

$$\begin{array}{ccc} N \otimes_R W & \xrightarrow{\alpha_N^P} & \text{Hom}_{-R}(V, N) \\ \downarrow \iota_N \otimes \text{id}_W & & \downarrow \\ M \otimes_R W & \xrightarrow{\alpha_M^P} & \text{Hom}_{-R}(V, M) \end{array}$$

By assumption α_N^P is injective and so $N \subset M$ is W -pure. By our choice M is an arbitrary R -module, hence ${}_R W$ is flat. If ${}_R W$ is finitely presented or R is left perfect, then ${}_R W$ is projective.

An important observation for α -pairings is

2.3. Lemma. Let $P = (V, W) \in \mathcal{P}_l^\alpha$. For every right R -module M and every R -submodule $N \subset M$ we have for arbitrary $\sum m_i \otimes w_i \in M \otimes_R W$:

$$\sum m_i \otimes w_i \in N \otimes_R W \iff \sum m_i \langle v, w_i \rangle \in N \text{ for all } v \in V. \tag{4}$$

Proof. By Remark 2.2, ${}_R W$ is flat and we get the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & N \otimes_R W & \xrightarrow{\iota_N \otimes \text{id}_W} & M \otimes_R W & \xrightarrow{\pi \otimes \text{id}_W} & M/N \otimes_R W \longrightarrow 0 \\ & & \downarrow \alpha_N^P & & \downarrow \alpha_M^P & & \downarrow \alpha_{M/N}^P \\ 0 & \longrightarrow & \text{Hom}_{-R}(V, N) & \xrightarrow{(V, \iota_N)} & \text{Hom}_{-R}(V, M) & \xrightarrow{(V, \pi)} & \text{Hom}_{-R}(V, M/N) \end{array}$$

Obviously $\sum m_i \langle v, w_i \rangle \in N$ for all $v \in V$ if and only if

$$\begin{aligned} \sum m_i \otimes w_i &\in \text{Ker}((V, \pi) \circ \alpha_M^P) = \text{Ker}(\alpha_{M/N}^P \circ (\pi \otimes \text{id}_W)) \\ &= \text{Ker}(\pi \otimes \text{id}_W) = N \otimes_R W. \quad \square \end{aligned}$$

2.4. Proposition.

- (1) Let $P = (V, W)$ be a left R -pairing.
 - (a) Let $W' \subset W$ be a left R -submodule and consider the induced left R -pairing $P' := (V, W')$. If $P' \in \mathcal{P}_l^\alpha$, then $W' \subset W$ is pure. If $P \in \mathcal{P}_l^\alpha$, then $P' \in \mathcal{P}_l^\alpha$ if and only if $W' \subset W$ is pure.
 - (b) Let $V' \subset V$ be a right R -submodule, $W' \subset W$ be a left R -submodule with $\langle V', W' \rangle = 0$ and consider the left R -subpairing $Q := (V/V', W')$ of P . If $P \in \mathcal{P}_l^\alpha$, then $Q \in \mathcal{P}_l^\alpha$ if and only if $W' \subset W$ is pure. In particular \mathcal{P}_l^α is closed under pure left R -subpairings.
- (2) Let $\Omega = (Y, W)$ be a left R -pairing, V be a right R -module, $\xi: V \rightarrow Y$ be an R -linear mapping, $P := (V, W)$ be the induced left R -pairing and consider the following statements:
 - (i) $\Omega \in \mathcal{P}_l^\alpha$ and P is dense;
 - (ii) $\Omega \in \mathcal{P}_l^\alpha$ and $\xi(V) \subset Y$ is dense w.r.t. $Y[\mathfrak{T}_s^r(W)]$;
 - (iii) $P \in \mathcal{P}_l^\alpha$;
 - (iv) $P \in \mathcal{P}_l^\alpha$ and $W \xrightarrow{\chi_P} V^*$ is an embedding.

The following implications are always true: (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv). If R_R is an injective cogenerator, then (i)–(iv) are equivalent.

Proof. (1) The result follows from the commutativity of the following diagram for every right R -module M

$$\begin{array}{ccc} M \otimes_R W' & \xrightarrow{\alpha_M^Q} & \text{Hom}_{-R}(V/V', M) \\ \downarrow \text{id}_M \otimes \iota_{W'} & \searrow \alpha_M^{P'} & \downarrow \\ M \otimes_R W & \xrightarrow{\alpha_M^P} & \text{Hom}_{-R}(V, M) \end{array}$$

(2) Consider for every right R -module M the commutative diagram

$$\begin{array}{ccc} M \otimes_R W & \xrightarrow{\alpha_M^\Omega} & \text{Hom}_{-R}(Y, M) \\ & \searrow \alpha_M^P & \downarrow (\xi, M) \\ & & \text{Hom}_{-R}(V, M) \end{array}$$

(i) \Rightarrow (ii) trivial.

(ii) \Rightarrow (iii) Let $\Omega \in \mathcal{P}_l^\alpha$ and assume that $\xi(V) \subset Y$ is dense. Let $\sum_{i=1}^n m_i \otimes w_i \in \text{Ker}(\alpha_M^P)$. By assumption for every $y \in Y$ there exists some $v_y \in V$, such that $\kappa_\Omega(y)(w_i) = \kappa_P(v_y)(w_i)$ for $i = 1, \dots, n$ and it follows then that

$$\begin{aligned} \alpha_M^\Omega \left(\sum_{i=1}^n m_i \otimes w_i \right) (y) &= \sum_{i=1}^n m_i \langle y, w_i \rangle = \sum_{i=1}^n m_i \langle v_y, w_i \rangle \\ &= \alpha_M^P \left(\sum_{i=1}^n m_i \otimes w_i \right) (v_y) = 0. \end{aligned}$$

So $\text{Ker}(\alpha_M^P) = \text{Ker}(\alpha_M^\Omega) = 0$, i.e., α_M^P injective. The R -module M is by our choice arbitrary and so $P \in \mathcal{P}_l^\alpha$.

(iii) \Rightarrow (iv) Trivial.

Let R_R be an injective cogenerator.

(iv) \Rightarrow (i) If $W \xrightarrow{\chi_P} V^*$ is an embedding, then it follows by Theorem 1.8(1) that $\overline{\kappa_P(V)} = \text{An Ke}(\kappa_P(V)) = \text{An}(V^\perp) = \text{An}\{0_W\} = {}^*W$, i.e., P is a dense left R -pairing. \square

Over Noetherian rings we have the following interesting observation:

2.5. Proposition. *Let V be a right R -module, $R[V]$ be the free right R -module with basis V , $W \subset V^*$ be a left R -submodule and consider the left R -pairing $P := (V, W)$. Assume R_R to be Noetherian.*

(1) *For every right R -module M the following mapping is injective*

$$\beta_M : M \otimes_R R^V \rightarrow M^V, \quad m \otimes f \mapsto [v \mapsto mf(v)], \quad (5)$$

i.e., $\tilde{P} := (R[V], R^V)$ is a left α -pairing.

(2) *Let M be an arbitrary right R -module. Then the canonical mapping $\alpha_M^P : M \otimes_R W \rightarrow \text{Hom}_{-R}(V, M)$ is injective if and only if $W \subset R^V$ is M -pure. If moreover V_R is projective, then α_M^P is injective if and only if $W \subseteq V^*$ is M -pure.*

(3) *The following statements are equivalent:*

- (i) $P \in \mathcal{P}_l^\alpha$;
- (ii) α_M^P is injective for every (finitely presented) right R -module M ;
- (iii) $W \subset R^V$ is pure.

Proof. (1) Let M be an arbitrary right R -module and write M as a direct limit of its finitely generated R -submodules $M = \lim_{\rightarrow \Lambda} M_\lambda$ (e.g., [13, 24.7]). For every $\lambda \in \Lambda$ the R -module M_λ is finitely presented in \mathcal{M}_R and so

$$\beta_{M_\lambda} : M_\lambda \otimes_R R^V \rightarrow M_\lambda^V$$

is an isomorphism (e.g., [13, 25.4]). Moreover, for every $\lambda \in \Lambda$ the restriction of β_M on M_λ coincides with β_{M_λ} and so the following mapping is injective:

$$\beta_M = \lim_{\rightarrow} \beta_{M_\lambda} : \lim_{\rightarrow} M_\lambda \otimes_R R^V \rightarrow \lim_{\rightarrow} M_\lambda^V \subset M^V.$$

Obviously $\tilde{P} \in \mathcal{P}_I^\alpha$ if and only if β_M is injective is for every $M \in \mathcal{M}_R$.

(2) The first statement follows by (1). If moreover V_R is projective, then the exact sequence $R[V] \rightarrow V \rightarrow 0$ splits, hence $V^* \subset R^V$ is pure (direct summand) and we are done.

(3) By [13, 34.5], $W \subset R^V$ is pure if and only if $W \subset R^V$ is M -pure for every finitely presented right R -module M . The result follows then from (2). \square

2.6. Definition. The ring R is called right (semi) hereditary iff every (finitely generated) right ideal is projective.

2.7. Lemma. Let R_R be Noetherian and hereditary and V be a right R -module. Then:

- (1) $\check{P} := (V, V^*) \in \mathcal{P}_I^\alpha$.
- (2) Let $W \subseteq V^*$ be a left R -module and $P := (W, V)$. Then $P \in \mathcal{P}_I^\alpha$ if and only if $W \subset V^*$ is a pure R -submodule.

Proof. Assume R_R to be Noetherian and hereditary. It follows then by [13, 26.6] that ${}_R R^\Lambda$ is flat for every set Λ . Moreover we have by [13, 39.13] that all R -cogenerated left R -modules are flat. Consider now $R[V]$, the free right R -module with basis V , and the exact sequence of right R -modules

$$0 \rightarrow \text{Ker}(\pi) \xrightarrow{\iota} R[V] \xrightarrow{\pi} V \rightarrow 0, \tag{6}$$

with ι the embedding map and π the canonical epimorphism. Then (6) induces the exact sequence of left R -modules

$$0 \rightarrow V^* \xrightarrow{\pi^*} R^V \xrightarrow{\iota^*} \text{Im}(\iota^*) \rightarrow 0. \tag{7}$$

Since $\text{Im}(\iota^*) \subseteq \text{Ker}(\pi)^*$, $\text{Im}(\iota^*)$ is an R -cogenerated left R -module, hence flat. Consequently $V^* \hookrightarrow R^V$ is pure (e.g., [13, 36.6]). By Proposition 2.5(1) the canonical mapping $\beta_M : M \otimes_R R^V \rightarrow M^V$ is injective for every $M \in \mathcal{M}_R$ and the result follows then from the commutativity of the following diagram

$$\begin{array}{ccccc}
 M \otimes_R W & \xrightarrow{\alpha_M^P} & \text{Hom}_R(V, M) & \hookrightarrow & M^V \\
 \text{id} \otimes \iota_W \downarrow & \nearrow \alpha_M^{\check{P}} & & \nearrow \beta_M & \\
 M \otimes_R V^* & \hookrightarrow & M \otimes_R R^V & &
 \end{array}
 \quad \square$$

2.8. Lemma. *Let V, W be R -bimodules.*

- (1) *If $P = (V, W)$, $P' = (V', W')$ are left α -pairings, then $P \otimes_l P' := (V' \otimes_R V, W \otimes_R W')$ is a left α -pairing with*

$$\kappa_{P \otimes_l P'}(v' \otimes v)(w \otimes w') = \langle v, w \langle v', w' \rangle \rangle = \langle \langle v', w' \rangle v, w \rangle.$$

- (2) *If $P = (V, W)$, $P' = (V', W')$ are right α -pairings, then $P \otimes_r P' := (V \otimes_R V', W' \otimes_R W)$ is a right α -pairing with*

$$\kappa_{P' \otimes_r P}(v \otimes v')(w' \otimes w) = \langle v, \langle v', w' \rangle w \rangle = \langle v \langle v', w' \rangle, w \rangle.$$

Proof. We prove (1). The proof of (2) is similar. For arbitrary $M \in \mathcal{M}_R$ consider the following commutative diagram

$$\begin{array}{ccc} M \otimes_R W \otimes_R W' & \xrightarrow{\alpha_M^{P \otimes P'}} & \text{Hom}_{-R}(V \otimes_R V', M) \\ \alpha_{M \otimes_R W}^{P'} \downarrow & & \uparrow \zeta^l \\ \text{Hom}_{-R}(V', M \otimes_R W) & \xrightarrow{(V', \alpha_M^P)} & \text{Hom}_{-R}(V', \text{Hom}_{-R}(V, M)) \end{array}$$

where ζ^l is the canonical isomorphism. By assumption the R -linear mappings $\alpha_{M \otimes_R W}^{P'}$ and α_M^P are injective and so $\alpha_M^{P \otimes P'}$ is injective. The last statement is obvious. \square

2.9. Corollary. *Let R_R be Noetherian.*

- (1) *Let X, X' be sets, $E \subseteq R^X$ be a right R -submodule and $E' \subseteq R^{X'}$ be a left R -submodule. If $E' \subseteq R^{X'}$ is E -pure, then the following mapping is injective:*

$$\delta: E \otimes_R E' \rightarrow (R^X)^{X'}, \quad f \otimes f' \mapsto [(x, x') \mapsto f(x) f'(x')]. \quad (8)$$

- (2) *Let W, W' be R -bimodules, $X \subset {}^*W$, $X' \subset {}^*W'$ be R -subbimodules and consider the canonical R -linear mappings*

$$\begin{aligned} \kappa: X' \otimes_R X &\rightarrow ({}^*(W \otimes_R W')) \quad \text{and} \\ \chi: W \otimes_R W' &\rightarrow (X' \otimes_R X)^*. \end{aligned}$$

If W_R is flat and $\text{Ke}(X)_R \subset W_R$ is pure, then

$$\text{Ke}(\kappa(X' \otimes_R X)) \simeq \text{Ke}(X) \otimes_R W' + W \otimes_R \text{Ke}(X'). \quad (9)$$

Proof. (1) Since R_R is coherent, ${}_R R^{X'}$ is flat in ${}_R \mathcal{M}$ by [13, 26.6]. The result follows then from Proposition 2.5(1).

(2) Consider the embeddings $E := W/\text{Ke}(X) \hookrightarrow X^*$, $E' := W'/\text{Ke}(X') \hookrightarrow R^{X'}$ and the commutative diagram

$$\begin{array}{ccc}
 W \otimes_R W' & \xrightarrow{\chi} & (X' \otimes_R X)^* \\
 \downarrow \pi \otimes \pi' & & \downarrow \iota \\
 W/\text{Ke}(X) \otimes_R W'/\text{Ke}(X') & \xrightarrow{\delta} & (X^*)^{X'} \\
 & \nearrow & \searrow \beta_{X^*} \\
 & W/\text{Ke}(X) \otimes_R R^{X'} & \xrightarrow{\beta_{X^*}} X^* \otimes_R R^{X'}
 \end{array}$$

It follows by assumptions that $W/\text{Ke}(X)$ is flat in \mathcal{M}_R and ${}_R R^{X'}$ is flat (e.g., [13, 36.5, 26.6]). Moreover β_{X^*} is injective by Lemma 2.5, hence δ is injective. It follows then by [5, II-3.6] that

$$\begin{aligned}
 \text{Ke}(\kappa(X \otimes_R X')) &:= \text{Ke}(\chi) = \text{Ke}(\delta \circ (\pi \otimes \pi')) \\
 &= \text{Ke}(\pi_X \otimes \pi_{X'}) = \text{Ke}(X) \otimes_R W' + W \otimes_R \text{Ke}(X'). \quad \square
 \end{aligned}$$

2.10. We say a left (respectively a right) R -module W satisfies the α -condition, if $({}^*W, W)$ (respectively (W^*, W)) satisfies the α -condition. Such modules were called *universally torsionless* by Garfinkel [7].

2.11. Locally projective modules. An R -module W is called *locally projective* (in the sense of Zimmermann-Huisgen [14]) iff for every diagram of R -modules

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F & \xrightarrow{\iota} & W & & \\
 & & \searrow g' \circ \iota & & \searrow g & & \\
 & & & & L & \xrightarrow{\pi} & N \longrightarrow 0
 \end{array}$$

with exact rows, F finitely generated as an R -module and every R -linear mapping $g : W \rightarrow N$ there exists an R -linear mapping $g' : W \rightarrow L$, such that the enststanding parallelogram is commutative. By [7, Theorem 3.2], [14, Theorem 2.1], W is locally projective if and only if W satisfies the α -condition. These modules are called *content modules* in [11]. It follows directly from the definition that every projective R -module is locally projective, hence satisfies the α -condition.

Before proceeding, we would like to remark that some of following results on the α -condition and locally projective modules appeared in the recent manuscript [6, 42.9–42.12].

2.12. Proposition. *Let W be a left R -module.*

- (1) *If ${}_R W$ is locally projective, then every pure left R -submodule $K \subset W$ is locally projective. If ${}_R R$ is W -injective, then every locally projective R -submodule of W is a pure left R -submodule.*

(2) Let R_R be Noetherian. Then ${}_R W$ is locally projective if and only if ${}_R W \subset_R R^{*W}$ is a pure R -submodule.

Proof. (1) Standard.

(2) This follows from Propositions 2.5(2). \square

2.13. Corollary. If R_R is an injective cogenerator, then for every left R -pairing (V, W) the following statements are equivalent:

- (i) ${}_R W$ is locally projective and P is dense.
- (ii) W satisfies the α -condition and P is dense.
- (iii) (V, W) is a left α -pairing.
- (iv) W satisfies the α -condition and $W \hookrightarrow V^*$.

If R a QF ring, then (i)–(iv) are moreover equivalent to:

- (v) ${}_R W$ is projective and $W \hookrightarrow V^*$.
- (vi) $W \subset R^V$ is a pure R -submodule.

Over semisimple rings we recover the characterizations of dense pairings over commutative base fields:

2.14. Corollary. Let $P = (V, W)$ be a left R -pairing. If R is semisimple, then

$$P \text{ is dense} \iff W \subset V^* \iff P \text{ is a left } \alpha \text{-pairing.}$$

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References

- [1] J.Y. Abuhlail, Rational modules for corings, *Comm. Algebra* 31 (12) (2003) 5793–5840.
- [2] J.Y. Abuhlail, Dualitätstheoreme für Hopf-Algebren über Ringen, Ph.D. Dissertation, Heinrich-Heine Universität, Düsseldorf, Germany, 2001, <http://www.ulb.uni-duesseldorf.de/diss/mathnat/2001/abuhlail.html>.
- [3] J.Y. Abuhlail, J. Gómez-Torrecillas, F. Lobillo, Duality and rational modules in Hopf algebras over commutative rings, *J. Algebra* 240 (2001) 165–184.
- [4] J. Berning, Beziehungen zwischen links-linearen Toplogien und Modulkategorien, Dissertation, Heinrich-Heine Universität, Düsseldorf, Germany, 1994.
- [5] N. Bourbaki, *Elements of Mathematics, Algebra I*, Hermann, Paris, 1974.
- [6] T. Brzeziński, R. Wisbauer, *Corings and Comodules*, London Mathematical Society Lecture Note Series, vol. 309, Cambridge University Press, Cambridge, 2003.
- [7] G. Garfinkel, Universally torsionless and trace modules, *J. Amer. Math. Soc.* 215 (1976) 119–144.
- [8] G. Köthe, *Topologische lineare Räume I*, Die Grundlagen der Mathematischen Wissenschaften, vol. 107, Springer, Berlin, 1966.

- [9] J. Kelley, I. Namioka, *Linear Topological Spaces*, Graduate Texts in Mathematics, vol. 36, Springer, Berlin, 1976, Reprint of the Van Nostrand edition.
- [10] L. Lambe, D. Radford, *Introduction to the Quantum Yang–Baxter Equation and Quantum Groups: An Algebraic Approach*, Mathematics and Its Applications, vol. 423, Kluwer Academic, Dordrecht, 1997.
- [11] J. Ohm, D. Rush, Content modules and algebras, *Math. Scand.* 31 (1972) 49–68.
- [12] D. Radford, Coreflexive coalgebras, *J. Algebra* 26 (1973) 512–535.
- [13] R. Wisbauer, *Foundations of Module and Ring Theory, a Handbook for Study and Research*, Algebra, Logic and Applications, vol. 3, Gordon and Breach, Philadelphia, PA, 1991.
- [14] B. Zimmermann-Huisgen, Pure submodules of direct products of free modules, *Math. Ann.* 224 (1976) 233–245.