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On the Egyptian method of decomposing $2/n$ into unit fractions

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Abstract

A fraction whose numerator is one is called a *unit fraction*. Unit fractions have been the source of one of the most intriguing mysteries about the mathematics of antiquity. Except for $2/3$, the ancient Egyptians expressed all fractions as sums of unit fractions. In particular, *The Rhind Mathematical Papyrus* (RMP) contains the decomposition of $2/n$ as the sum of unit fractions for odd n ranging from 5 to 101. The way $2/n$ was decomposed has been widely debated and no general method that works for all n has ever been discovered. In this paper we provide an elementary procedure that reproduces the decompositions as found in the RMP.

Résumé

Une fraction dont le numérateur est égale à 1 est appelée *fraction unitée*. Les fractions unitées ont été la source de l'une des plus intriguants mystères des mathématiques de l'antiquité. Apart la fraction $2/3$ les egyptiens de l'antiquité ont exprimé toutes les fractions comme la somme de fractions unitées. En particulier *Le Papyrus Mathématique de Rhind* (The Rhind Mathematical Papyrus) contient la décomposition de la fraction $2/n$ comme la somme de fractions unitées pour n impair variant entre 5 et 101. La façon d'exprimer ces fractions unitées a été largement débatue mais personne n'a réussi à donner une méthode générale pour exprimer toutes ces fractions. Dans ce papier nous proposons une procédure élémentaire qui reproduit les décompositions trouvées dans le Papyrus Mathématique de Rhind.

MSC: 01A16

Keywords: Rhind papyrus; Ahmes; 2:n-table; Egyptian fractions

1. Background

In 1858 the Scottish antiquary Henry Rhind purchased an ancient papyrus roll in the Egyptian city of Luxor. Ahmes, the scribe of the papyrus, tells us that it was copied around 1650 BCE, but its content belonged to an older period, possibly 1800 BCE or earlier. The papyrus contains a series of mathematical problems written in the cursive hieratic script, suitable for writing on papyri, rather than the more elaborate hieroglyphic form, reserved for carving on stones. In this ancient script (written from right to left), the reciprocal of a natural number is indicated by placing a dot over the number. These reciprocals, commonly called *unit fractions* by modern writers, played a crucial role in ancient Egyptian arithmetic. Except for two-thirds, which has a symbol of its own, all fractions were decomposed into unit fractions. The unit fractions, along with two-thirds, are collectively known as *Egyptian fractions*. In modern texts, it is customary to write $\overline{n}$ for the reciprocal of n and $\overline{\overline{3}}$ for two-thirds. Also, 2:n is the standard notation for what Ahmes writes as "Call 2 out of n." A more descriptive translation (see van der Waerden, 1980, 259) would be "What

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part is 2 of n?" The relevance of $2{:}n$ stems from the fact that duplication was, as we shall see, one of the Egyptians' basic mathematical operations, and so the double of $\overline{n}$ was one of the most *prevalent* types of fractions they had to break into unit fractions.

The Rhind Mathematical Papyrus (RMP) opens by expressing $2{:}n$ for odd n from 5 to 101 as the sum of two, three, or four unit fractions. The Egyptians avoided the *trivial* equality $2{:}n = \overline{n} + \overline{n}$ because it leads to a futile amassment of unit fractions. Rather they decomposed $2{:}n$ into a minimal number of carefully picked unit fractions. In Table 1 (henceforth referred to as "the $2{:}n$-table" or simply "the table"), we list the decompositions as they appear in the Rhind Papyrus [Chace et al., 1927, Plates 2–33]. A decomposition of $\overline{\overline{3}}$ is added for completeness.[1] The absence of mathematical signs from the table is in accordance with the Egyptian way of writing mathematical texts. However, in the original the answer is written in a slightly different form. For example, the result of dividing 2 by 47 is given by Ahmes as

$$47 \qquad \mathbf{\overline{30}} \qquad 1\ \overline{2}\ \overline{15} \qquad \mathbf{\overline{141}} \qquad \overline{3} \qquad \mathbf{\overline{470}} \qquad \overline{10}$$

Observe that the boldfaced (red in the original) numbers add up to $2/47$, giving the required decomposition. The answer is verified as follows: First, take $\overline{30}$ of 47. This yields $1+\overline{2}+\overline{15}$, which is $\overline{3}+\overline{10}$ short of 2. Since $3 \times 47 = 141$ and $10 \times 47 = 470$, or equivalently $\overline{141}$ of 47 is $\overline{3}$ and $\overline{470}$ of 47 is $\overline{10}$, the decomposition $2{:}47 = \overline{30} + \overline{141} + \overline{470}$ follows immediately.

Let n be the number to be decomposed, that is, the number for which we wish to express $2{:}n$ as the sum of unit fractions. Now, define $\overline{a}$ to be the first unit fraction in the decomposition of n, M to be the result of taking $\overline{a}$ of n, and R to be the complement of M to 2. Then for $n = 47$, we have $\overline{a} = \overline{30}$, $M = 1 + \overline{2} + \overline{15}$, and $R = \overline{3} + \overline{10}$. From this, we see that decomposing n is a straightforward exercise provided that a, M, and R have been found. But before we say how this may have been done, let us review the sort of mathematics that was available to the Egyptians during the times of Ahmes. This will be important because it would give us an idea of the level of mathematical sophistication that can be employed in reconstructing the table.[2]

The Egyptians used base 10 to express their numbers. Their symbols for 1, 10, and 100 are shown in Fig. 1. They also had symbols for the other powers of 10 up to 1,000,000. The numbers from 1 to 9 were expressed by grouping together the corresponding number of ones; see [Gillings, 1972, 5; van der Waerden, 1954, 17–18] for more on the Egyptian number system. Starting from right to left, 11 was written as 10 followed by 1. For example, the symbols in Fig. 1, written sequentially as a single number, amount to 111. Clearly, the number 10, its multiples, and its powers played a central role in such a number system. Indeed, following the $2{:}n$-table, Ahmes decomposes $n{:}10$ for n from 2 to 9. Besides 10, twelve and its multiples were also of special utility. It seems that the ancient Egyptians were the first to divide the day into 24 hours, 12 for daytime and 12 for nighttime [Neugebauer, 1969, 81]. Twelve was a number of choice because it is the smallest number that has 2, 3, 4, and 6 as divisors, and so $m/12$ is a unit fraction when m is one of those divisors. It follows that a unit fraction whose denominator is a multiple of 12 is desirable because it facilitates breaking other nonunit fractions into unit fractions.

The RMP clearly shows that the Egyptian method of multiplication was based on duplication and addition. For example, in RPM 32 the calculation of 12×12 proceeds like this:

1	12
2	24
/4	48
/8	96
Result:	144

[1] At the beginning of the RMP, the result of 2 divided by 3 was written as $\overline{\overline{3}}$. But elsewhere in the papyrus, the equality $\overline{\overline{3}} = \overline{2} + \overline{6}$ was often used. The general rule for finding $\overline{\overline{3}}$ of the reciprocal of an odd fraction is given in problem 61B of the Rhind Papyrus (RMP 61B). To get $\overline{\overline{3}}$ of $\overline{5}$, RMP 61B states: "Make its 2 times and its 6 times; this is $\overline{\overline{3}}$ of it. One does the same for every odd fraction that may occur." That is, $\overline{\overline{3}}$ of $\overline{n}$ is $\overline{2n} + \overline{6n}$.

[2] A portion of the $2{:}n$-table covering the odd numbers from 3 to 21 appears in fragment UC 32159 of the Lahun Papyrus, a contemporary of the RMP. The decompositions given are identical to those in the RMP; see [Imhausen and Ritter, 2004]. The agreement between the two tables shows that the decompositions used were not arbitrarily chosen.

Table 1
The division of 2 by odd n from $n = 3$ to $n = 101$

n	a	M	R	2:n
3	2	$1\ \overline{2}$	$\overline{2}$	$\overline{2}\ \overline{6}$
5	3	$1\ \overline{\overline{3}}$	$\overline{3}$	$\overline{3}\ \overline{15}$
7	4	$1\ \overline{2}\ \overline{4}$	$\overline{4}$	$\overline{4}\ \overline{28}$
9	6	$1\ \overline{2}$	$\overline{2}$	$\overline{6}\ \overline{18}$
11	6	$1\ \overline{\overline{3}}\ \overline{6}$	$\overline{6}$	$\overline{6}\ \overline{66}$
13	8	$1\ \overline{2}\ \overline{8}$	$\overline{4}\ \overline{8}$	$\overline{8}\ \overline{52}\ \overline{104}$
15	10	$1\ \overline{2}$	$\overline{2}$	$\overline{10}\ \overline{30}$
17	12	$1\ \overline{3}\ \overline{12}$	$\overline{3}\ \overline{4}$	$\overline{12}\ \overline{51}\ \overline{68}$
19	12	$1\ \overline{2}\ \overline{12}$	$\overline{4}\ \overline{6}$	$\overline{12}\ \overline{76}\ \overline{114}$
21	14	$1\ \overline{2}$	$\overline{2}$	$\overline{14}\ \overline{42}$
23	12	$1\ \overline{\overline{3}}\ \overline{4}$	$\overline{12}$	$\overline{12}\ \overline{276}$
25	15	$1\ \overline{\overline{3}}$	$\overline{3}$	$\overline{15}\ \overline{75}$
27	18	$1\ \overline{2}$	$\overline{2}$	$\overline{18}\ \overline{54}$
29	24	$1\ \overline{6}\ \overline{24}$	$\overline{2}\ \overline{6}\ \overline{8}$	$\overline{24}\ \overline{58}\ \overline{174}\ \overline{232}$
31	20	$1\ \overline{2}\ \overline{20}$	$\overline{4}\ \overline{5}$	$\overline{20}\ \overline{124}\ \overline{155}$
33	22	$1\ \overline{2}$	$\overline{2}$	$\overline{22}\ \overline{66}$
35	30	$1\ \overline{6}$	$\overline{\overline{3}}\ \overline{6}$	$\overline{30}\ \overline{42}$
37	24	$1\ \overline{2}\ \overline{24}$	$\overline{3}\ \overline{8}$	$\overline{24}\ \overline{111}\ \overline{296}$
39	26	$1\ \overline{2}$	$\overline{2}$	$\overline{26}\ \overline{78}$
41	24	$1\ \overline{\overline{3}}\ \overline{24}$	$\overline{6}\ \overline{8}$	$\overline{24}\ \overline{246}\ \overline{328}$
43	42	$1\ \overline{42}$	$\overline{2}\ \overline{3}\ \overline{7}$	$\overline{42}\ \overline{86}\ \overline{129}\ \overline{301}$
45	30	$1\ \overline{2}$	$\overline{2}$	$\overline{30}\ \overline{90}$
47	30	$1\ \overline{2}\ \overline{15}$	$\overline{3}\ \overline{10}$	$\overline{30}\ \overline{141}\ \overline{470}$
49	28	$1\ \overline{2}\ \overline{4}$	$\overline{4}$	$\overline{28}\ \overline{196}$
51	34	$1\ \overline{2}$	$\overline{2}$	$\overline{34}\ \overline{102}$
53	30	$1\ \overline{\overline{3}}\ \overline{10}$	$\overline{6}\ \overline{15}$	$\overline{30}\ \overline{318}\ \overline{795}$
55	30	$1\ \overline{\overline{3}}\ \overline{6}$	$\overline{6}$	$\overline{30}\ \overline{330}$
57	38	$1\ \overline{2}$	$\overline{2}$	$\overline{38}\ \overline{114}$
59	36	$1\ \overline{2}\ \overline{12}\ \overline{18}$	$\overline{4}\ \overline{9}$	$\overline{36}\ \overline{236}\ \overline{531}$
61	40	$1\ \overline{2}\ \overline{40}$	$\overline{4}\ \overline{8}\ \overline{10}$	$\overline{40}\ \overline{244}\ \overline{488}\ \overline{610}$
63	42	$1\ \overline{2}$	$\overline{2}$	$\overline{42}\ \overline{126}$
65	39	$1\ \overline{\overline{3}}$	$\overline{3}$	$\overline{39}\ \overline{195}$
67	40	$1\ \overline{2}\ \overline{8}\ \overline{20}$	$\overline{5}\ \overline{8}$	$\overline{40}\ \overline{335}\ \overline{536}$
69	46	$1\ \overline{2}$	$\overline{2}$	$\overline{46}\ \overline{138}$
71	40	$1\ \overline{2}\ \overline{4}\ \overline{40}$	$\overline{8}\ \overline{10}$	$\overline{40}\ \overline{568}\ \overline{710}$
73	60	$1\ \overline{6}\ \overline{20}$	$\overline{3}\ \overline{4}\ \overline{5}$	$\overline{60}\ \overline{219}\ \overline{292}\ \overline{365}$
75	50	$1\ \overline{2}$	$\overline{2}$	$\overline{50}\ \overline{150}$
77	44	$1\ \overline{2}\ \overline{4}$	$\overline{4}$	$\overline{44}\ \overline{308}$
79	60	$1\ \overline{4}\ \overline{15}$	$\overline{3}\ \overline{4}\ \overline{10}$	$\overline{60}\ \overline{237}\ \overline{316}\ \overline{790}$
81	54	$1\ \overline{2}$	$\overline{2}$	$\overline{54}\ \overline{162}$
83	60	$1\ \overline{3}\ \overline{20}$	$\overline{4}\ \overline{5}\ \overline{6}$	$\overline{60}\ \overline{332}\ \overline{415}\ \overline{498}$
85	51	$1\ \overline{\overline{3}}$	$\overline{3}$	$\overline{51}\ \overline{255}$
87	58	$1\ \overline{2}$	$\overline{2}$	$\overline{58}\ \overline{174}$
89	60	$1\ \overline{3}\ \overline{10}\ \overline{20}$	$\overline{4}\ \overline{6}\ \overline{10}$	$\overline{60}\ \overline{356}\ \overline{534}\ \overline{890}$
91	70	$1\ \overline{5}\ \overline{10}$	$\overline{\overline{3}}\ \overline{30}$	$\overline{70}\ \overline{130}$
93	62	$1\ \overline{2}$	$\overline{2}$	$\overline{62}\ \overline{186}$
95	60	$1\ \overline{2}\ \overline{12}$	$\overline{4}\ \overline{6}$	$\overline{60}\ \overline{380}\ \overline{570}$
97	56	$1\ \overline{2}\ \overline{8}\ \overline{14}\ \overline{28}$	$\overline{7}\ \overline{8}$	$\overline{56}\ \overline{679}\ \overline{776}$
99	66	$1\ \overline{2}$	$\overline{2}$	$\overline{66}\ \overline{198}$
101	101	1	$\overline{2}\ \overline{3}\ \overline{6}$	$\overline{101}\ \overline{202}\ \overline{303}\ \overline{606}$

Note. Here M is the result of taking $\overline{a}$ of n and R is the complement of M to 2.

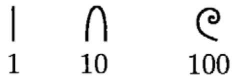

Fig. 1. Hieroglyphic symbols for 1, 10, and 100.

Starting with one, each number in the left column is twice the number above it. The right column consists of the products of 12 with the corresponding numbers in the left column. The strokes near 4 and 8 indicate the numbers to be added. Since $4 + 8 = 12$, the final answer is $48 + 96 = 144$. This doubling process was the most common way to multiply two numbers. Sometimes the process was accelerated by using multiples of 10. In working out 2:23 Ahmes computes 12×23 by writing

1	23
/10	230
/ 2	46
Result:	276

Division was carried out in a similar fashion. For example, to divide 15 by 5 the scribe would say something like: "Multiply 5 so as to get 15." The calculation is done as follows:

/1	5
/2	10
Result:	3

When the division of m by n did not yield a whole number, the Egyptians resorted to unit fractions. Following [van der Waerden, 1980, 260], we illustrate the Egyptian method of division by computing 19:8 and 16:3. The calculations are parts of problems 24 and 25 of the RMP:

<u>19:8</u>	1	8	<u>16:3</u>	/1	3
	/2	16		2	6
	$\bar{2}$	4		/4	12
	/$\bar{4}$	2		$\bar{\bar{3}}$	2
	/$\bar{8}$	1		/$\bar{3}$	1
	Result:	2 $\bar{4}$ $\bar{8}$		Result:	5 $\bar{3}$

To divide 19 by 8 one operates on 8 to reach 19. Since the double of 8 is 3 less than 19, we must operate on 8 to obtain 3. This is done by taking a half, a quarter, and an eighth of 8, yielding 4, 2, and 1. Since $2 + 1 = 3$, the division of 19 by 8 yields $2 + \bar{4} + \bar{8}$. In modern terms, $19/8$ is first written as $2 + 3/8$ and then $3/8$ is broken into $\bar{4}$ and $\bar{8}$. A similar process is applied to 16:3. In this case, one might think that taking $\bar{\bar{3}}$ of 3 before taking $\bar{3}$ of 3 is an oversight on the part of Ahmes, but in reality this was more the rule than the exception.[3] From these and other examples, we find that taking the kth part of n was the basic tool used in calculating $m{:}n$, where the most common values of $\bar{k}$ are those that can be obtained from $\bar{2}$ and $\bar{\bar{3}}$ (and to a lesser extent $\overline{10}$) by the process of halving.

2. Modern methods for reconstructing the 2:n-table

Ever since it was discovered, many writers (professionals and amateurs) have attempted to reconstruct the 2:n-table of the RMP. The usual flaw with most of these attempts is that by using modern symbolism to explain the table the specific structure of the ancient text is lost, and consequently a simple manipulation of the modern form of the source may not be easily applied to the ancient form, which more often than not makes the whole process questionable.

[3] The Egyptians constantly took two-thirds of a number knowing that $\bar{\bar{3}}$ is the reciprocal of $1 + \bar{2}$. In other words, if $2/3$ of x is y, then $x = y + y/2$. This equality is evident in RMP 33, where $\bar{\bar{3}}$ of 42 is 28 and $\overline{28}$ of 42 is $1 + \bar{2}$.

Therefore, one should be overly cautious when ancient mathematical texts are to be analyzed by means of modern notation.

Various authors [Boyer, 1989, 13; Knorr, 1982, 140; and others] have noted that for an odd integer n one can always find a 2-term decomposition of 2:n using the equality

$$2:n = \frac{1}{(n+1)/2} + \frac{1}{n(n+1)/2}. \tag{1}$$

If n is replaced by a prime p then this is the only nontrivial 2-term decomposition of 2:p; see [Bruins, 1981, 283]. Yet the Egyptian decomposition of 2:n does not agree with (1) except for a few prime values of n (the remaining primes are decomposed as the sum of three or four unit fractions). This led some modern writers [Brown, 1995; Gardner, 2005] to propose a method in which 2:pq is decomposed as $\overline{q}$ times the decomposition of 2:p. For example, applying (1) to $p = 3$ yields $\overline{\overline{3}} = \overline{2} + \overline{6}$ and thus the decomposition of every multiple of 3 can be carried out as

$$2:3q = \frac{1}{q} \times \left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{2q} + \frac{1}{6q}. \tag{2}$$

If $\overline{2} + \overline{6}$ is accepted as the standard decomposition of $\overline{\overline{3}}$, then it is true that the Egyptian decomposition of 2:p for every prime p up to 11 satisfies (1), and by choosing an appropriate p the decomposition

$$2:pq = \frac{1}{q}\left[\frac{1}{(p+1)/2} + \frac{1}{p(p+1)/2}\right] \tag{3}$$

covers every composite number n in the table. In other words, the method argues that the Egyptians used (1) to decompose the primes 3, 5, 7, and 11, and then decomposed the multiples of those primes according to (3). That the Egyptians stopped using this process at $p = 11$ was interpreted as a sign that they had used some sort of prime sieve, like that of Eratosthenes [Brown, 1995]. This view was taken in light of the fact that 11 is the first prime whose square is greater than 101, the last entry of the table. One problem with this argument is that if n is a composite number in the table then n can always be written as pq with p a prime less than or equal to *seven*. Another problem is that the method does not give a convincing explanation of why the prime 23, even though it is greater than 11, is also decomposed according to (1).

Equation (3) yields the decomposition found in the RMP for all $n = pq$ except 35, 91, and 95. In all cases, p is taken to be the smallest prime factor of n except for $n = 55$, where p was taken to be 11 instead of 5. As for 2:35 and 2:91, they are decomposed according to

$$2:pq = \frac{2}{p+q} \times \left(\frac{1}{p} + \frac{1}{q}\right). \tag{4}$$

In [Brown, 1995], it is suggested that this is related to the arithmetic and harmonic means[4] of p and q. Finally, 2:95 was decomposed as $\overline{5}$ times the decomposition of 19. This yields

$$2:95 = \overline{5} \times \left(\overline{12} + \overline{76} + \overline{114}\right) = \overline{60} + \overline{380} + \overline{570}.$$

For the remaining primes ($p \geqslant 13$), a method that was first discovered by F. Hultsch in 1895 and then rediscovered by E.M. Bruins in 1945 does the job [Gardner, 2005]. It amounts to finding a number a (usually *highly composite*) such that $p/2 < a < p$ and $2a - p$ is equal to the sum of two or three divisors of a. Since

$$\frac{2}{p} - \frac{1}{a} = \frac{2a-p}{ap} \quad \text{or} \quad \frac{2}{p} = \frac{1}{a} + \frac{2a-p}{ap}, \tag{5}$$

[4] For the numbers p and q, the arithmetic mean is $A = (p+q)/2$ and the harmonic mean is $H = 2/(\overline{p} + \overline{q})$. It follows that $2/pq = 2/AH$.

the decomposition can be easily completed once the divisors of a have been found. For example, to find a decomposition of 2:37, take a to be the highly composite number 24. Then $2a - p = 2 \times 24 - 37 = 11$. Since 24 is divisible by 8 and 3, whose sum is 11, we can write

$$2:37 = \frac{1}{24} + \frac{1}{37} \times \left(\frac{8}{24} + \frac{3}{24}\right) = \overline{24} + \overline{37} \times (\overline{3} + \overline{8}) = \overline{24} + \overline{111} + \overline{296},$$

agreeing with the decomposition of Ahmes. The problem with this method is that it does not specify how the number a was chosen. Moreover, it uses (modern) mathematical techniques that are not explicitly mentioned in extant Egyptian sources.

A more source-based method is summarized by B. van der Waerden in his article *The* (2:n) *Table in the Rhind Papyrus* [van der Waerden, 1980]. The method divides the 50 entries in the table into five groups, the first of which is decomposed according to (2). The second group is decomposed by using the Egyptian method of division (see previous section), and the third group is decomposed by multiplying the denominators of a decomposition in the second group by an appropriate number. The fourth group is decomposed using the so-called *auxiliary* numbers, and the fifth group consists of the exceptional cases 35, 91, and 101. Even though the method is based on Egyptian techniques, I think that it is unnecessary to divide the table into such a large number of groups.

Another attempt to systematically decompose 2:n was given by R.J. Gillings in his Book *Mathematics in the Time of the Pharaohs* [Gillings, 1972]. By considering all possible decompositions of 2:n, Gillings argued that Ahmes made his choice according to a canon of five *precepts*. In short, Precept 1 eliminates denominators with more than three digits; Precept 2 excludes decompositions of more than four terms; Precept 3 forbids the trivial decomposition; Precept 4 asserts that the smallness of the first number is the main consideration; and Precept 5 presumes that even numbers are preferred to odd numbers. Although the first three precepts are generally held to be true, the overall method drew Gillings into a debate with M. Bruckheimer and Y. Salomon [Bruckheimer and Salomon, 1977; Gillings, 1978]. For one thing, Gillings gives a total of 22,295 possible decompositions, while the number given by Bruckheimer and Salomon is approximately 28,000. Other minor and not so minor shortcomings on the part of Gillings are also cited. But E.M. Bruins went a step further in his criticism of Gillings' method by challenging the precepts themselves. According to Bruins [1975, 249], Ahmes had no predilection for even numbers, which contradicts Precept 5. On that point, I believe that it is Gillings rather than Bruins who was the one on the right track. However, in [Bruins, 1981, 287] a more serious criticism of Precept 4 is given by citing numerous cases where the precept is violated. In this respect, the reader will see that the truth is more like the opposite of Gillings' Precept 4.

More recently, A. Imhausen, based on the work of J. Ritter, has developed a method that stresses the algorithmic structure of Egyptian mathematics, see [Imhausen, 2002, 2003; Ritter, 1995]. For a given problem, the method establishes a numerical and a symbolical algorithm. While the numerical algorithm stays close to the source by preserving the numbers used in the original problem, the symbolical algorithm provides the structure of the problem and thus makes it readily comparable to other problems. The method is promising but it has only been applied to problem texts and not to table texts. Applying the method to table texts, the 2:n-table in this case, is problematic, since the decomposition of 2:n does not yield a unique answer. This difficulty can be easily overcome if some other piece of information, in particular the first term a, is known a priori[5] (I will show that it is this first number a, more than anything else, that holds the key to unlocking the 2:n-table). Unfortunately, nowhere in the RMP does Ahmes tell us how a was chosen. For this reason, finding a will be at the core of the method for decomposing 2:n that I will propose in the following section.

3. A new method for reconstructing the 2:n-table

Although the Egyptians did not allow the trivial decomposition $\overline{n} + \overline{n}$ or a decomposition with an exceedingly large denominator, they still had many possible ways to decompose 2:n. Even today, the way 2:n was decomposed has not been fully understood. The lack of original sources makes it extremely difficult to determine the exact process through which the ancient Egyptians duplicated the odd fractions from $\overline{3}$ to $\overline{101}$. However, it is the opinion of this author that

[5] If a is also known, then the method can be applied, but its impact will be greatly reduced since the ensuing numerical algorithm is more or less given by Ahmes after each decomposition.

the techniques used in the method described below are more closely related to the original procedure than any of the previously proposed methods for composing the 2:n-table.

3.1. Basic concepts

Since only odd values of n appear in the table, it is safe to assume that the Egyptians were aware of the fact that 2:2n is the same as $\overline{n}$. It follows that a unit fraction with an even denominator can be easily duplicated, and hence that an even unit fraction is preferable to one with an odd denominator. Not counting 2/3, the table has 103 even fractions and only 24 odd fractions. It can also be assumed that the Egyptians did not allow the denominator of a unit fraction to get arbitrarily large. For now, let us adopt Gillings' Precept 1 that no denominator should be as large as one thousand.[6] This holds for all entries in the table, where the greatest term (denominator) is 890. Indeed, only 13 terms exceed 500, 10 of which are even.

In RMP 16, we find the identity $\overline{2}+\overline{3}+\overline{6}=1$. Multiplying both sides of the identity by $\overline{n}$ leads to the equality

$$\overline{n}=\overline{2n}+\overline{3n}+\overline{6n}. \tag{6}$$

RMP 17–20 leave no doubt that the Egyptians were aware of this equality, which is equivalent to saying that a half, a third and a sixth of any number add up to the number itself.[7] Adding $\overline{n}$ to both sides of (6), one obtains the decomposition

$$2:n=\overline{n}+\overline{2n}+\overline{3n}+\overline{6n}. \tag{7}$$

A particular case of (7) can be clearly seen in the last entry of the table, where 2:101 is decomposed as

$$\overline{101}+\overline{202}+\overline{303}+\overline{606}.$$

In fact, Ahmes worked out this decomposition by calculating $\overline{2}$ of $\overline{101}$, $\overline{3}$ of $\overline{101}$, and $\overline{6}$ of $\overline{101}$ before adding the results to $\overline{101}$. Since the first fraction of the decomposition is $\overline{101}$ and the last three fractions add up to $\overline{101}$, we clearly see that the Egyptians refrained from using the trivial decomposition $\overline{n}+\overline{n}$.

It is immediately clear to anyone reading the Rhind Papyrus that RMP 16–20 form a set of closely related problems and that the same is true for RMP 21–23. The latter set deals with problems of a different nature, called problems in completion. For instance, RMP 22 says "Complete $\overline{\overline{3}}+\overline{30}$ to 1." The solution can be reworded as follows: $\overline{\overline{3}}$ of 30 is 20 and $\overline{30}$ of 30 is 1, making a total of 21. Since 30 exceeds 21 by 9, multiply by 30 to get 9. That is,

1	30
/$\overline{10}$	3
/ $\overline{5}$	6
Result:	9

Since $3+6=9$, we must add $\overline{5}+\overline{10}$ to complete the whole. For a proof, add $\overline{\overline{3}}$, $\overline{5}$, $\overline{10}$, and $\overline{30}$ to get 1. As parts of 30, these fractions are 20, 6, 3, and 1, a total of 30.

RPM 22 gives us a deep insight into Egyptian arithmetic. What Ahmes has done is equivalent to writing $\overline{\overline{3}}+\overline{30}$ as 21/30, and since $30-21=9$, he then breaks 9/30 into $\overline{5}+\overline{10}$. Note that it was easy to break 9/30 since 9 is the sum of 6 and 3, both of which are divisors of 30. Extant Egyptian sources are replete with examples where the non-Egyptian fraction m:n is decomposed by writing m as the sum of distinct divisors of n. Another common way used by the Egyptians to break a non-Egyptian fraction is to write it as $\overline{\overline{3}}$ plus a unit fraction. For example, in the

[6] Quoting [Bruins, 1981, 283]: "It is clear that the Rhind decompositions are expressed using the number symbols for 1, 10, 100 only, i.e., all parts are greater than $\overline{1000}$."

[7] In RMP 18, we have $\overline{6}+\overline{9}+\overline{18}=\overline{3}$. Also, by ignoring the red auxiliaries in RMP 19, one can clearly see that a total of $\overline{6}$ at the bottom of the second column is given as the sum of $\overline{12}$, $\overline{18}$, and $\overline{36}$ on top of it. Doing the same in RMP 20, one obtains the equality $\overline{24}+\overline{36}+\overline{72}=\overline{12}$.

Table 2
Egyptian or reducible fractions of the form $(k-1)/k$

k	2	3	4	6	12
$(k-1)/k$	$\overline{2}$	$\overline{\overline{3}}$	$\overline{2}\ \overline{4}$	$\overline{\overline{3}}\ \overline{6}$	$\overline{\overline{3}}\ \overline{4}$

decomposition of 2:53 the answer to taking $\overline{30}$ of 53 is given as $1+\overline{\overline{3}}+\overline{10}$. Since $53/30 = 1 + 23/30$, Ahmes could easily find the answer by writing 23/30 as $(20+3)/30 = \overline{\overline{3}} + \overline{10}$. These two ways of breaking a non-Egyptian fraction were constantly used by Egyptian scribes. Formally, a non-Egyptian fraction $m{:}n$ will be called *reducible* if it can be written as the sum of two unit fractions or as the sum of a unit fraction and $\overline{\overline{3}}$, where the denominator of each unit fraction is a divisor of n. When the reducible fraction $m{:}n$ is not written as the sum of two Egyptian fractions, it will be denoted by m/n.

The Egyptian expression for $\overline{\overline{3}}$ literally means "the two part," making $\overline{3}$ "the third part" that completes the whole. Originally, the Egyptians also had a symbol for 3/4, which was later replaced by $\overline{2}+\overline{4}$; see [van der Waerden, 1954, 19–20]. If we agree to call the fraction 3/4 "the three part" and its complement $\overline{4}$ "the fourth part," and so on, then for $2 \leqslant k \leqslant 101$, $\overline{k}$ is the complement of an Egyptian or reducible fraction only if k is one of the numbers listed in Table 2. Observe that 5/6 was decomposed as $(4+1)/6 = \overline{\overline{3}} + \overline{6}$ although it could have been broken as $(3+2)/6 = \overline{2} + \overline{3}$. In fact, both forms are used in the RMP. For instance, in computing 2:11, Ahmes writes $\overline{6}$ of 11 as $1+\overline{\overline{3}}+\overline{6}$, while in working out 2:17 he writes $\overline{6}$ of 17 as $2+\overline{2}+\overline{3}$. Similarly, he breaks 7/10 as $\overline{\overline{3}}+\overline{30}$ in the n:10-table following the decomposition of 2:101 and as $\overline{2}+\overline{5}$ in RMP 54. This flexibility in expressing the same fraction in two different forms should be on the mind of anyone trying to reconstruct the 2:n-table.

Before we proceed with our method of deciphering the table, a few remarks are in order. First, we divide the entries in the table into two groups, G_1 and G_2 (this division is artificial and is only done to make our task easier). The group G_1 consists of the 29 entries that are expressed as the sum of two unit fractions along with $n = 95$. We include 95 in G_1 because its last two fractions, $\overline{380}$ and $\overline{570}$, add up to $\overline{228}$, a unit fraction different from $\overline{95}$. The group G_2 consists of the remaining 20 entries, which are expressed as the sums of three or four unit fractions. Second, our method handles both groups without having to distinguish between prime and composite numbers. Moreover, the method of decomposing the elements of G_2 will be a natural extension of the method used to decompose the elements of G_1. Third, we will provide a plain rule to determine the first fraction in each decomposition. Finally, we will only use techniques that are, in some form or another, explicitly mentioned in the RMP.

3.2. The group G_1

Let n be an element of G_1. Then finding an acceptable (2-term) decomposition of 2:n is equivalent to finding two numbers a and b such that $2/n = \overline{a} + \overline{b}$. Since using the same fraction twice is not allowed, we can assume that $a < n < b$. On the other end, a must be greater than $n/2$, since otherwise $\overline{a}$ will be greater than $2/n$. Therefore, we must have $n/2 < a < n$. Let M and R be as defined in Section 1, that is (using modern notation), $M = n/a$ and $R = 2 - M$. If Q is the fractional part of M, then $Q = (n-a)/a$ and R is the complement of Q to 1. Since the decomposition of n is completely determined by a, Q and R, we will naturally concentrate on these parts of the decomposition. In fact, we shall show that the vast majority of the elements of G_1 are decomposed as $2{:}n = \overline{a} + \overline{kn}$, where a is the largest number for which $(k-1)/k$ (the reduced form of Q) belongs to the set of Egyptian fractions $\{\overline{2}, \overline{\overline{3}}\}$ or the set of reducible fractions $\{3/4, 5/6, 11/12\}$.

It is clear from the definitions of Q and R that the easiest numbers to decompose are those for which $Q = R = \overline{2}$. These correspond to the multiples of 3. In fact, if n is a multiple of 3 and a is taken to be $\overline{\overline{3}}$ of n, then $n{:}a = 1 + \overline{2}$ and so $Q = R = \overline{2}$. Since $\overline{2n}$ of n is $\overline{2}$, we obtain the decomposition

$$2 : n = \overline{a} + \overline{2n}, \quad a = 2n/3. \tag{8}$$

There is ample evidence that the Egyptians were aware of this way of finding 2:n when n is a multiple of 3. This view is supported by the fact that all multiples of 3 are decomposed according to (8). However, we shall see that even if the Egyptians did not use (8) to decompose the multiples of 3, our general method will in no way be affected.

The next numbers to decompose are those for which $Q = \overline{\overline{3}}$ and $R = \overline{3}$. These correspond to the multiples of 5. In this case, taking a to be 3 times $\overline{5}$ of n yields $Q = \overline{\overline{3}}$ and $R = \overline{3}$. Hence, we have the decomposition

$$2 : n = \overline{a} + \overline{3n}, \quad a = 3n/5. \tag{9}$$

Although Ahmes may have been aware of this decomposition, we will see that he did not use it for all multiples of 5 not covered by (8). One drawback of this decomposition is that it produces odd unit fractions, and we mentioned earlier that the Egyptians preferred even fractions because they can easily be duplicated.

Equations (8) and (9) are special cases of a more general method that decomposes all the elements of G_1. If n is an element of G_1 not covered by (8) or (9), that is, Q is not an Egyptian fraction, then the method amounts to finding the largest a that yields a reducible Q whose complement R is a unit fraction. In other words, starting from $n-1$ downward, let a be the first number such that $\overline{a}$ of n is equivalent to $1 + (k-1)/k$ for $k = 4$, 6, or 12. Since $\overline{kn}$ of n is $\overline{k}$, we have $\overline{a}$ of n plus $\overline{kn}$ of n equal to $1 + (k-1)/k + 1/k = 2$. It follows that

$$2 : n = \overline{a} + \overline{kn}. \tag{10}$$

It is no coincidence that n is an element of G_1 *if and only if* there exists a first number a such that $(n-a)/a$ is equivalent to $(k-1)/k$, where k is one of the entries in Table 2. These are the only values of k for which the fraction $(k-1)/k$ is Egyptian or reducible. They lead to the decomposition of the multiples of the primes 3, 5, 7, 11, and 23, respectively. This shows that the decomposition process need not differentiate between prime and composite values of n. It follows that the relation between 11 being the largest prime decomposed using (1) and 101 being the last entry in the table (see Section 2) is somewhat contrived.

We illustrate our procedure with two examples. First, we take $n = 15$. Then we must have $8 \leqslant a \leqslant 14$. But only $a = 10$ and $a = 9$ yield an Egyptian or reducible Q whose complement R is a unit fraction. The corresponding decompositions are

$*10$	$\overline{2}$	$\overline{2}$	$\overline{10}\ \overline{30}$
9	$\overline{\overline{3}}$	$\overline{3}$	$\overline{9}\ \overline{45}$

The four columns represent a, Q, R, and the resulting set of unit fractions; the asterisk $(*)$ indicates the decomposition given by Ahmes. Since 15 is divisible by 3, the chosen decomposition could have been obtained using (8). By doing so, no further decomposition need be considered. But even if the decomposition $\overline{9} + \overline{45}$ was also considered it would be ignored because it consists of odd fractions. Next, we take $n = 77$. Then the only acceptable decompositions are

$*44$	$\overline{2}\ \overline{4}$	$\overline{4}$	$\overline{44}\ \overline{308}$
42	$\overline{\overline{3}}\ \overline{6}$	$\overline{6}$	$\overline{42}\ \overline{462}$

Since both decompositions consist of even terms, Ahmes chose, as always, the one with the largest a.

Stopping after the first (largest) a is found, the search process terminates with the desired decomposition for all elements of G_1 except 35, 55, 91, and 95, which we call *irregular*. For each of the irregular elements, Table 3 lists the (regular) decomposition produced by the method described above, followed by the decomposition found in the RMP. The last column lists all other 2-term decompositions such that $n/2 < a < n$ and the second term b is less than 1000. Observe that for each of the multiples of 5 in Table 3, our procedure yields a pair of odd unit fractions satisfying (9), while the decomposition found in the RMP consists of even fractions. It could be that this is why the regular decompositions were overlooked. As for the other elements of G_1 with odd decompositions, namely 5, 25, 65, and 85, no even 2-term decomposition exists for any a in the proper range. But there is more. The four entries in Table 3 have something quite subtle in common. They are the only nonmultiples of 3 that have a decomposition where both Q and R can be expressed as the sum of at most two Egyptian fractions and n/R is a whole number. The meaning of the last statement will be clear as we take a closer look at the irregular entries. In each case, we will find that the choice made by Ahmes is hard to surpass.

Table 3
The irregular elements of G_1

n	Regular	RMP	Other decompositions
35	$\overline{21}\ \overline{105}$	$\overline{30}\ \overline{42}$	$\overline{20}\ \overline{140}$, $\overline{18}\ \overline{630}$
55	$\overline{33}\ \overline{165}$	$\overline{30}\ \overline{330}$	$\overline{40}\ \overline{88}$
91	$\overline{52}\ \overline{364}$	$\overline{70}\ \overline{130}$	$\overline{49}\ \overline{637}$
95	$\overline{57}\ \overline{285}$	$\overline{60}\ \overline{380}\ \overline{570}$	$\overline{60}\ \overline{228}$, $\overline{50}\ \overline{950}$

Take $n = 35$. From Table 3 we see that there are four 2-term decompositions. However, the decomposition $\overline{18} + \overline{630}$ will not be considered, since it yields $R = 17{:}18$, which is not reducible. Therefore, the acceptable decompositions are

$$\begin{array}{rllll} *30 & \overline{6} & & 5/6 & \overline{30}\ \overline{42} \\ 21 & \overline{\overline{3}} & & \overline{3} & \overline{21}\ \overline{105} \\ 20 & \overline{2} & \overline{4} & \overline{4} & \overline{20}\ \overline{140} \end{array}$$

The regular decomposition ($a = 21$) was dismissed because it is the sum of two odd fractions. Since the other two decompositions consist of even terms, Ahmes chose the decomposition with the largest first term ($a = 30$). Had the remainder $5/6$ of the chosen decomposition been written as $\overline{2} + \overline{3}$, we would get the decomposition $\overline{30} + \overline{70} + \overline{105}$. Instead Ahmes treated $5/6$ like a *single* fraction, and this is why he went to some length to explain how the second denominator was obtained. He wrote (boldface indicates red)

$$\begin{array}{lllll} 35 & \mathbf{30} & 1\ \overline{6} & \mathbf{42} & \overline{\overline{3}}\ \overline{6} \\ \mathbf{6} & 7 & & 5 & \end{array}$$

The auxiliary numbers 6, 7, and 5 in the second row are not shown in any other decomposition. They are used to find the number b that solves the equation $\overline{b}$ of 35 is $5/6$. The extra piece of information is needed because up to this point R has been the sum of unit fractions and so finding the remaining terms was straightforward. For instance, if $R = \overline{k}$ then $b = kn$, and the same is true when R is the sum of two or three parts. But in this case, $R = 5/6$ and so $1/R = 6/5$. Solving for b, we get

$$b = \frac{6}{5} \times 35 = 6 \times 7 = 42.$$

The exact way of calculating b may never be known, but it is reasonable to say that this is the closest Ahmes got to using a fraction in the modern sense.

Next, we consider $n = 91$, since it is similar to the case of $n = 35$. In this case, the decomposition $\overline{49} + \overline{637}$ is ignored because it leads to the nonreducible $Q = 42{:}49$. The remaining decompositions are

$$\begin{array}{rlll} *70 & \overline{5}\ \overline{10} & 7/10 & \overline{70}\ \overline{130} \\ 52 & \overline{2}\ \overline{4} & \overline{4} & \overline{52}\ \overline{364} \end{array}$$

Again, the chosen decomposition may be expressed as follows (Ahmes did not write the auxiliary numbers 10, 13, and 7):

$$\begin{array}{lllll} 91 & \mathbf{70} & 1\ \overline{5}\ \overline{10} & \mathbf{130} & \overline{\overline{3}}\ \overline{30} \\ \mathbf{10} & 13 & & 7 & \end{array}$$

By writing $R = 7/10$ as $\overline{2} + \overline{5}$ instead of $\overline{\overline{3}} + \overline{30}$, Ahmes could have obtained the decomposition $\overline{70} + \overline{182} + \overline{455}$. But as in the case of $n = 35$, there is no benefit in choosing a 3-term decomposition with an odd fraction over a 2-term decomposition with even denominators.

We have seen that in working out 2:35, Ahmes wrote the numbers 5, 6, and 7 below 42, 35, and 30. Since $5 \times 42 = 6 \times 35 = 7 \times 30 = 210$, it could be that he had found the least common multiple of the three numbers. The same idea is (implicitly) used in the decomposition of 91, but in no other decomposition. Moreover, in the process of computing 2:91, Ahmes wrote the word *find* before taking $\overline{70}$ of 91, and again before taking $\overline{130}$ of 91. It is the only decomposition in which the word *find* is written twice.[8] For both 35 and 91, we see a conspicuous alteration of the normal procedure. In fact, 35 and 91 are the only elements of G_1 for which a is greater than $\overline{\overline{3}}$ of n and b is not a multiple of n.

Now for $n = 55$, the possible decompositions are

$$\begin{array}{llll} 40 & \overline{4}\ \overline{8} & 5/8 & \overline{40}\ \overline{88} \\ 33 & \overline{\overline{3}} & \overline{3} & \overline{33}\ \overline{165} \\ *30 & \overline{\overline{3}}\ \overline{6} & \overline{6} & \overline{30}\ \overline{330} \end{array}$$

Ahmes picked the decomposition $\overline{30} + \overline{330}$ over the regular decomposition $\overline{33} + \overline{165}$ because the former consists of even denominators. Of course, Ahmes could have chosen the decomposition with the largest first term ($\overline{40} + \overline{88}$ in this case) as he had done for 35 and 91. However, one advantage of the decomposition $\overline{30} + \overline{330}$ over the decomposition $\overline{40} + \overline{88}$ is that 330 is a multiple of 55 while 88 is not. Since the second denominator of every regular decomposition is a multiple of n, it makes sense to replace the odd decomposition of 55 by a regular-like decomposition with even terms. Furthermore, by writing $R = 5/8$ as $\overline{2} + \overline{8}$, we see that the decomposition $\overline{40} + \overline{88}$ is equivalent to the even regular-like decomposition $\overline{40} + \overline{110} + \overline{440}$. From the two even regular-like decompositions, clearly $\overline{30} + \overline{330}$ is preferable to $\overline{40} + \overline{110} + \overline{440}$.

Lastly, we consider $n = 95$. In this case, the decomposition $\overline{50} + \overline{950}$ is dismissed because it does not produce a reducible Q, and so the acceptable decompositions are

$$\begin{array}{llll} 60 & \overline{2}\ \overline{12} & 5/12 & \overline{60}\ \overline{228} \\ 57 & \overline{\overline{3}} & \overline{3} & \overline{57}\ \overline{285} \end{array}$$

The regular decomposition $\overline{57} + \overline{285}$ is ignored because it consists of odd fractions. Ahmes' only remaining choice is $\overline{60} + \overline{228}$. But since $5/12$ can be written as the sum of the two even fractions $\overline{4}$ and $\overline{6}$, Ahmes chose the equivalent 3-term decomposition $\overline{60} + \overline{380} + \overline{570}$. Unlike the cases of 35 and 91, in this case both 380 and 570 are even multiples of 95 and so it was worthwhile to choose the 3-term decomposition. Ahmes was stuck with this decomposition because he could not find another regular-like decomposition of two even terms, as he did for $n = 55$.

The above analysis forces us to appreciate the ingenious techniques that must have been used in decomposing the irregular entries. The argument that similar techniques were used by the Egyptians can be strengthened by analyzing the three multiples of 3 for which a similar situation occurs.[9] The corresponding decompositions are listed in Table 4. For example, the second term of the irregular decomposition $2{:}15 = \overline{12} + \overline{20}$ could be obtained by dividing 15 by 3 and then multiplying the result by 4. Now, writing $3/4$ as $\overline{2} + \overline{4}$, we get the equivalent decomposition $\overline{12} + \overline{30} + \overline{60}$. Ahmes did not use the last decompositions, since 2:15 has the even regular decomposition $\overline{10} + \overline{30}$. This is in total agreement

Table 4
Multiples of 3 for which n/R is a whole number

n	a	Q	R	$2{:}n$
15	12	$\overline{4}$	3/4	$\overline{12}\ \overline{20}$
45	36	$\overline{4}$	3/4	$\overline{36}\ \overline{60}$
75	60	$\overline{4}$	3/4	$\overline{60}\ \overline{100}$

[8] Excluding the multiples of 3, Ahmes wrote the word *find* before taking $\overline{a}$ of n for every n from 43 to 89. He did the same for each entry between 90 and 100, but only for $n = 91$ did he also write the word *find* before taking $\overline{b}$ of n.

[9] The decompositions $2{:}39 = \overline{24} + \overline{104}$ and $2{:}51 = \overline{30} + \overline{170}$ are not counted because their first terms, 24 and 30, are smaller than 26 and 34, the respective first terms of the regular decompositions.

with the decomposition of $n = 55$. Of course, it could be that Ahmes did not notice the irregular decomposition because, as a multiple of 3, 15 may have been decomposed according to (8).

3.3. The group G_2

If n is an element of G_2, then n cannot have a 2-term decomposition where Q and R satisfy the conditions that Q is reducible and R is a unit fraction. As a natural next step, Ahmes kept the condition that Q is reducible but allowed R to be the sum of two or three unit fractions. If still no decomposition was found, then Ahmes allowed Q to be the sum of three or four unit fractions.[10] As a last resort, he decomposed $2{:}n$ according to (7). For every n, we shall first list the possible 3-term decompositions and then, if necessary, the 4-term decompositions (decompositions with parts less then $\overline{1000}$ will not be listed). By carefully examining these decompositions, we will show that except in extreme cases Ahmes did not choose a decomposition if it has a term greater than $10n$. As in the case of G_1, the chosen decomposition is usually the one with the largest a, and a decomposition of even terms (unless it has a part of R less than $\overline{10}$) will always be preferred to a decomposition of the same order with one or more odd fractions.

If we apply the above rules to G_2, then the desired decomposition is obtained in almost every case. Because of the many possible situations that could arise, we will make sure to analyze each and every element of G_2. By doing so, the underlying procedure will become more clearly visible. Starting with $n = 13$, the possible 3-term decompositions are

10	$\overline{5}\ \overline{10}$	$\overline{2}\ \overline{5}$	$\overline{10}\ \overline{26}\ \overline{65}$
$*8$	$\overline{2}\ \overline{8}$	$\overline{4}\ \overline{8}$	$\overline{8}\ \overline{52}\ \overline{104}$

Since the first decomposition has an odd term, the second decomposition was picked because it has only even fractions.

For $n = 17$, we have the three decompositions

$*12$	$\overline{3}\ \overline{12}$	$\overline{3}\ \overline{4}$	$\overline{12}\ \overline{51}\ \overline{68}$
12	$\overline{3}\ \overline{12}$	$\overline{2}\ \overline{12}$	$\overline{12}\ \overline{34}\ \overline{204}$
10	$\overline{2}\ \overline{5}$	$\overline{5}\ \overline{10}$	$\overline{10}\ \overline{85}\ \overline{170}$

Observe that the first two decompositions have the same first number, $a = 12$. Now, the second decomposition was not chosen because Ahmes did not permit R to have parts smaller than $\overline{10}$. Since the other two decompositions have an odd fraction each, Ahmes preferred the one with the larger first term (or smaller last denominator).

For $n = 19$, we have

$*12$	$\overline{2}\ \overline{12}$	$\overline{4}\ \overline{6}$	$\overline{12}\ \overline{76}\ \overline{114}$
12	$\overline{2}\ \overline{12}$	$\overline{3}\ \overline{12}$	$\overline{12}\ \overline{57}\ \overline{228}$

Since Q and thus R (respectively, $7/12$ and $5/12$) are the same in both decompositions, Ahmes chose to break R as $\overline{4} + \overline{6}$ rather than $\overline{3} + \overline{12}$ because the former consists of even numbers. In addition, both terms of $\overline{4} + \overline{6}$ are less than 10 while the second term of $\overline{3} + \overline{12}$ is greater than 10. This is important, since in the decomposition of 17 Ahmes wrote $Q = 5/12$ as $\overline{3} + \overline{12}$. Similarly, he wrote $R = 7/12$ as $\overline{3} + \overline{4}$ in the decomposition of 17 but wrote $Q = 7/12$ as $\overline{2} + \overline{12}$ in the decomposition of 19. All of this shows that keeping the terms of R less than 10 was an essential part of determining how $2{:}n$ is to be decomposed. On the other hand, Ahmes was more flexible in breaking Q, since it would have no effect on the eventual decomposition of $2{:}n$. For this reason, we only listed one form of $Q = 5/12$ in the decomposition of $n = 17$, while we listed both forms of $R = 5/12$ in the decompositions of $n = 19$.

One may wonder why Ahmes did not decompose 2:19 as

10	$\overline{\overline{3}}\ \overline{5}\ \overline{30}$	$\overline{10}$	$\overline{10}\ \overline{190}$

[10] When we say that Q (or R) is the sum of unit fractions, we mean that its numerator is equal to the sum of distinct divisors of its denominator.

This is even more surprising knowing that $Q = 9{:}10 = \overline{\overline{3}} + \overline{5} + \overline{30}$ is an entry of the n:10-table. Similarly, 2:13 can be decomposed as

$$\begin{array}{llll} 7 & \overline{\overline{3}}\ \overline{7}\ \overline{21} & \overline{7} & \overline{7}\ \overline{91} \end{array}$$

It could be that Ahmes did not use these decompositions because nowhere in the table was Q allowed to be written as $\overline{\overline{3}}$ plus two unit fractions. More importantly, the last term of Q in both decompositions is equal to $3a$. Since every term of Q different from $\overline{\overline{3}}$ is a divisor of a for every entry in the table, we believe that this is the most likely reason for rejecting these decompositions. Interestingly, these decompositions of 13 and 19 are found in the (Greek) Akhmîm papyrus of around 400 CE; see [Knorr, 1982, 144]. However, even in the Akhmîm papyrus, the decomposition $2{:}17 = \overline{9} + \overline{153}$ cannot be found. In this case, there is no easy way to break $Q = 8{:}9$. For instance, taking $\overline{\overline{3}}$ of 9 yields 6 with a remainder of 2:9, an unwelcome return to the 2:n-table.

For $n = 29$, we have

$$\begin{array}{llll} 20 & \overline{4}\ \overline{5} & \overline{2}\ \overline{20} & \overline{20}\ \overline{58}\ \overline{580} \\ 18 & \overline{2}\ \overline{9} & \overline{3}\ \overline{18} & \overline{18}\ \overline{87}\ \overline{522} \end{array}$$

Since the first decomposition is the sum of even numbers, it should have been the one picked by Ahmes. But the second term of R is larger than 10 in both decompositions and so none of them was acceptable. Consequently, Ahmes searched for a third term beside a, obtaining the decompositions

$$\begin{array}{rlll} *24 & \overline{6}\ \overline{24} & \overline{2}\ \overline{6}\ \overline{8} & \overline{24}\ \overline{58}\ \overline{174}\ \overline{232} \\ 24 & \overline{6}\ \overline{24} & \overline{2}\ \overline{4}\ \overline{24} & \overline{24}\ \overline{58}\ \overline{116}\ \overline{696} \\ 20 & \overline{2}\ \overline{20} & \overline{4}\ \overline{5}\ \overline{10} & \overline{20}\ \overline{116}\ \overline{145}\ \overline{290} \end{array}$$

Obviously, the first decomposition is the one to choose.

For $n = 31$, we have

$$\begin{array}{rlll} *20 & \overline{2}\ \overline{20} & \overline{4}\ \overline{5} & \overline{20}\ \overline{124}\ \overline{155} \\ 18 & \overline{\overline{3}}\ \overline{18} & \overline{6}\ \overline{9} & \overline{18}\ \overline{186}\ \overline{279} \end{array}$$

Since each of the two decompositions has an odd number, the first one is chosen because it has a larger first term.

For $n = 37$, the only possible decomposition is

$$\begin{array}{rlll} *24 & \overline{2}\ \overline{24} & \overline{3}\ \overline{8} & \overline{24}\ \overline{111}\ \overline{296} \end{array}$$

As expected, this is the decomposition found in the RMP.

For $n = 41$, we have

$$\begin{array}{rlll} *24 & \overline{\overline{3}}\ \overline{24} & \overline{6}\ \overline{8} & \overline{24}\ \overline{246}\ \overline{328} \\ 24 & \overline{\overline{3}}\ \overline{24} & \overline{4}\ \overline{24} & \overline{24}\ \overline{164}\ \overline{984} \end{array}$$

The choice in this case is plain.

For $n = 43$, Ahmes did not choose one of the decompositions[11]

$$\begin{array}{rlll} 30 & \overline{3}\ \overline{10} & \overline{2}\ \overline{15} & \overline{30}\ \overline{86}\ \overline{645} \\ 24 & \overline{\overline{3}}\ \overline{8} & \overline{8}\ \overline{12} & \overline{24}\ \overline{344}\ \overline{516} \end{array}$$

[11] The decomposition $\overline{24} + \overline{258} + \overline{1032}$ is not listed because its smallest part is less than $\overline{1000}$. Such decompositions will always be ignored.

because in both cases the largest term of R is larger than 10. Looking for higher order decompositions, we find

$*42$	$\overline{42}$	$\overline{2}\ \overline{3}\ \overline{7}$	$\overline{42}\ \overline{86}\ \overline{129}\ \overline{301}$
36	$\overline{6}\ \overline{36}$	$\overline{2}\ \overline{4}\ \overline{18}$	$\overline{36}\ \overline{86}\ \overline{172}\ \overline{774}$
30	$\overline{3}\ \overline{10}$	$\overline{3}\ \overline{6}\ \overline{15}$	$\overline{30}\ \overline{129}\ \overline{258}\ \overline{645}$
28	$\overline{2}\ \overline{28}$	$\overline{4}\ \overline{7}\ \overline{14}$	$\overline{28}\ \overline{172}\ \overline{301}\ \overline{602}$

It is true that the chosen decomposition has four terms, one of which is odd, but it has the advantage that it is the only decomposition for which the largest term of R is less than 10. Moreover, $Q = \overline{42}$ is the simplest of all cases.

For $n = 47$, we have

$*30$	$\overline{2}\ \overline{15}$	$\overline{3}\ \overline{10}$	$\overline{30}\ \overline{141}\ \overline{470}$
28	$\overline{\overline{3}}\ \overline{84}$	$\overline{4}\ \overline{14}$	$\overline{28}\ \overline{188}\ \overline{658}$

Although the chosen decomposition has an odd fraction, it was picked because the remainder R of the other decomposition has a part less than $\overline{10}$. Strictly speaking, the second decomposition should not even be listed since $Q = \overline{\overline{3}} + \overline{84}$ has a term greater than $a = 28$ and so it is not reducible.

For $n = 53$, the sole 3-term decomposition is

$*30$	$\overline{\overline{3}}\ \overline{10}$	$\overline{6}\ \overline{15}$	$\overline{30}\ \overline{318}\ \overline{795}$

Since 15, the largest term of R, is greater than 10, Ahmes should have searched for a 4-term decomposition. He would obtain

48	$\overline{12}\ \overline{48}$	$\overline{2}\ \overline{3}\ \overline{16}$	$\overline{48}\ \overline{106}\ \overline{159}\ \overline{848}$

But the largest term of R in this decomposition is 16, forcing Ahmes to accept the previous decomposition. There is one more possibility that Ahmes could have considered. Namely,

36	$\overline{3}\ \overline{12}\ \overline{18}$	$\overline{4}\ \overline{6}\ \overline{9}$	$\overline{36}\ \overline{212}\ \overline{318}\ \overline{477}$

However, this would mean that Q must be broken into three unit fractions, something that Ahmes had not done so far. Also, it would increase the number of terms from 3 to 4. It seems that Ahmes made a reasonable choice despite the fact that the largest term of R in the chosen decomposition is greater than 10.

The only other case where a number greater than 10 was used in breaking R is in the decomposition of 23, a member of G_1. In that case, the possible choices (including the possibility where Q is broken into three unit fractions) are

20	$\overline{10}\ \overline{20}$	$\overline{2}\ \overline{4}\ \overline{10}$	$\overline{20}\ \overline{46}\ \overline{92}\ \overline{230}$
18	$\overline{6}\ \overline{9}$	$\overline{2}\ \overline{6}\ \overline{18}$	$\overline{18}\ \overline{46}\ \overline{138}\ \overline{414}$
16	$\overline{4}\ \overline{8}\ \overline{16}$	$\overline{2}\ \overline{16}$	$\overline{16}\ \overline{46}\ \overline{368}$
$*12$	$\overline{\overline{3}}\ \overline{4}$	$\overline{12}$	$\overline{12}\ \overline{276}$

Clearly, the last decomposition is the most attractive even though its last term is slightly larger than the last term of the first decomposition. In addition, the chosen decomposition has a reducible Q of the form $(k-1)/k$, an important property that no decomposition of any element of G_2 could have.

For $n = 59$, no 3-term or 4-term decompositions with a reducible Q exist. Hence, Ahmes searched for a decomposition where Q may be broken as the sum of three unit fractions. This yields the 3-term decomposition

$*36$	$\overline{2}\ \overline{12}\ \overline{18}$	$\overline{4}\ \overline{9}$	$\overline{36}\ \overline{236}\ \overline{531}$

Observe that Ahmes did not break Q into three unit fractions until it became absolutely necessary to do so (see $n = 53$). But having done this once, he did not hesitate doing it again.

For $n = 61$, there are no 3-term decompositions, and the 4-term decompositions are

48	$\overline{4}\ \overline{48}$	$\overline{2}\ \overline{6}\ \overline{16}$	$\overline{48}\ \overline{122}\ \overline{366}\ \overline{976}$
45	$\overline{3}\ \overline{45}$	$\overline{3}\ \overline{5}\ \overline{9}$	$\overline{45}\ \overline{183}\ \overline{305}\ \overline{549}$
*40	$\overline{2}\ \overline{40}$	$\overline{4}\ \overline{8}\ \overline{10}$	$\overline{40}\ \overline{244}\ \overline{488}\ \overline{610}$

The first decomposition is dismissed since 16 (the last term of R) is larger than 10, and the second is discarded because it has odd numbers. The third decomposition is the one given by Ahmes.

For $n = 67$, the only 3-term or 4-term decomposition is

$$*40 \qquad \overline{\overline{3}}\ \overline{120} \qquad \overline{5}\ \overline{8} \qquad \overline{40}\ \overline{335}\ \overline{536}$$

Ahmes picked this decomposition but he wrote $Q = 27/40$ as $\overline{2} + \overline{8} + \overline{20}$ instead of $\overline{\overline{3}} + \overline{120}$. Again, Ahmes avoided the latter form of Q because its second term is not a divisor of 40, a necessary condition for the reducibility of Q.

For $n = 71$, the sole 3-term decomposition is

$$42 \qquad \overline{\overline{3}}\ \overline{42} \qquad \overline{6}\ \overline{7} \qquad \overline{42}\ \overline{426}\ \overline{497}$$

Unfortunately, this is not the decomposition found in the RMP. But by now Ahmes had allowed Q to be written as the sum of three unit fractions, which gave him the decomposition

$$*40 \qquad \overline{2}\ \overline{4}\ \overline{40} \qquad \overline{8}\ \overline{10} \qquad \overline{40}\ \overline{568}\ \overline{710}$$

It is this even-term decomposition that appears in the table. Of course, it could be that Ahmes picked this decomposition because he may have broken $Q = \overline{\overline{3}} + \overline{42}$ of the other decomposition as $\overline{2} + \overline{7} + \overline{21}$ or $\overline{2} + \overline{6} + \overline{42}$, both of which have the same number of terms as Q of the chosen decomposition. It turned out that allowing Q to be written as the sum of three unit fractions from 59 on would lead to the same decompositions as when Q is reducible, except in this case.

For $n = 73$, no 3-term decomposition exists, and so we are left with the 4-term decompositions

*60	$\overline{6}\ \overline{20}$	$\overline{3}\ \overline{4}\ \overline{5}$	$\overline{60}\ \overline{219}\ \overline{292}\ \overline{365}$
60	$\overline{6}\ \overline{20}$	$\overline{2}\ \overline{5}\ \overline{12}$	$\overline{60}\ \overline{146}\ \overline{365}\ \overline{876}$

The first decomposition is clearly the better choice.

For $n = 79$, there are no 3-term decompositions and the 4-term decompositions are

*60	$\overline{4}\ \overline{15}$	$\overline{3}\ \overline{4}\ \overline{10}$	$\overline{60}\ \overline{237}\ \overline{316}\ \overline{790}$
60	$\overline{4}\ \overline{15}$	$\overline{2}\ \overline{10}\ \overline{12}$	$\overline{60}\ \overline{158}\ \overline{790}\ \overline{948}$

The first decomposition agrees with the one in the table. Note that the second decomposition was dismissed despite the fact that it consists of even numbers. This is so because the largest term of R is 12, a number greater than 10.

For $n = 83$, Ahmes' search for a 3-term decomposition was fruitless. Looking for higher order decompositions he found

*60	$\overline{3}\ \overline{20}$	$\overline{4}\ \overline{5}\ \overline{6}$	$\overline{60}\ \overline{332}\ \overline{415}\ \overline{498}$
60	$\overline{3}\ \overline{20}$	$\overline{3}\ \overline{5}\ \overline{12}$	$\overline{60}\ \overline{249}\ \overline{415}\ \overline{996}$

Clearly, the first decomposition is the one to choose.

For $n = 89$, there are no 3-term nor 4-term decompositions unless we break Q into three unit fractions. This yields the decomposition

$$*60 \qquad \overline{3}\ \overline{10}\ \overline{20} \qquad \overline{4}\ \overline{6}\ \overline{10} \qquad \overline{60}\ \overline{356}\ \overline{534}\ \overline{890}$$

found in the RMP.

For $n = 97$, we must break Q into four unit fractions in order to obtain the lone decomposition

$$*56 \qquad \overline{2}\ \overline{8}\ \overline{14}\ \overline{28} \qquad \overline{7}\ \overline{8} \qquad \overline{56}\ \overline{697}\ \overline{776}$$

Again, this is the decomposition given by Ahmes.

For $n = 101$, there are no 3-term nor 4-term decompositions, even if we break Q into four unit fractions. This forced Ahmes to choose the *default* decomposition $\overline{101} + \overline{202} + \overline{303} + \overline{606}$, obtained from (7) by setting $n = 101$. It should be mentioned that 101 has the decomposition

$$60 \qquad \overline{\overline{3}}\ \overline{60} \qquad \overline{4}\ \overline{15} \qquad \overline{60}\ \overline{404}\ \overline{1515}$$

which could have been chosen by Ahmes. In this case, having the largest term of R equal to 15 would have been acceptable (as in the case of 53) had it not forced the smallest part to be less than $\overline{1000}$. This not only is in agreement with our rules for decomposing the elements of G_2, but also explains why 101 was the last number considered by Ahmes. Of course, it could be that after finding a decomposition for each odd number from 3 to 99, the scribe gave the decomposition of 101 as a prototype of how to decompose every odd number beyond one hundred.[12] The use of the powers of 10 as upper bounds is totally in line with the Egyptian way of doing mathematics.

Finally, suppose that we allow Q to be the sum of up to four Egyptian fractions, where the denominator of each unit fraction is a divisor of a. Then we would still get the same chosen decomposition for every element of G_2, provided that the number of terms is not increased. In addition, applying the same rules to G_1 does not lead to any change in the regular decompositions, proving that the table can and should be treated as a single entity.

4. The 2:*n*-table revisited

A better understanding of the table as a whole can be reached by looking at the set of decompositions chosen by Ahmes as a subset of the set of all possible decompositions. We will see that the number of choices available for the ancient Egyptians is but a small fraction of the number of choices at the disposal of a modern calculator. We will also see that the Egyptian way of expressing 2:n holds the advantage over its decimal equivalent that it always admits a finite (unit-fraction) expansion.

Let n be an odd number less than 100 and a be the first term in the decomposition of n. Then starting from n downward, the first choice for a is $n - 1$ and the last choice is $(n + 1)/2$, making a total of $(n - 1)/2$ choices. Summing up over the odd numbers from 3 to 99, we get a grand total of 1225 choices for a. Now, call a decomposition of at most four parts, none smaller than $\overline{1000}$, *acceptable* if $Q = (n - a)/a$ can be written as the sum of up to four unit fractions or as the sum of a unit fraction and $\overline{\overline{3}}$. Then the 1225 choices for a produce a set of only 255 acceptable decompositions, far less than the roughly 28,000 possible decompositions produced by a modern computer. More importantly, the set of acceptable decompositions contains the decomposition given by Ahmes for every number except 35 and 91. For these exceptional numbers, Ahmes gives the decompositions $\overline{30} + \overline{42}$ and $\overline{70} + \overline{130}$, while the set of acceptable decompositions contains the equivalent decompositions $\overline{30} + \overline{70} + \overline{105}$ and $\overline{70} + \overline{182} + \overline{455}$. Moreover, if we decompose the multiples of 3 according to (8), then we are left with only 143 acceptable decompositions for the 32 nonmultiples of 3, an average of less than five decompositions per number. It follows that once the procedure is understood then reconstructing the table is no longer the difficult task it appeared to be. An experienced scribe can certainly complete the job in a day or two. The scribe need not consider every value of a, since he will soon realize that a number with too few divisors, especially a prime number, does not make a good choice for a. In fact, a is a multiple of either 10 or 12 for every element of G_2 except $n = 13$, 43, 97, and 101. It happens that for $n = 97$ and $n = 101$, one cannot find an acceptable decomposition whose first term is a multiple of 10 or 12.

[12] Observe that the condition that no part should be less than $\overline{1000}$ is contained in the condition that no term should be larger than $10n$, provided that we stop at $n = 99$.

Let a and n be as above. Then one can argue that the Egyptians applied the following systemic method to break 2:n into unit fractions. As a first step, the scribe tries to find a number k greater than 1 and less than 11 satisfying the equation

$$2:n=\overline{a}+\overline{kn}. \tag{11a}$$

If the previous step fails to produce a decomposition of n, the scribe searches for a second number l such that $k < l \leqslant 10$ and

$$2:n=\overline{a}+\overline{kn}+\overline{ln}. \tag{11b}$$

If still no decomposition is found, the scribe introduces a third number m, where $k < l < m \leqslant 10$ and

$$2:n=\overline{a}+\overline{kn}+\overline{ln}+\overline{mn}. \tag{11c}$$

In all cases (except $n = 71$), a was chosen so that $Q = (n - a)/a$ would be equivalent to an Egyptian or reducible fraction, and only when this could not be done did the scribe allow Q to be the sum of three or four unit fractions.

It is remarkable that every entry in the table up to 100, apart from 23, 35, 53 and 91, satisfies exactly one of (11a) to (11c). This means that the Egyptians decomposed n in such a way that k, l, and m are integers greater than 1 and no larger than 10, the underlying base. As for 35 and 91, they are the only values of n having a decomposition where a term other than a is a noninteger multiple of n. Their respective values of k are $6/5$ and $10/7$. On the other hand, 23 and 53 are the only entries having a decomposition with a term greater than $10n$. For $n = 23$ the last (second) term is $12n$, while for $n = 53$ the last (third) term is $15n$.

Looking at the 2:n-table in this way, we see that the Egyptians used a number system quite similar to our decimal system, but they avoided the use of infinite sums. Rather than writing

$$\frac{2}{n}=\frac{a_1}{10}+\frac{a_2}{10^2}+\frac{a_3}{10^3}+\frac{a_4}{10^4}+\cdots, \quad 0 \leqslant a_i \leqslant 9, \tag{11}$$

they expressed $2/n$ (excluding $n = 23$, 35, 53 and 91) as

$$\frac{2}{n}=\frac{1}{a}+\frac{1}{kn}+\frac{1}{ln}+\frac{1}{mn}, \quad 2 \leqslant k < l < m \leqslant 10, \tag{12}$$

where the third and fourth terms are used only if needed. The Egyptians preferred their system because it gives a finite yet exact representation of $2/n$. One might become more sympathetic to their point of view by knowing that among the entries in the table only $2/5$ and $2/25$ have finite decimal expansions. We think that taking this new insight into consideration should drastically change the way Egyptian arithmetic has been perceived.

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