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Longest cycles in threshold graphs

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Abstract

The length of a longest cycle in a threshold graph is obtained in terms of a largest matching in a specially structured bipartite graph. It can be computed in linear time. As a corollary, Hamiltonian threshold graphs are characterized. This characterization yields Golumbic's characterization and sharpens Minty's characterization. It is also shown that a threshold graph has cycles of length $3, \ldots, l$ where l is the length of a longest cycle.

1. Introduction

We consider finite loopless undirected graphs with no multiple edges. Terms not defined here can be found in [2, 8]. For a given graph G, we consider the largest length (number of vertices or edges) of a cycle in G. If the length of a longest cycle equals the number of vertices, the graph is called *Hamiltonian*. It is well known that finding the length of a longest cycle and recognizing a Hamiltonian graph are NP-complete problems [6]. Minty, as reported in Chvátal and Hammer [4], Golumbic [8, Ex. 10.6], and Harary and Peled [10] have characterized Hamiltonian threshold graphs. A *threshold graph* is a graph having a hyperplane separating the characteristic vectors of the stable sets of vertices from the characteristic vectors of the other sets. These graphs have been extensively studied, and possess many beautiful properties (see [1, 4, 9, 12, 14–16]) as well as many extensions.

We show below that the length of a longest cycle of a threshold graph G is equal to the size of a largest matching in a specially structured bipartite graph obtained from G. Moreover, because of the special structure, namely nested neighborhoods, this matching can be obtained in linear time. A longest cycle of G can also be constructed from the matching in linear time. The characterization also leads to a characterization

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of Hamiltonian threshold graphs. We use this characterization to prove Golumbic's characterization and sharpen Minty's.

Section 2 defines threshold graphs and gives their known structure. This structure is used to obtain the length of a longest cycle. Section 3 characterizes Hamiltonian threshold graphs.

2. Longest cycles

Definition 2.1. The *LCL* (largest cycle length) of a graph G is the largest length of a simple cycle of G. We denote the LCL by l, and call a cycle of length l an l-cycle.

Definition 2.2 ([4]). A graph $G=(V,E)$ is called a *threshold graph* when there exist nonnegative reals w_v, $v \in V$ and t such that

$$\sum\{w(v)\colon v \in S\} \leqslant t \text{ if and only if } S \text{ is a stable set } S \subseteq V. \tag{1}$$

To paraphrase this definition, G is a threshold graph whenever one can assign vertex weights such that a set of vertices is stable if and only if its total weight does not exceed a certain threshold (t). Yet another interpretation is that G is a threshold graph if and only if some hyperplane strictly separates the characteristic vectors of the stable sets of G from the characteristic vectors of the nonstable sets. In other words, the Boolean function that selects the stable sets of vertices is a threshold function [13]. The following definition and notation are used below to describe the structure of threshold graphs:

Definition 2.3 ([8]). Let $G=(V,E)$ be a graph whose distinct positive vertex-degrees are $\delta_1 < \cdots < \delta_m$, and let $\delta_0 = 0$ (even if no vertex of degree 0 exists). Let $D_i = \{v \in V\colon \deg(v) = \delta_i\}$ for $i = 0, \ldots, m$. The sequence $D_0, \ldots, D_m$ is called the *degree partition* of G.

For a graph $G=(V,E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V\colon uv \in E\}$, and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For $U, W \subseteq V$, we define $N_W(U) = \bigcup\{W \cap N(v)\colon v \in U\}$ and $N(U) = N_V(U)$. Among the many characterizations of threshold graphs, we shall use the following ones, which describe their structure and their degree sequence.

Theorem 2.4 ([4,5,8,9,11,12]). *Let G be a graph with $\delta_0, \ldots, \delta_m$ and $D_0, \ldots, D_m$ as in Definition 2.3. Then the following are equivalent:*

(1) *G is a threshold graph;*

(2) *G does not have an induced subgraph isomorphic to a 4-cycle C_4, its complement $2K_2$, or a path P_4 on 4 vertices;*

(3) *for each* $v \in D_k$

$$N(v) = \bigcup_{j=1}^{k} D_{m+1-j} \quad k = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, \tag{2}$$

$$N[v] = \bigcup_{j=1}^{k} D_{m+1-j} \quad k = \left\lfloor \frac{m}{2} \right\rfloor + 1, \ldots, m. \tag{3}$$

in other words, if $i + j > m$, *then all possible edges between* D_i and D_j *are present; if* $i + j \leqslant m$, *non of these edges is presents.*

It follows from Eqs. (2) and (3) that for a threshold graph $G = (V, E)$, the set V is partitioned into a stable set

$$S = D_0 \cup \cdots \cup D_{\lfloor m/2 \rfloor} \tag{4}$$

and a clique

$$K = D_{\lfloor m/2 \rfloor + 1} \cup \cdots \cup D_m. \tag{5}$$

A graph whose vertex set is partitioned into a stable set S and a clique K (not necessarily of the form (4) and (5) above) is called a *split graph* and denoted $G(S, K)$.

Lemma 2.5. *Every split graph* $G(S, K)$ *possessing cycles has an l-cycle containing all of* K.

Proof. Let C be an l-cycle. If C has no vertices of S, then C must contain K, because K is a clique and C is a longest cycle. If C has a vertex $s \in S$, let a, b be the vertices adjacent to s in C. Since S is stable, $a, b \in K$. If C excludes some vertices of K, drop s and add all of them between a and b, to obtain a cycle at least as large as C — hence another l-cycle — containing all of K. □

In the following we always assume that G is a threshold graph with degree partition $D_0, \ldots, D_m$ as in Definition 2.3 and S, K as in Eqs. (4) and (5).

Definition 2.6. A matching M of G from S into K is called *special* if either $M = \emptyset$ or the following holds:

- if m is odd, M misses some vertex $x \in D_m$;
- if m is even, M matches some vertex $y \in D_{m/2}$ with some vertex $x \in D_m$.

Lemma 2.7. *If* M *is a special matching of a threshold graph* G, *then* G *has a cycle of length* $|K| + |M|$, *provided* $|K| + |M|$ *is at least* 3.

Proof. The result is trivial if $M = \emptyset$. So let $|M| = k \geqslant 1$. Let $s_1, \ldots, s_k$ be the vertices of S that are saturated by M, listed in nondecreasing degrees, so that $N(s_1) \subseteq \cdots \subseteq N(s_k)$. If m is even, we may assume that $s_k = y$.

The required cycle starts from s_1 along the matching edge, proceeds to s_2, then along the matching edge, ..., then to s_k. If m is odd, it proceeds from s_k along the matching edge into K, visits the remaining vertices of K in any order ending at x, and returns to s_1. If m is even, then $s_k = y$ is adjacent to every vertex to K, so the cycle proceeds from y into K, visits the remaining vertices of K in any order ending with x, and returns to s_1 (thus it does not use the matching edge yx if $|K| > k$ and $k \neq 1$). □

Lemma 2.8. *If C is any cycle containing all of K, then there is a special matching of size $|C| - |K|$, where $|C|$ is the length of C.*

Proof. If $|C| = |K|$ then $M = \emptyset$ is the required matching. Now assume that $|C| > |K|$. Orient the edges of C along the cycle. Let M be the set of edges of C oriented from S into K. Then M is a matching of size $|C| - |K|$. If M is already special, we are done, so assume it is not. To complete the proof, we shall obtain a special matching M^* with $|M^*| = |M|$. Since M is not special, M saturates D_m if m is odd, and M has no edges from $D_{m/2}$ to D_m if m is even. We now distinguish two cases.

Case 1: All vertices of K are saturated by M. Then C must alternate between S and K. Further, m must be even (for otherwise the vertices of $D_{(m+1)/2}$ cannot be saturated by M), and M saturates some vertex $y \in D_{m/2}$ (because the vertices of $D_{(m/2)+1}$ can only be matched to vertices of $D_{m/2}$). Let x be any vertex of D_m. Then there is a path P in C between y and x, whose edges are alternatingly in M and not in M, that begins and ends with edges of M. This P with the edge yx of G forms an alternating cycle A. Swapping matching and nonmatching edges along A, we obtain the desired special matching M^*.

Case 2: Some vertex of K is not saturated by M. Let $k > m/2$ be the largest index such that D_k contains an unsaturated vertex u. If m is odd then $k > (m+1)/2$, since when the cycle leaves $D_{(m+1)/2}$ it enters D_j for some $j > (m+1)/2$, causing a vertex of D_j to be unsaturated. We assert that for each $j > k$, M has an edge oriented from the set

$$E_j = D_{m+2-j} \cup \cdots \cup D_{\lfloor m/2 \rfloor},$$

into the set

$$F_j = D_j \cup \cdots \cup D_m.$$

Observe that for any $v \in D_{j-1}$, $N(v) \cap S = E_j$. To prove the assertion, consider the cut (P_j, Q_j), where $P_j = D_{m+2-j} \cup \cdots \cup D_{j-1}$ and $Q_j = V - P_j$. Since $u \in P_j \cap K$ and C visits every vertex of K, C visits P_j. Therefore, C has an edge ab oriented from some vertex $a \in P_j$ to some vertex $b \in Q_j$. Clearly $b \notin Q_j - F_j$ since G has no edges between P_j and $Q_j - F_j$, and hence ab must be an edge of M and $a \in E_j$. This proves the assertion.

We shall now show that there exists an even alternating path P from u to some vertex $x \in D_m$. Indeed, by the assertion, M has an edge $a_1 b_1$ with $a_1 \in E_{k+1}$, $b_1 \in F_{k+1}$. If $b_1 \in D_m$, the desired path is ua_1b_1. If not, let $b_1 \in D_{j_1}$ with $k < j_1 < m$. By the assertion again, M has an edge a_2b_2 with $a_2 \in E_{j_1+1}$ and $b_2 \in F_{j_1+1}$. If $b_2 \in D_m$, the desired path is $ua_1b_1a_2b_2$. Otherwise we continue in the same way until the path P ends in D_m.

Now swap matching and nonmatching edges along P to obtain a matching M' such that $|M'| = |M|$ and M' misses some vertex $x \in D_m$. If m is even, pick any vertex $y \in D_{m/2}$, add the edge yx to M' and drop either the matching edge at y, if y is saturated by M', or any other edge of M' if it is not. The resulting matching is the required special matching M^*. □

Theorem 2.9. *For a threshold graph G having cycles and with degree partition $D_0, \ldots, D_m$, the LCL equals $|K| + |M|$, where $K = D_{\lfloor m/2 \rfloor + 1} \cup \cdots \cup D_m$ and M is a largest special matching of G.*

Proof. Follows from Lemmas 2.5, 2.7, 2.8. □

To find a largest special matching in a threshold graph, drop any one vertex $x \in D_m$, and for even m drop any one vertex $y \in D_{m/2}$. Then find a largest matching in the remaining graph using only edges between S and K. If m is even, add to the matching the edge yx. Such a matching can be easily obtained by the following greedy algorithm: Arrange the vertices of S as $s_1, \ldots, s_r$ with nondecreasing degrees; for i running from 1 to r, if s_i has any unmatched neighbor t_i, add the edge s_it_i to the matching and increase i. Given the (unsorted) degree sequence of G, this can be implemented in time $O(|V|)$.

3. Hamiltonicity

Theorem 3.1. *A threshold graph with degree partition $D_0, \ldots, D_m$ is Hamiltonian if and only if it has a special matching of size $|S|$, where $S = D_0 \cup \cdots \cup D_{\lfloor m/2 \rfloor}$.*

Proof. This follows directly from Theorem 2.9. □

We shall now derive Golumbic's necessary and sufficient conditions for a threshold graph to be Hamiltonian.

Theorem 3.2 ([8]). *A threshold graph G with degree partition $D_0, \ldots, D_m$ is Hamiltonian if and only if*

(1) $D_0 = \emptyset$;

(2) $\sum_{j=1}^{k} |D_j| < \sum_{j=1}^{k} |D_{m-j+1}| \quad k = 1, \ldots, \lfloor (m-1)/2 \rfloor$;

(3) *if m is even, then* $\sum_{j=1}^{m/2} |D_j| \leqslant \sum_{j=1}^{m/2} |D_{m-j+1}|$.

Proof. By Theorem 3.1 and Hall's theorem, G is Hamiltonian if and only if

- for m odd, $\forall U \subseteq S$, $|N_{K-x}(U)| \geqslant |U|$, where x is any vertex of D_m;

- for m even, $\forall U \subseteq S - y$, $|N_{K-x}(U)| \geqslant |U|$, where x is any vertex of D_m and y is any vertex of $D_{m/2}$.

Conditions 1, 2, 3 of the theorem are special cases of Hall's conditions with $U = D_0 \cup \cdots \cup D_k$, $k = 0, \ldots, \lfloor (m-1)/2 \rfloor$, and in addition for m even with $U = D_0 \cup \cdots \cup D_{m/2} - y$. On the other hand, these special cases imply the full Hall's conditions by the nesting of neighborhoods (Eq. (2)). □

Chvátal [3] has given a sufficient condition for a general graph to be Hamiltonian: Let $d_1 \leqslant \cdots \leqslant d_n$ be the degrees of a graph G. Then G is Hamiltonian if

$$d_j \leqslant j < \frac{n}{2} \Rightarrow d_{n-j} \geqslant n - j. \tag{6}$$

The necessity of (6) in the case of threshold graphs has been proved by Minty as reported in Chvátal and Hammer [4]. We sharpen this result as follows:

Theorem 3.3. *Let $d_1 \leqslant \cdots \leqslant d_n$ be the degrees of a threshold graph G. Then G is Hamiltonian if and only if there is no j with $d_j \leqslant j < n/2$.*

Proof. Label the vertices as $1, \ldots, n$ so that vertex j has degree d_j.

'Only if': Assume if possible that $d_j \leqslant j < n/2$ and there exists a special matching of size $|S|$.

Case 1: $j \in S$. If m is even and $j = y$, then $N_K(j) = K$, hence

$$d_j = |K| \overset{(*)}{\geqslant} |S| \geqslant j,$$

where $(*)$ follows from the existence of the special matching. Hence by the assumption $d_j \leqslant j$, we have $d_j = j = |K| = |S| = n/2$, contradicting $j < n/2$. Therefore m is odd, or m is even and $j \neq y$. Then we have

$$d_j \overset{(*)}{=} |N_K(\{1, \ldots, j\})| = 1 + |N_{K-x}(\{1, \ldots, j\})| \overset{(\circ)}{\geqslant} 1 + j,$$

where $(*)$ follows from the nesting property (Eq. (2)), and $(\circ)$ follows from the existence of the special matching. This contradicts the assumption $d_j \leqslant j$.

Case 2: $j \in K$. Then $d_j \geqslant |\{j+1, \ldots, n\}| = n - j > n/2$, contradicting the assumption $d_j \leqslant j < n/2$.

'If': This part follows from Chvátal's general sufficiency condition (6), but the proof below does not use his result. Since no special matching of size $|S|$ exists, Hall's conditions are violated. Therefore, there exists a subset $U \subseteq S$ ($y \notin U$ in case m is even) such that $|N_{K-x}(U)| < |U|$. Let j be the largest element of U. We may extend U to $\{1, \ldots, j\}$ because this does not change $N_{K-x}(U)$. Thus there exists an element $j \in S$ such that

$$|N_{K-x}(j)| = |N_{K-x}(\{1, \ldots, j\})| < j \tag{7}$$

and therefore $d_j \leqslant j$. Choose the smallest j satisfying (7). We now have

$$d_j - 1 = |N_{K-x}(j)| \geqslant |N_{K-x}(j-1)| \overset{(*)}{\geqslant} j - 1 \geqslant d_j - 1,$$

where (*) follows from the minimality of j. Therefore $d_j = j$. It remains to show that $j < n/2$. If $j \geqslant n/2$, then $d_j \geqslant n/2$, and since $|S| \geqslant j$, $|K| \geqslant d_j$, and $|S| + |K| = n$, we have $j = d_j = n/2 = |S| = |K|$. Therefore $N_K(j) = K$ and j is adjacent to x. By the minimality of j, there exists a matching of $\{1, \ldots, j-1\}$ into $K - x$, which can be augmented by the edge jx to form a special matching, a contradiction. Therefore $j < n/2$. □

Remark. The 'only if' part of the above theorem can just as easily be proved using the existence of a Hamiltonian cycle and not the special matching.

We conclude with the following result.

Theorem 3.4. *Let G be a graph with LCL l and with no induced P_4 or C_4. Then G has cycles of length* $3, \ldots, l$.

Proof. If C is any cycle of length $k + 1 \geqslant 4$, consider four consecutive vertices a, b, c, d along C. Then either ac or bd is an edge of G, or otherwise G has a P_4 or C_4. Therefore by removing b or c from C, we obtain a cycle of length k. □

In particular, Theorem 3.4 applies to threshold graphs by condition 2 of Theorem 2.4. Thus a Hamiltonian threshold graph $G = (V, E)$ is also pancyclic (i.e., has cycles of all lengths $3, \ldots, |V|$). We note that the graphs without induced P_4 or C_4 are studied by Golumbic [7] under the name 'trivially prefect graphs'.

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