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Exponential Sums with Monomials

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1. INTRODUCTION

The exponential sums of type

$$\sum_{m} e(f(m)), \tag{1.1}$$

where m ranges over integers from an interval and $f(m)$ is a smooth function, play a central role in analytic number theory. General methods of estimating such sums were established by H. Weyl [9], J. G. van der Corput [2], and I. M. Vinogradov [8]. Later, the exponential sums in several variables

$$\sum_{m_1} \cdots \sum_{m_j} e(f(m_1, ..., m_j)) \tag{1.2}$$

were introduced to enhance these methods as well as for their own importance. Our main interest in this paper is to estimate the exponential sums in which f is a monomial function,

$$f(m_1, ..., m_j) = x m_1^{\alpha_1} \cdots m_j^{\alpha_j}. \tag{1.3}$$

We regard such sums as a special case of bilinear forms

$$\mathscr{B}_{\varphi\psi}(\mathscr{X}, \mathscr{Y}) = \sum_{r} \sum_{s} \varphi_r \psi_s e(x_r y_s), \tag{1.4}$$

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where $\mathscr{X} = (x_r)$, $\mathscr{Y} = (y_s)$ are finite sequences of real numbers with

$$|x_r| \leqslant X, \qquad |y_s| \leqslant Y, \tag{1.5}$$

say, and φ_r, ψ_s are complex numbers. Our arguments are based on the following general inequality of [1].

PROPOSITION 1. *We have*

$$|\mathscr{B}_{\varphi\psi}(\mathscr{X}, \mathscr{Y})|^2 \leqslant 20(1+XY)\mathscr{B}_\varphi(\mathscr{X}, Y)\mathscr{B}_\psi(\mathscr{Y}, X) \tag{1.6}$$

with

$$\mathscr{B}_\varphi(\mathscr{X}, Y) = \sum_{|x_{r_1} - x_{r_2}| \leqslant Y^{-1}} |\varphi_{r_1}\varphi_{r_2}|$$

and $\mathscr{B}_\psi(\mathscr{Y}, X)$ defined similarly.

We begin by applying (1.6) directly. Next we introduce a number of innovations to combine with (1.6) giving deeper results. Finally, we illustrate how to use these results for estimating sums which occur in sieve problems for short intervals.

Notation and conventions:

$[x] = \max\{k \in \mathbb{Z}; k \leqslant x\}$.

$\|x\| = \min\{|x-k|; k \in \mathbb{Z}\}$.

$e(z) = \exp(2\pi i z)$.

$f \ll g$ means $|f| \leqslant cg$ with some positive constant c.

$f = O(g)$ means $f \ll g$.

$f \sim g$ means $c_1 f < g < c_2 g$ with some positive, unspecified constants c_1, c_2.

The constants implied in the symbols O and $\ll$ may depend, without mentioning, on those implied in the relevant relations $f \sim g$.

$\|\varphi\|$ stands for the l_2-norm of the sequence $\varphi = (\varphi_r)$, i.e.,

$$\|\varphi\| = \left(\sum_r |\varphi_r|^2\right)^{1/2}.$$

▮ indicates the end of a proof.

2. Direct Results

By Proposition 1 we immediately obtain

COROLLARY 1. *Suppose that the sequences $\mathscr{X}$ and $\mathscr{Y}$ are A-spaced and B-spaced, respectively, i.e.,* $|x_{r_1}-x_{r_2}|\geqslant A$ *and* $|y_{s_1}-y_{s_2}|\geqslant B$ *for* $r_1\neq r_2$ *and* $s_1\neq s_2$, *respectively. We then have*

$$|\mathscr{B}_{\varphi\psi}(\mathscr{X},\mathscr{Y})|\leqslant 5(1+XY)^{1/2}\left(1+\frac{1}{AY}\right)^{1/2}\left(1+\frac{1}{BX}\right)^{1/2}\|\varphi\|\,\|\psi\|.$$

Yet, this result is ineffective since the spacing problem is not resolved. We easily handle that in the following situation.

THEOREM 1. *Let f and g be smooth functions such that*

$$f\sim F,\qquad f'\sim FM^{-1},$$
$$g\sim G,\qquad g'\sim GN^{-1},$$

with F, G, M, N positive, and let φ_m, ψ_n *be complex numbers. We then have*

$$S_{\varphi\psi}(M,N)=\sum_{m\sim M}\sum_{n\sim N}\varphi_m\psi_n e(f(m)\,g(n))$$
$$\ll (FG)^{-1/2}(FG+M)^{1/2}(FG+N)^{1/2}\|\varphi\|\,\|\psi\|.$$

Proof. It follows from Corollary 1 and from the relations

$$|f(m_1)-f(m_2)|\sim|m_1-m_2|\,M^{-1}F,\quad |g(n_1)-g(n_2)|\sim|n_1-n_2|\,N^{-1}G. \;\blacksquare$$

Next we investigate the spacing of binomial points $m^\alpha n^\beta$.

LEMMA 1. *Let* $\alpha\beta\neq 0$, $\Delta>0$, $M\geqslant 1$ *and* $N\geqslant 1$. *Let* $\mathscr{A}(M,N;\Delta)$ *be the number of quadruples* $(m,\tilde{m},n,\tilde{n})$ *such that*

$$\left|\left(\frac{\tilde{m}}{m}\right)^\alpha-\left(\frac{\tilde{n}}{n}\right)^\beta\right|<\Delta,$$

with $M\leqslant m,\ \tilde{m}<2M$ *and* $N\leqslant n,\ \tilde{n}<2N$. *We then have*

$$\mathscr{A}(M,N;\Delta)\ll MN\log 2MN+\Delta M^2N^2.$$

Proof. We divide the solutions into classes each one having fixed values $(m,\tilde{m})=\mu$ and $(n,\tilde{n})=\nu$, say. In a given class the points $(\tilde{m}/m)^\alpha$ are spaced by $c(\alpha)\mu^2M^{-2}$ and the points $(\tilde{n}/n)^\beta$ are spaced by $c(\beta)\nu^2N^{-2}$, where $c(\alpha)$

and $c(\beta)$ are positive constants. Therefore, by the Dirichlet box principle each class contains

$$\begin{aligned}&\ll \min\{\mu^{-2}M^2(1+\Delta\nu^{-2}N^2), \nu^{-2}N^2(1+\Delta\mu^{-2}M^2)\}\\&=\min\{(\mu^{-2}M^2, \nu^{-2}N^2\}+\Delta\mu^{-2}\nu^{-2}M^2N^2\end{aligned}$$

points. Summing over μ and ν we complete the proof. ∎

Proposition 1 and Lemma 1 yield an estimate for the exponential sum (1.2) in which f is a quadrinomial function.

THEOREM 2. *Let $\alpha_j \neq 0$, $M_j \geqslant 1$ for $j=1, 2, 3, 4$, $X>0$, and $\varphi_{m_1m_2}$, $\psi_{m_3m_4}$ be complex numbers with $|\varphi_{m_2m_2}| \leqslant 1$, $|\psi_{m_3m_4}| \leqslant 1$. We then have*

$$\begin{aligned}S_{\varphi\psi}(M_1, M_2, M_3, M_4) &= \sum_{m_j \sim M_j} \varphi_{m_1m_2}\psi_{m_3m_4} e\left(x\,\frac{m_1^{\alpha_1}m_2^{\alpha_2}m_3^{\alpha_3}m_4^{\alpha_4}}{M_1^{\alpha_1}M_2^{\alpha_2}M_3^{\alpha_3}M_4^{\alpha_4}}\right)\\&\ll \{(xM_1M_2M_3M_4)^{1/2}+M_1M_2(M_3M_4)^{1/2}\\&\quad+(M_1M_2)^{1/2}M_3M_4+x^{-1/2}M_1M_2M_3M_4\}\\&\quad\times \log 2M_1M_2M_3M_4.\end{aligned}$$

Proof. We apply (1.6) for the two sequences $\mathscr{X}=(xm_1^{\alpha_1}m_2^{\alpha_2}M_1^{-\alpha_1}M_2^{-\alpha_2})$ and $\mathscr{Y}=(m_3^{\alpha_3}m_4^{\alpha_4}M_3^{-\alpha_3}M_4^{-\alpha_4})$ getting

$$S_{\varphi\psi} \ll ((1+x)S_\varphi S_\psi)^{1/2},$$

where

$$S_\varphi = \#\{(m_1, m_2, \tilde{m}_1, \tilde{m}_2);\, |m_1^{\alpha_1}m_2^{\alpha_2}-\tilde{m}_1^{\alpha_1}\tilde{m}_2^{\alpha_2}| \ll x^{-1}M_1^{\alpha_1}M_2^{\alpha_2}\}$$

and S_φ is defined similarly. Hence S_φ is bounded by the number of solutions to

$$\left|\left(\frac{\tilde{m}_1}{m_1}\right)^{\alpha_1}-\left(\frac{\tilde{m}_2}{m_2}\right)^{\alpha_2}\right| \ll \frac{1}{x},$$

so, by Lemma 1 we have

$$S_\varphi \ll M_1M_2 \log 2M_1M_2 + x^{-1}M_1^2M_2^2.$$

Similarly

$$S_\psi \ll M_3M_4 \log 2M_3M_4 + x^{-1}M_3^2M_4^2.$$

These estimates and (1.6) yield the assertion of Theorem 2 provided $x \geqslant 1$. If $x<1$ the assertion is trivial. ∎

Remarks. Theorem 2 is useful in the range $1 \ll x \ll M_1 M_2 M_3 M_4$. The sharpest bound is attained for $M_1 M_2 M_3 M_4 \sim x^2$. The reason that Theorem 2 is trivial for $x \ll 1$ is that the exponential factor does not oscillate and the reason that it is also trivial for $x \gg M_1 M_2 M_3 M_4$ is that the exponential factor oscillates too rapidly. In the latter case the method fails because the number of terms is too small to control the oscillations. In order to overcome this problem, in the next section, we appeal to Weyl's method which creates additional points of summation and at the same time it reduces the oscillatory behaviour of the exponential factor. There is a double price for this operation. First is that Weyl's method shifts the argument of the exponential function destroying its monomial character and consequently it causes serious problems about the spacing of the resulting points. Second is the use of Cauchy's inequality which halves the final saving.

3. Two Combinations with the Weyl Shift

Weyl's method depends on the following inequality:

Lemma 2. *Let $L > K$, $Q > 0$, and z_k be complex numbers. We then have*

$$\left| \sum_{K \leqslant k < L} z_k \right|^2 \leqslant \left(2 + \frac{L-K}{Q}\right) \sum_{|q| < Q} \left(1 - \frac{|q|}{Q}\right) \sum_{K \leqslant k-q,\, k+q < L} z_{k-q} \bar{z}_{k+q}.$$

Proof. It is similar to that of Lemma 5.10 in [6]. ∎

Our aim is to prove the following:

Theorem 3. *Let α, α_1, α_2 be real constants such that $\alpha \neq 1$ and $\alpha\alpha_1\alpha_2 \neq 0$. Let M, M_1, M_2, $x \geqslant 1$, and φ_m, $\psi_{m_1 m_2}$ be complex numbers with $|\varphi_m| \leqslant 1$ and $|\psi_{m_1 m_2}| \leqslant 1$. We then have*

$$\begin{aligned} S_{\varphi\psi}(M, M_1, M_2) &= \sum_{m \sim M} \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} \varphi_m \psi_{m_1 m_2} e\left(x \frac{m^\alpha m_1^{\alpha_1} m_2^{\alpha_2}}{M^\alpha M_1^{\alpha_1} M_2^{\alpha_2}}\right) \\ &\ll \{x^{1/4} M^{1/2} (M_1 M_2)^{3/4} + M^{7/10} M_1 M_2 \\ &\quad + M (M_1 M_2)^{3/4} + x^{-1/4} M^{11/10} M_1 M_2\} (\log 2 M M_1 M_2)^2. \end{aligned}$$

Proof. By Lemma 2 we get that

$$|S_{\varphi\psi}|^2 \ll Q^{-1} M M_1 M_2 (M M_1 M_2 + |S(Q_0)| \log 2Q)$$

for any $Q \leqslant \frac{1}{3}M$ and some $Q_0 \leqslant Q$, where

$$S(Q_0) = \sum_{q \sim Q_0} (1 - qQ^{-1}) \sum_m \sum_{m_1} \sum_{m_2} \varphi_{m+q} \bar{\varphi}_{m-q} e\left(x \frac{t(m, q) m_1^{\alpha_1} m_2^{\alpha_2}}{M^\alpha M_1^{\alpha_1} M_2^{\alpha_2}} \right)$$

and

$$t(m, q) = (m+q)^\alpha - (m-q)^\alpha \sim Q_0 M^{\alpha - 1}.$$

By (1.6) we obtain

$$S(Q_0) \ll (\mathscr{A} \mathscr{B} x Q_0 M^{-1})^{1/2}. \tag{3.1}$$

Here $\mathscr{A}$ is the number of quadruples $(m_1, m_2, \tilde{m}_1, \tilde{m}_2)$ such that

$$\left| \left(\frac{\tilde{m}_1}{m_1} \right)^{\alpha_1} - \left(\frac{\tilde{m}_2}{m_2} \right)^{\alpha_2} \right| \ll (xQ_0)^{-1} M,$$

thus by Lemma 1 we have

$$\mathscr{A} \ll M_1 M_2 \log 2M_1 M_2 + (xQ_0)^{-1} M M_1^2 M_2^2. \tag{3.2}$$

It remains to estimate $\mathscr{B}$ which stands for the number of quadruples $(m, \tilde{m}, q, \tilde{q})$ such that

$$|t(m, q) - t(\tilde{m}, \tilde{q})| \ll x^{-1} M^\alpha.$$

We shall consider this problem in the next section. From Proposition 2 we obtain

$$\mathscr{B} \ll Q_0 M (1 + x^{-1} M^2)(\log 2M)^4 \tag{3.3}$$

provided $3Q_0 \leqslant M^{3/5}$. Combining (3.1), (3.2), and (3.3) we infer

$$S(Q_0) \ll Q_0 (x M_1 M_2)^{1/2} (1 + x^{-1} Q_0^{-1} M M_1 M_2)^{1/2} \\ \times (1 + x^{-1} M^2)^{1/2} (\log 2MM_1 M_2)^{5/2}.$$

Note that the worst value for Q_0 is $Q_0 = Q$. Setting $Q = \frac{1}{3} M^{3/5}$ we conclude the proof. ∎

Next we estimate the bilinear forms of type

$$S_{\varphi\psi}(M_1, M_2) = \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} \varphi_{m_1} \psi_{m_2} e\left(x \frac{m_1^{\alpha_1} m_2^{\alpha_2}}{M_1^{\alpha_1} M_2^{\alpha_2}} \right)$$

by alternated application of the Weyl shift.

THEOREM 4. *Let α_1, α_2 be real constants different from 0 and 1. Let $M_1, M_2, x \geqslant 1$, and $\varphi_{m_1}, \psi_{m_2}$ be complex numbers with $|\varphi_{m_1}| \leqslant 1$ and $|\psi_{m_2}| \leqslant 1$. We then have*

$$S_{\varphi\psi}(M_1, M_2) \ll \{x^{1/8}(M_1 M_2)^{3/4} + M_1^{4/5} M_2 + M_1 M_2^{17/20} + x^{-1/8}(M_1 M_2)^{21/20}\}(\log 2M_1 M_2)^2.$$

Proof. By Lemma 2 we get that

$$|S_{\varphi\psi}|^2 \ll Q_1^{-1} M_1^2 M_2^2 + Q_1^{-1} M_1 M_2 (\log 2M_1) \left| \sum_{q_1 \sim Q_1^*} \left(1 - \frac{q_1}{Q_1^*}\right) \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} \varphi_{m_1 + qq_1} \bar{\varphi}_{m_1 - q_1} e(xt(m_1, q_1) m_2^{\alpha_2} M_1^{-\alpha_1} M_2^{-\alpha_2}) \right|$$

for any $Q_1 \leqslant \frac{1}{3} M_1$ and some $Q_1^* \leqslant Q_1$. Another application of Lemma 2 gives

$$|S_{\varphi\psi}|^4 \ll Q_1^{-2} M_1^4 M_2^4 + Q_1^{-2} Q_2^{-1} Q_1^* M_1^3 M_2^3 (Q_1^* M_1 M_2 + S)(\log 2M_1 M_2)^3,$$

where

$$S \ll \sum_{\substack{m_1 \sim M_1 \\ m_2 \sim M_2}} \sum_{\substack{q_1 \sim Q_1^* \\ q_2 \sim Q_2^*}} \left(1 - \frac{q_2}{Q_2}\right) e(xt(m_1, q_1) t(m_2, q_2) M_1^{-\alpha_1} M_2^{-\alpha_2})$$

for any $Q_2 \leqslant \frac{1}{3} M_2$ and some $Q_2^* \leqslant Q_2$. By Proposition 1 we obtain

$$S \ll (\mathscr{B}_1 \mathscr{B}_2 x Q_1^* Q_2^* M_1^{-1} M_2^{-1})^{1/2},$$

where $\mathscr{B}_1$ is the number of solutions to the inequality

$$|t(m_1, q_1) - t(\tilde{m}_1, \tilde{q}_1)| \ll (xQ_2^*)^{-1} M_1^{\alpha_1} M_2$$

in integers $m_1, \tilde{m}_1 \sim M_1$ and $q_1, \tilde{q}_1 \sim Q_1^*$ and $\mathscr{B}_2$ is defined similarly. Assuming that $3Q_1 \leqslant M_1^{3/5}$ and $3Q_2 \leqslant M_2^{3/5}$ we are entitled to use Proposition 2 giving

$$\mathscr{B}_1 \ll Q_1^* M_1 \left(1 + \frac{M_1^2 M_2}{xQ_2^*}\right) (\log 2M_1)^4.$$

A similar inequality holds for $\mathscr{B}_2$. From both results we get

$$S \ll x^{1/2} Q_1^* Q_2^* \left(1 + \frac{M_1^2 M_2}{x Q_2^*}\right)^{1/2} \left(1 + \frac{M_1 M_2^2}{x Q_1^*}\right)^{1/2} (\log 2M_1 M_2)^4.$$

Now observe that the worst values for Q_1^*, Q_2^* are $Q_1^* = Q_1$ and $Q_2^* = Q_2$. Setting $Q_1 = \frac{1}{3} M_1^{3/5}$ and $Q_2 = \frac{1}{3} M_2^{3/5}$ the resulting inequalities finally yield

$$\begin{aligned} S_{\varphi\psi} &\ll M_1 M_2 (M_1^{-3/10} + M_2^{-3/20}) \log 2M_1 M_2 \\ &\quad + x^{1/8} (M_1 M_2)^{3/4} (1 + x^{-1} M_1^2 M_2^{2/5})^{1/8} \\ &\quad \times (1 + x^{-1} M_1^{2/5} M_2^2)^{1/8} (\log 2M_1 M_2)^2 \\ &\ll (M_1 M_2^{17/20} + M_1^{4/5} M_2 + x^{1/8} M_1^{3/4} M_2^{3/4} + x^{-1/8} M_1^{21/20} M_2^{21/20}) \\ &\quad \times (\log 2M_1 M_2)^2. \quad \blacksquare \end{aligned}$$

4. The Spacing Problem

In this section we investigate the distribution of real numbers of type

$$t(m, q) = (m+q)^\alpha - (m-q)^\alpha$$

with $\alpha \neq 0, 1$ where m, q range over integers with $M \leqslant m < 2M$, $Q \leqslant q < 2Q$, and $3Q < M$. Note that

$$|t(m, q)| \sim M^{\alpha - 1} Q = T,$$

say. Let $\mathscr{B}(M, Q, \Delta)$ be the number of quadruples $(m, \tilde{m}, q, \tilde{q})$ such that

$$|t(m, q) - t(\tilde{m}, \tilde{q})| < \Delta T. \tag{4.1}$$

Our aim is to prove the following:

Proposition 2. *If $Q \leqslant M^{2/3}$ we have*

$$\mathscr{B}(M, Q, \Delta) \ll (MQ + \Delta M^2 Q^2 + M^{-2} Q^6)(\log 2M)^2, \tag{4.2}$$

the constant implied in $\ll$ depends on α only.

The proof depends on three lemmas.

Lemma 3. *Let $\mathscr{C}(A, B, M, \Delta)$ be the number of integers m with $M \leqslant m < 2M$ such that*

$$\|Am - Bm^{-1}\| < \Delta.$$

If $0 < B < \Delta M^2$ *we have*

$$\mathscr{C}(A, B, M, \Delta) \ll \Delta M \sum_{0 \leqslant s < \Delta^{-1}} (1 + \|sA\| M)^{-1} + \Delta B^{-1/2} M^{3/2} \sum_{\substack{0 < s < \Delta^{-1} \\ \|sA\| < 2sBM^{-2}}} s^{-1/2},$$

where the constant implied in $\ll$ *is absolute.*

Proof. We assume that $\Delta \leqslant \frac{1}{4}$ because otherwise the assertion of Lemma 3 is trivial. For $S > 0$ we have the identiy

$$\sum_{|s| < S} \left(1 - \frac{|s|}{S}\right) e(sx) = \frac{1 - \{S\}}{S} \left(\frac{\sin \pi x[S]}{\sin \pi x}\right)^2 + \frac{\{S\}}{S} \left(\frac{\sin \pi x[S+1]}{\sin \pi x}\right)^2.$$

Hence the sum is positive for all real x and it is $\geqslant \frac{1}{4} S$ if $\|x\| < (4S)^{-1}$. From this observation it follows that

$$\mathscr{C}(A, B, M\Delta) \ll S^{-1} \sum_{0 \leqslant s < S} \left| \sum_{M \leqslant m < 2M} e(Asm - Bsm^{-1}) \right|$$

with $S = (4\Delta)^{-1}$. The constant term $(s = 0)$ contributes $O(\Delta M)$. For $1 \leqslant s \leqslant S$ the innermost sum is equal to (see Lemma 4.8 in [6])

$$\int_M^{2M} e(\pm \|As\| \xi - Bs\xi^{-1})\, d\xi + O(1).$$

By Lemma 4.4 of [6] we have

$$\int_M^{2M} \ll (sB)^{-1/2} M^{3/2}$$

and by Lemma 4.2 of [6] we have

$$\int_M^{2M} \ll \|sA\|^{-1}$$

unless $\|sA\| < 2sBM^{-2}$. Finally we have the trivial bound

$$\int_M^{2M} \ll M.$$

Gathering together the above estimates we complete the proof of Lemma 3. ∎

LEMMA 4. *Let* $A(q/\tilde{q})=(q/\tilde{q})^{\gamma}$ *with* $\gamma \neq 0$ *and* $B(q, \tilde{q}) \sim |q-\tilde{q}| Q$ *for* $Q \leqslant q, \tilde{q} < 2Q$. *Let* $\mathscr{D}(M, Q, \Delta)$ *be the number of triplets* $(m, q, \tilde{q})$ *of integers* $m, q, \tilde{q}$ *in* $M \leqslant m < 2M$, $Q \leqslant q, \tilde{q} < 2Q$ *such that*

$$\|A(q/\tilde{q})m - B(q, \tilde{q})m^{-1}\| < \Delta.$$

If $Q < M^{2/3}$ *we have*

$$\mathscr{D}(M, Q, \Delta) \ll (MQ + \Delta MQ^2 + Q^{8/3})(\log 2M)^4.$$

Proof. Without loss of generality we can assume that $Q^{-1} < \Delta < 1$. First we count the triplets $(m, q, \tilde{q})$ with $|B(q, \tilde{q})| < \Delta M$ by a crude argument. We have

$$\|A(q/\tilde{q})m\| < 2\Delta$$

which implies

$$\left|\left(\frac{q}{\tilde{q}}\right)^{\gamma} - \frac{\tilde{m}}{m}\right| \ll \Delta M^{-1}.$$

Therefore, by Lemma 1 we conclude that the number of such triplets is

$$O(MQ \log 2M + \Delta MQ^2).$$

For counting the remaining triplets, i.e., those $(m, q, \tilde{q})$ with $|B(q, \tilde{q})| \geqslant \Delta M$, we appeal to Lemma 3, giving

$$O(\Delta MQ^2 + \mathscr{D}_1(M, Q, \Delta) + \mathscr{D}_2(M, Q, \Delta)),$$

where $\mathscr{D}_1$ and $\mathscr{D}_2$ are defined by

$$\mathscr{D}_1(M, Q, \Delta) = \Delta M \sum_{0 < s < \Delta^{-1}} \sum\sum_{Q \leqslant q, \tilde{q} < 2Q} (1 + \|sA(q/\tilde{q})\| \, M)^{-1}$$

and

$$\mathscr{D}_2(M, Q, \Delta) = \Delta M^{3/2} Q^{-1/2} \sum_{0 < s < \Delta^{-1}}^{*} \sum^{*}\sum^{*}_{Q \leqslant q, \tilde{q} < SQ} (s\,|q-\tilde{q}|)^{-1/2},$$

where * means that the summation is restricted by two inequalities

$$\Delta M \ll |q-\tilde{q}| \, Q,$$

and

$$\|sA(q/\tilde{q})\| \ll s\,|q-\tilde{q}| \, QM^{-2}.$$

First, we estimate $\mathscr{D}_1$. We split the range of summation into subsets defined by

$$S \leqslant s < 2S \qquad \text{and} \qquad \|sA(q/\tilde{q})\| < \delta,$$

where $1 \leqslant S < \Delta^{-1}$ and $M^{-1} < \delta < 1$. The complete splitting can be arranged with at most $O((\log 2M)^2)$ subsets, so

$$\mathscr{D}_1(M, Q, \Delta) \ll \Delta\delta^{-1}\mathscr{A}(Q, S, \delta S^{-1})(\log 2M)^2$$

for some relevant S and δ. Since $Q \leqslant M$ we conclude by Lemma 1 that

$$\mathscr{D}_1(M, Q, \Delta) \ll MQ(\log 2M)^3.$$

Now we estimate $\mathscr{D}_2$ in a similar way, so we split the range of summation into subsets defined by

$$S \leqslant s < 2S \qquad \text{and} \qquad R \leqslant |q - \tilde{q}| < 2R,$$

where $1 \leqslant S < \Delta^{-1}$ and $\Delta MQ^{-1} \ll R \ll Q$. We obtain

$$\mathscr{D}_2(M, Q, \Delta) = \Delta M^{3/2}(QRS)^{-1/2}\mathscr{A}(Q, S, M^{-2}QR)(\log 2M)^2$$

for some relevant S and R. By Lemma 1 we conclude that

$$\mathscr{D}_2(M, Q, \Delta) \ll (MQ + (\Delta M)^{-1/2}Q^3)(\log 2M)^4.$$

Hence

$$\mathscr{D}(M, Q, \Delta) \ll \Delta MQ^2 + (MQ + (\Delta M)^{-1/2}Q^3)(\log 2M)^4.$$

Finally, since $\mathscr{D}(M, Q, \Delta)$ is non-decreasing in Δ we can replace Δ on the right-hand side by $\Delta + M^{-1}Q^{2/3}$, completing the proof of Lemma 4. ∎

Now we are ready to prove Proposition 2. Clearly (4.1) implies

$$|s(m, q) - s(\tilde{m}, \tilde{q})| \ll \Delta T^{1/(\alpha-1)} = \Delta MQ^{1/(\alpha-1)},$$

where

$$s(m, q) = \left(\frac{t(m, q)}{2\alpha}\right)^{1/(\alpha-1)} = q^{1/(\alpha-1)} mf\left(\frac{q}{m}\right)$$

with

$$f(u) = \left(\frac{(1+u)^\alpha - (1-u)^\alpha}{2\alpha u}\right)^{1/(\alpha-1)} = 1 + \frac{\alpha-2}{6}u^2 + O(u^4).$$

Hence

$$(q^{1/(\alpha-1)}m-\tilde{q}^{1/(\alpha-1)}\tilde{m})+\frac{\alpha-2}{6}(q^{(2\alpha-1)/(\alpha-1)}m^{-1}-\tilde{q}^{(2\alpha-1)/(\alpha-1)}\tilde{m}^{-1})$$
$$\ll \Delta MQ^{1/(\alpha-1)}+M^{-3}Q^{4+1/(\alpha-1)}.$$

This gives the first approximation to $\tilde{m}$ in terms of $m, q, \tilde{q}$, namely,

$$\tilde{m}=\left(\frac{q}{\tilde{q}}\right)^{1/(\alpha-1)}m+O(\Delta M+M^{-1}Q^2),$$

which inserted into the last term yields the second (stronger) approximation

$$|A(q/\tilde{q})m-\tilde{m}+B(q,\tilde{q})m^{-1}|\ll \Delta M+M^{-3}Q^4,$$

where

$$A(q/\tilde{q})=(q/\tilde{q})^{1/(\alpha-1)}$$

and

$$B(q,\tilde{q})=\frac{\alpha-2}{6}(q^2A(q/\tilde{q})-\tilde{q}^2A(\tilde{q}/q)).$$

The second approximation implies

$$\|A(q/\tilde{q})m+B(q/\tilde{q})m^{-1}\|\ll \Delta M+M^{-3}Q^4 \tag{4.3}$$

and that for given $m, q, \tilde{q}$ the number of $\tilde{m}$'s is bounded by $O(1+\Delta M)$. Now by Lemma 4 we obtain

$$\mathscr{B}(M,Q,\Delta)\ll(1+\Delta M)(MQ+\Delta M^2Q^2+M^{-2}Q^6+Q^{8/3})(\log 2M)^4.$$

Since $Q^{8/3}=(MQ)^{2/3}(M^{-2}Q^6)^{1/3}\leqslant MQ+M^{-2}Q^6$ the last term can be omitted. If $\Delta M<1$ we obtain (4.2), otherwise the trivial bound yields

$$\mathscr{B}(M,Q,\Delta)\ll(1+\Delta M)MQ^2\ll \Delta M^2Q^2.$$

This completes the proof of Proposition 2. ∎

5. A Combination of Weyl's Shift with Poisson's Summation

In this section we anticipate the use of Lemma 2 by an application of Poisson's summation and the stationary phase method to the sum

$$S_{\varphi\psi}(M_1, M_2, M_3, M_4) = \sum_{m_j \sim M_j} \varphi_{m_1 m_2} \psi_{m_3} e\left(x \frac{m_1^{\alpha_1} m_2^{\alpha_2} m_3^{\alpha} m_4^{-\alpha}}{M_1^{\alpha_1} M_2^{\alpha_2} M_3^{\alpha} M_4^{-\alpha}}\right).$$

We appeal to a special case of the van der Corput lemma

Lemma 5. *Let $X>0$, $M>0$, $\mu>1$, and $\alpha \neq 0, 1$. We then have*

$$\sum_{M<m<\mu M} m^{-1/2} e(\alpha^{-1} m^{\alpha} M^{-\alpha} X) = \gamma \sum_{N<n<\nu N} n^{-1/2} e(-\beta^{-1} n^{\beta} N^{-\beta} X) + O(M^{-1/2} \log(2+M) + N^{-1/2} \log(2+N))$$

with $\beta = \alpha/(\alpha-1)$, $\nu = \mu^{\alpha-1}$, $MN = X$, and some γ depending on α alone. The constant implied in the symbol O depends at most on α and μ. If $\nu < 1$ $(\alpha < 1)$ the range of summation in n is understood to be $\nu N < n < N$.

Proof. This result is a special case of Theorem 4.9 of [6], except that we claim a better error term, for which see [7]. ∎

In our applications of Lemma 5 the parameter X will depend multiplicatively on several variables; so will N. In order to separate the dependence from the range of summation we appeal to the following formula

Lemma 6. *Let $0 < L \leqslant N < \nu N < \lambda L$ and let a_l be complex numbers with $|a_l| \leqslant 1$. We then have*

$$\sum_{N<n<\nu N} a_n = \frac{1}{2\pi} \int_L^L \left(\sum_{L<l<\lambda L} a_l l^{-it} \right) N^{it} (\nu^{it} - 1) t^{-1} \, dt + O(\log(2+L)),$$

where the constant implied in O depends on λ only. ∎

Combining Lemmas 5 and 6 we obtain

Lemma 7. *Let $X>0$, $M>0$, $\mu>1$, and $\alpha \neq 0, 1$. We then have*

$$\sum_{M<m<\mu M} m^{-1/2} e(\alpha^{-1} m^{\alpha} M^{-\alpha} X)$$
$$= \frac{\gamma}{2\pi} \int_{-L}^{L} \left(\sum_{\lambda^{-1} L < l < \lambda L} l^{-1/2} \left(\frac{X}{lM}\right)^{it} e\left(-\beta^{-1} \left(\frac{lM}{X}\right)^{\beta} X\right)\right) \frac{\mu^{i(\alpha-1)t} - 1}{t} \, dt$$
$$+ O(M^{-1/2} \log(2+M) + L^{-1/2} \log(2+L)),$$

where $\beta=\alpha/(\alpha-1)$, $\lambda=2(\mu^{\alpha-1}+\mu^{1-\alpha})$, *and* L *is any number with* $\frac{1}{2}<LMX^{-1}<2$, *the constant implied in the symbol* O *depends at most on* α *and* μ.

Now by Lemma 7 with α replaced by $-\alpha$ we obtain

$$\begin{aligned}S_{\varphi\psi} &\ll x^{-1/2}M_4 \sum_{m_1\sim M_1}\sum_{m_2\sim M_2}\sum_{m_3\sim M_3}\sum_{l\sim L}\tilde{\varphi}_{m_1m_2}\tilde{\psi}_{m_3}\zeta_l \\ &\quad\times e\left((\tilde{a}-1)x\frac{m_1^{\beta_1}m_2^{\beta_2}m_3^{\beta}l^{\beta}}{M_1^{\beta_1}M_2^{\beta_2}M_3^{\beta}L^{\beta}}\right) \\ &\quad+O(M_1M_2M_3(1+x^{-1/2}M_4)(\log xM_4),\end{aligned}$$

where $\tilde{\varphi}_{m_1m_2}$ and $\tilde{\psi}_{m_3}$ are some complex numbers with $|\tilde{\varphi}_{m_1m_2}|<1$, $|\tilde{\psi}_{m_3}|<\log 2x$, ζ_l is a complex number with $|\zeta_l|\leqslant 1$, and $\beta_1=\alpha_1/(\alpha+1)$, $\beta_2=\alpha_2/(\alpha+1)$, $\beta=\alpha/(\alpha+1)$, $\tilde{\alpha}=(\alpha+1)\alpha^{\beta}$, $L=M_4^{-1}x$. Since the exponents in m_3 and l are equal it permits us to treat lm_3 as one variable, $m=lm_3$ say, with the multiplicity bounded by the divisor function $\tau(m)\ll m^{\varepsilon}$. Theorem 3 is applicable with $M=LM_3=xM_3M_4^{-1}$ giving

THEOREM 5. *Let* $\alpha, \alpha_1, \alpha_2$ *be real constants such that* $\alpha\neq 1$ *and* $\alpha\alpha_1\alpha_2\neq 0$. *Let* M_1, M_2, M_3, M_4, $x\geqslant 1$, *and* $\varphi_{m_1m_2}, \psi_{m_3}$ *be complex numbers with* $|\varphi_{m_1m_2}|\leqslant 1$ *and* $|\psi_{m_3}|\leqslant 1$. *We then have*

$$\begin{aligned}S_{\varphi\psi}(M_1M_2M_3M_4) &\ll \{x^{1/2}(M_1M_2)^{3/4}M_3+x^{7/20}M_1M_2M_3^{11/10}M_4^{-1/10} \\ &\quad+x^{1/4}(M_1M_2)^{3/4}(M_3M_4)^{1/2}+x^{1/5}M_1M_2M_3^{7/10}M_4^{3/10} \\ &\quad+x^{-1/2}M_1M_2M_3M_4\}(xM_1M_2M_3M_4)^{\varepsilon}\end{aligned}$$

for any $\varepsilon>0$, *the constant implied in* $\ll$ *depends on* α, α_1, α_2 *and* ε *only.*

6. A SPECIAL SUM

In this section we estimate sums of type

$$S_{\chi\varphi\psi}(H, M, N)=\sum_{h\sim H}\sum_{m\sim M}\sum_{n\sim N}\chi(h)\varphi_m\psi_n e\left(x\frac{hn^{-1}m^{\alpha}}{HN^{-1}M^{\alpha}}\right),$$

where $\chi(h)$ is an additive character, i.e., $\chi(h)=e(\xi h)$, say. Such sums occur in applications of modern sieve methods. The nature of the monomial hn^{-1} will be exploited effectively by a refined version of the argument established in [5]. We prove

THEOREM 6. *Let $\alpha \neq 0, 1$ and $H, M, N, x \geqslant 1$. Let $\chi(h)$ be an additive character and φ_m, ψ_n be complex numbers with $|\varphi_m| \leqslant 1$ and $|\psi_n| \leqslant 1$. We then have*

$$S_{\chi\varphi\psi}(H, M, N) \ll (HMN)^{1/2}[(H+N)^{1/2}(x^{1/8}H^{-1/6}M^{1/12}N^{1/6} + x^{1/8}H^{-1/8}N^{3/8} + N^{1/2} + N^{1/4}M^{1/8})x^{1/8} + M^{1/2} + x^{-1/4}M^{1/2}N]\mathscr{L}^4,$$

where the constant implied in $\ll$ depends on α only and we set $\mathscr{L} = \log 2HMNx$.

The proof is rather long. We begin by applying Cauchy's inequality

$$|S_{\chi\varphi\psi}|^2 \ll M^{3/2} \sum_{m \sim M} m^{-1/2} \left| \sum_{h \sim H} \sum_{n \sim N} \chi(h)\psi_n e\left(x \frac{hn^{-1}m^\alpha}{HN^{-1}M^\alpha}\right)\right|^2$$
$$= M^{3/2} \sum_{n_1 n_2}\sum \psi_{n_1}\bar{\psi}_{n_2} \sum_k \omega_{n_1 n_2}(k) \sum_m m^{-1/2} e\left(y\frac{km^\alpha}{n_1 n_2}\right),$$

say, where $y = xH^{-1}NM^{-\alpha}$ and $\omega_{n_1 n_2}(k)$ is defined by

$$\omega_{n_1 n_2}(k) = \sum_{\substack{h_1 \sim H\ h_2 \sim H \\ h_1 n_2 - h_2 n_1 = k}}\sum \chi(h_1)\bar{\chi}(h_2).$$

The terms with $k = 0$ contribute trivially $O(HM^2N\mathscr{L})$. The remaining k fall into dyadic intervals $|k| \sim K$ with $1 \ll K \ll HN$. For a technical convenience we wish to work with n_1, n_2 coprime. To make that restriction we observe that $\omega_{n_1 n_2}(k) = \omega_{n_1^* n_2^*}(k^*)$, where $k^* = k/d$, $n_1^* = n_1/d$, $n_2^* = n_2/d$ with $d = (n_1, n_2)$, and we get

$$|S_{\chi\varphi\psi}|^2 \ll \mathscr{L}^3 M^{3/2} \sum_{d \sim D} \sum_{n_1 \sim N_1} \sum_{\substack{n_2 \sim N_2 \\ (n_1, n_2) = 1}} \left| \sum_{r \sim R} \omega_{n_1 n_2}(r) \sum_{m \sim M} m^{-1/2} e\left(y \frac{rm^\alpha}{dn_1 n_2}\right)\right|$$
$$+ HM^2N\mathscr{L}^3$$

for some K with $1 \ll K \ll HN$ and for some D with $1 \leqslant D \leqslant \min\{K, N\}$, where $R = D^{-1}K$ and $N_1 = N_2 = D^{-1}N$. Hence by Lemma 7 we obtain

$$|S_{\chi\varphi\psi}|^2 \ll \mathscr{L}^3 M^2 \left(\frac{HN}{xK}\right)^{1/2} \sum_{d \sim D} \sum_{(n_1, n_2) = 1}\sum \sum_{l \sim L} \left| \sum_{r \sim R} r^{it}\omega_{n_1 n_2}(r) e(wr^{-\beta}R^\beta)\right|$$
$$+ \mathscr{L}^3 HMN(M + N^2 + x^{-1/2}MN^2), \tag{6.1}$$

where $L = |\alpha|\, xK(HMN)^{-1}$, t is a real number, $\beta = (\alpha - 1)^{-1}$, and

$$w = (1-\alpha)\frac{xK}{HN}\left(\frac{l}{L}\right)^{\alpha\beta}\left(\frac{dn_1 n_2}{DN_1 N_2}\right)^{\beta}.$$

Now, our nearest aim is to build a single variable $c = dn_1 n_2$, say, out of the three variables d, n_1, n_2. We begin by the Fourier analysis of the arithmetic function $\omega_{n_1 n_2}(r)$ to separate the variable r from the parameters n_1, n_2.

LEMMA 8. *Let $H_1 \geqslant H_1' \geqslant 1$, $H_2 \geqslant H_2' \geqslant 1$, ξ_1, ξ_2 be real numbers and n_1, n_2 be positive integers with $(n_1, n_2) = 1$. We have*

$$\omega(r) = \mathop{\sum\sum}_{\substack{H_1' \leqslant h_1 < H_1 H_2' \leqslant h_2 \leqslant H_2 \\ h_1 n_1 - h_2 n_2 = r}} e(\xi_1 h_1 + \xi_2 h_2)$$

$$= \int_0^1 \hat{\omega}(\theta) e(\theta r)\, d\theta \tag{6.2}$$

with

$$\int_0^1 |\hat{\omega}(\theta)|\, d\theta \ll \left(1 + \frac{H_1 H_2}{n_1 n_2}\right)^{1/2} (\log 2H_1 H_2)^2 \tag{6.3}$$

and the constant implied in $\ll$ is absolute.

Proof. We have (6.2) with

$$\hat{\omega}(\theta) = \left(\sum_{H_1' \leqslant h_1 < H_1} e((\xi_1 - \theta n_1) h_1)\right)\left(\sum_{H_2' \leqslant h_2 < H_2} e((\xi_2 + \theta n_2) h_2)\right).$$

Since

$$\left|\sum_{M_1 \leqslant m < M} e(\xi m)\right| = \sum_{m=-\infty}^{\infty} \alpha(m) e(\xi m)$$

with

$$\alpha(m) = \alpha(M_1, M; m) \ll \left(1 + \frac{m^2}{M^2}\right)^{-1} \log 2M,$$

it gives

$$|\hat{\omega}(\theta)| = \left(\sum_{h_1=-\infty}^{\infty} \alpha_1(h_1) e((\xi_1 - \theta n_1) h_1)\right)\left(\sum_{h_2=-\infty}^{\infty} \alpha_2(h_2) e((\xi_2 + \theta n_2) h_2)\right)$$

with

$$\alpha_j(h_j) \ll (1 + h_j^2 H_j^{-2})^{-1} \log 2H_j.$$

Hence

$$\int_0^1 |\hat{\omega}(\theta)|\, d\theta = \sum_{\substack{h_1=-\infty \\ h_1 n_1 = h_2 n_2}}^{\infty} \sum_{h_2=-\infty}^{\infty} \alpha_1(h_1)\alpha_2(h_2)e(\xi_1 h_1 + \xi_2 h_2)$$

$$\ll \sum_{h=-\infty}^{\infty} (1 + h^2 n_2^2 H_1^{-2})^{-1} (1 + h^2 n_1^2 H_2^{-2})^{-1} (\log 2H_1 H_2)^2$$

$$\ll \left(1 + \min\left\{\frac{H_1}{n_2}, \frac{H_2}{n_1}\right\}\right) (\log 2H_1 H_2)^2$$

and this yields (6.3). ∎

By Lemma 8 and by the Cauchy–Schwarz inequality we deduce that

$$\left|\sum_{r \sim R} r^{it}\omega_{n_1 n_2}(r)e(wr^{-\beta}R^{\beta})\right|^2$$
$$\leqslant \left(\int_0^1 |\hat{\omega}(\theta)|\, d\theta\right) \int_0^1 |\hat{\omega}(\theta)| \left|\sum_{r \sim R} r^{it}e(\theta r + wr^{-\beta}R^{\beta})\right|^2 d\theta.$$

Next, by Lemma 2 we obtain

$$\left|\sum_{r \sim R} r^{it}e(\theta r + wr^{-\beta}R^{\beta})\right|^2$$
$$\ll \frac{R^2}{Q} + \frac{R}{Q} \sum_{1 \leqslant q < Q} \left|\sum_{\substack{r+q \sim R \\ r-q \sim R}} \left(\frac{r+q}{r-q}\right)^{it} e(wR^{\beta}[(r+q)^{-\beta} - (r-q)^{-\beta}])\right|$$

with any $Q < \frac{1}{3}R$. Note that the bound does not depend on θ and that

$$\int_0^1 |\hat{\omega}(\theta)|\, d\theta \ll (1 + DHN^{-1})\mathscr{L}^2.$$

Combining the above results with (6.1) we conclude that

$$|S_{\chi\varphi\psi}|^2 \ll \mathscr{L}^5 M^2 \left(\frac{HN}{xK}\right)^{1/2} \left(1 + \frac{DH}{N}\right)$$
$$\times \sum_{d \sim D} \sum_{n_1 \sim N_1} \sum_{n_2 \sim N_2} \sum_{l \sim L} \left(\frac{R^2}{Q} + \frac{R}{Q} \sum_{1 \leqslant q < Q} \left|\sum_{\substack{r+q \sim R \\ r-q \sim R}} \left(\frac{r+q}{r-q}\right)^{it}\right.\right.$$
$$\left.\left.\times e(wR^{\beta}[(r+q)^{-\beta} - (r-q)^{-\beta}])\right|\right)^{1/2}$$
$$+ \mathscr{L}^3 HMN(M + N^2 + x^{-1/2}MN^2).$$

Note that the variables d, n_1, n_2 occur only in w as a product $c = dn_1n_2$. Given $c \sim DN_1N_2 = D^{-1}N^2 = C$, say, the number of representations of c in the form $c = dn_1n_2$ is bounded by the divisor function $\tau_3(c)$. By Cauchy's inequality we obtain

$$|S_{\chi\varphi\psi}|^4 \ll \mathscr{L}^{12} D^{-1} M^3 (DH+N)^2 \left(\frac{xNK^3}{HMQD^3} + \frac{K}{QD} \sum_{1 \leqslant q < Q} S_q\right)$$
$$+ \mathscr{L}^6 (HMN)^2 \, (M + N^2 + x^{-1/2} MN^2)^2,$$

where

$$S_q = \sum_{c \sim C} \sum_{l \sim L} \left| \sum_{\substack{r+q \sim R \\ r-q \sim R}} \left(\frac{r+q}{r-q}\right)^{it} e(wR^\beta[(r+q)^{-\beta} - (r-q)^{-\beta}])\right|$$

with $w = (1-\alpha)(xK/HN)(l/L)^{\alpha\beta}(c/C)^\beta$ and $\beta = (\alpha-1)^{-1}$. It remains to estimate S_q for each q individually. We appeal to Proposition 1. The arguments are similar to those used in the proof of Theorem 2 if we treat

$$\delta(r) = (r+q)^{-\beta} - (r-q)^{-\beta} \sim qr^{-\alpha\beta}$$

as a monomial in r of degree $-\alpha\beta$. More to the point we have

$$|\delta(r) - \delta(\tilde{r})| \sim q \, |r^{-\alpha\beta} - \tilde{r}^{-\alpha\beta}|,$$

so the spacing problems for the points $\delta(r)$ and for the points $r^{\alpha\beta}$ are equivalent. From these remarks it is clear that Proposition 1 produces a bound for S_q, the same as that in Theorem 2 with the following parameters; $M_1 = L$, $M_2 = C$, $M_3 = R$, $M_4 = 1$, and with x replaced by $\mathscr{x} = qxDH^{-1}N^{-1}$. We obtain

$$\mathscr{L}^{-1} S_q \ll (\mathscr{x} M_1 M_2 M_3 M_4)^{1/2} + M_1 M_2 (M_3 M_4)^{1/2}$$
$$+ (M_1 M_2)^{1/2} M_3 M_4 + \mathscr{x}^{-1/2} M_1 M_2 M_3 M_4$$
$$\ll \frac{xK}{H}\left(\frac{q}{DM}\right)^{1/2} + \frac{xN}{HN}\left(\frac{K}{D}\right)^{3/2} + \left(\frac{xN}{HM}\right)^{1/2}\left(\frac{K}{D}\right)^{3/2} + \left(\frac{x}{qH}\right)^{1/2} \frac{K^2 N^{3/2}}{MD^{5/2}}.$$

Hence

$$|S_{\chi\varphi\psi}|^4 \ll \mathscr{L}^{13} D^{-1} M^3 (DH+N)^2 \, (W_1 Q^{-1} + W_2 Q^{-1/2} + W_3 + W_4 Q^{1/2})$$
$$+ \mathscr{L}^6 (HMN)^2 \, (M + N^2 + x^{-1/2} MN^2)^2,$$

where $W_1=(xN/HM)(K/D)^3$, $W_2=(xN^3K^6/D^7HM^2)^{1/2}$, $W_3=(xN/HM)(K/D)^{5/2}+(xN/HM)^{1/2}(K/D)^{5/2}$, $W_4=xK^2/HM^{1/2}D^{3/2}$. We choose

$$Q=\min\left\{\frac{K}{3D}, \frac{W_2}{W_4}+\left(\frac{W_1}{W_4}\right)^{2/3}\right\}$$

giving

$$\begin{aligned}
&W_1Q^{-1}+W_2Q^{-1/2}+W_3+W_4Q^{1/2}\\
&\quad\ll W_1\left(\frac{D}{K}\right)+W_2\left(\frac{D}{K}\right)^{1/2}+W_3+W_1^{1/3}W_4^{2/3}+W_2^{1/2}W_4^{1/2}\\
&\quad\ll\frac{xNK^2}{HMD^2}+\left(\frac{xN^3K^5}{HM^2D^6}\right)^{1/2}+\frac{xN}{HM}\left(\frac{K}{D}\right)^{5/2}\\
&\qquad+\left(\frac{xN}{HM}\right)^{1/2}\left(\frac{K}{D}\right)^{5/2}+\frac{x}{H}\left(\frac{NK^7}{M^2D^6}\right)^{1/3}+\left(\frac{xN}{HM}\right)^{3/4}\left(\frac{K}{D}\right)^{5/2}.
\end{aligned}$$

Here the first term is majorized by the third term and the sixth term is equal to the geometric mean of the third and the fourth terms. Thus we obtain

$$\begin{aligned}
|S_{\chi\varphi\psi}|^4\ll\mathscr{L}^{13}D^{-1}M^3(DH+N)&\left\{\left(\frac{xN^3K^5}{HM^2D^6}\right)^{1/2}+\frac{xN}{HM}\left(\frac{K}{D}\right)^{5/2}\right.\\
&\left.+\left(\frac{xN}{HM}\right)^{1/2}\left(\frac{K}{D}\right)^{5/2}+\frac{x}{H}\left(\frac{NK^7}{M^2D^6}\right)^{1/3}\right\}\\
&+\mathscr{L}^6(HMN)^2\,(M+N^2+x^{-1/2}MN^2)^2.
\end{aligned}$$

Now observe that the worst case is $D\sim 1$ and $K\sim HN$ giving

$$\begin{aligned}
|S_{\chi\varphi\psi}|^4\ll\mathscr{L}^{13}M^2(H+N)^2\,&\{x^{1/2}H^2N^4+xH^{3/2}N^{7/2}\\
&+x^{1/2}H^2M^{1/2}N^3+xH^{4/3}M^{1/3}N^{8/3}\}\\
&+\mathscr{L}^6(HMN)^2\,(M+N^2+x^{-1/2}MN^2)^2.
\end{aligned}$$

Here the last but one term is majorized by the first term, so it can be omitted. This completes the proof of Theorem 6. ∎

7. Exponential Sums Related to Short Intervals

A considerable attention in analytic number theory is given to the distribution of special sequences in short intervals

$$\mathscr{A}=\{n;\, x-y<n\leqslant x\}$$

with $1 \leqslant y \leqslant \frac{1}{2}x$. A powerful approach to many such problems is offered by the sieve method which depends on estimates for the remainder terms

$$r_d = \left[\frac{x}{d}\right] - \left[\frac{x-y}{d}\right] - \frac{y}{d}.$$

The modern version of the linear sieve [4] requires estimates for bilinear form

$$R(M, N) = \sum_{1 \leqslant m < M} \sum_{1 \leqslant n < N} a_m b_n r_{mn} \tag{7.1}$$

with real coefficients a_m, b_n subject to $|a_m| \leqslant 1$, $|b_n| \leqslant 1$. One needs the upper bound

$$R(M, N) \ll yx^{-\varepsilon} \tag{7.2}$$

Clearly (7.2) holds true if $MN \leqslant yx^{-\varepsilon}$ because $|r_d| \leqslant 1$. However, one can do better by taking into account a cancellation of terms in (7.1) due to the variation in sign of r_d. The following lemma transforms the problem to estimating exponential sums of type

$$S_x(H, M, N) = \sum_{1 \leqslant h \leqslant H} \sum_{1 \leqslant m < M} \sum_{1 \leqslant m < N} \frac{a_m b_n}{mn} e\left(\frac{hx}{mn}\right).$$

LEMMA 9. *Let $M, N \geqslant 1$ and $|a_m| \leqslant 1$, $|b_n| \leqslant 1$. We have*

$$|R(M, N)| \leqslant 2 \int_{x-2y}^{x+y} |S_x(H, M, N)| \, dx + O(yx^{-\varepsilon})$$

with $H = MNy^{-1}x^{3\varepsilon}$, the constant implied in O depending on ε alone.

Proof. Let $f(x)$ be a smooth function supported in the interval $[x - y - yx^{-2\varepsilon}, x + yx^{-2\varepsilon}]$ such that $f(x) = 1$ in $[x - y, x]$ and

$$f^{(j)}(x) \ll (yx^{-2\varepsilon})^{-j}$$

for any $j \geqslant 0$, the constant implied in $\ll$ depending on j only. We then have

$$\begin{aligned}\sum\sum_{x-y<lmn \leqslant x}\sum a_m b_n &= \sum_l \sum_m \sum_n a_m b_n f(lmn) + O(yx^{-\varepsilon}) \\ &= \sum_m \sum_n \frac{a_m b_n}{mn} \sum_h \hat{f}\left(\frac{h}{mn}\right) + O(yx^{-\varepsilon})\end{aligned}$$

by Poisson's summation, where $\hat{f}(t)$ is the Fourier transform of $f(x)$. The constant term $(h=0)$ contributes

$$\hat{f}(0)\sum_m\sum_n \frac{a_m b_n}{mn} = y\sum_m\sum_n \frac{a_m b_n}{mn} + O(yx^{-\varepsilon}).$$

For $|h| > H = MNy^{-1}x^{3\varepsilon}$ we have $\hat{f}(h/mn) \ll h^{-2}$, so the terms with $|h| > H$ contribute $O(yx^{-\varepsilon})$. Combining these results we complete the proof. ∎

By Theorem 6 and Lemma 9 we infer

THEOREM 7. *Let $2 \leqslant y \leqslant x^{1/2}$ and $|a_m| \leqslant 1$, $|b_n| \leqslant 1$. We then have*

$$\sum_{M \leqslant m < 2M} \sum_{N \leqslant n < 2N} a_m b_n r_{mn} \ll yx^{-\varepsilon} \tag{7.3}$$

provided

$$M < yx^{-\varepsilon'} \tag{7.4}$$

$$N^6 < My^7x^{-3-\varepsilon'}, \tag{7.5}$$

$$M^2N^4 < yx^{1-\varepsilon'}, \tag{7.6}$$

where $\varepsilon' = 48\varepsilon$, the constant implied in $\ll$ depending on ε alone.

Proof. We have to show that

$$\sum_{h\sim H}\sum_{m\sim M}\sum_{n\sim N} a_m b_n e\left(\frac{hx}{mn}\right) \ll MNx^{-2\varepsilon}$$

for any H with $1 \leqslant H < MNy^{-1}x^{3\varepsilon}$ and $x-2y < x < x+y$. Since $H < N$ we obtain by Theorem 6 that the sum is

$$\begin{aligned}\ll (HMN)^{1/2}\Bigg[&N^{1/2}\left(\left(\frac{xH}{MN}\right)^{1/8} H^{-1/6}M^{1/12}N^{1/6}\right.\\ &\left.+\left(\frac{xH}{MN}\right)^{1/8} H^{-1/8}N^{3/8} + N^{1/2} + N^{1/4}M^{1/8}\right)\left(\frac{xH}{MN}\right)^{1/8}\\ &+ M^{1/2} + \left(\frac{MN}{xH}\right)^{1/4} M^{1/2}N\Bigg](\log x)^4.\end{aligned}$$

Now observe that the worst value of H is $H = MNy^{-1}x^{3\varepsilon}$ giving

$$\begin{aligned}\ll y^{-1/2}MN[&x^{1/4}(yM)^{-1/12}N^{1/2} + x^{1/4}(yM)^{-1/8}N^{3/4}\\ &+ x^{1/8}y^{-1/8}N + x^{1/8}y^{-1/8}N^{3/4}M^{1/8}\\ &+ M^{1/2} + x^{-1/4}y^{1/4}M^{1/2}N]x^{2\varepsilon} \ll MNx^{-2\varepsilon}\end{aligned}$$

provided (7.4), (7.5), (7.6) hold as well as

$$N^6 < My^5x^{-2-\varepsilon'}, \tag{7.7}$$

$$N^8 < y^5x^{-1-\varepsilon'}. \tag{7.8}$$

$$MN^6 < y^5x^{-1-\varepsilon'}. \tag{7.9}$$

But (7.7)–(7.9) follow from (7.4)–(7.6) under the hypothesis $y \leqslant x^{1/2}$. ∎

The bilinear form (7.1) was investigated in various papers. It was proved in [3, 5] that (7.2) holds true subject to (7.4) and

$$M^2N^4 < y^5x^{-1-\varepsilon'}. \tag{7.10}$$

Combining (7.5) and (7.10) we conclude

COROLLARY. *Let* $x^{7/19} < y < x^{11/23}$ *and* $|a_m| \leqslant 1$, $|b_n| \leqslant 1$. *We then have* (7.2) *provided*

$$M < yx^{-\varepsilon'} \tag{7.4}$$

and

$$N < y^{19/16}x^{-7/16-\varepsilon'}. \tag{7.11}$$

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