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Diameters in Graphs

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ABSTRACT

A diameter critical graph has the property that the addition of any edge decreases the diameter. All such graphs are determined for a given vertex connectivity and the edge number is given.

1. DIAMETER CRITICAL GRAPHS

We shall investigate some properties of graphs with respect to their diameters. The graphs in question are connected with single edges and no loops. (The graph terminology used is that of Ore [1].) Let G denote such a graph; its vertex set is V, the number of vertices and edges respectively:

$$\nu_v = n = |V|, \qquad \nu_e = |G|.$$

The *distance* $d = d(a, b)$ between the vertices a and b is the number of edges in any shortest graph arc (*geodesic*) connecting them. The diameter of G is

$$d_0 = \max d(a, b), \qquad a, b \in V$$

and a *diametral arc is any geodesic* $D(a, b)$ of length d_0.

If one adds a new edge E to G connecting two vertices in V the diameter cannot increase but it may decrease. We say that G is *diameter critical* (d.c.) when the addition of any edge decreases the diameter. Every graph is contained in some d.c. graph with the same vertex set and diameter. In a complete graph (simplex) $S_n = S(n)$ on n vertices, $d_0 = 1$ and it may be considered to be a d.c. graph. A d.c. graph with $d_0 = 2$ has the form $G = S_n - E$, which means that G is the sum of two simplexes S_{n-1} with a common intersection S_{n-2}. One of our main problems is to determine all d.c. graphs.

2. Determination of the Diameter Critical Graphs

Let

$$D(a_0, a_d) = (a_0, a_1)(a_1, a_2) \cdots (a_{d-1}, a_d), \qquad d = d_0 \tag{2.1}$$

be a diametral arc in G. Denote by A_i the set of vertices in G having distance i from a_0; hence $a_i \in A_i$ in (2.1). This defines a disjoint decomposition

$$V = a_0 + A_1 + \cdots + A_d \tag{2.2}$$

with

$$n = \alpha_0 + \alpha_1 + \cdots + \alpha_d, \qquad \alpha_i = | A_i |, \qquad \alpha_0 = 1. \tag{2.3}$$

The section graph

$$G_i = G(A_i) \tag{2.4}$$

is the *i-distance graph* with respect to a_0. G consists of these graphs together with the edges

$$E_i = (v_{i-1}, v_i), \qquad v_{i-1} \in A_{i-1}, \qquad v_i \in A_i \tag{2.5}$$

connecting two consecutive graphs G_i. Each $v_i \in A_i$ is the end-point of at least one edge (2.5).

Suppose from now on that G is a d.c. graph. The addition of an edge F connecting two vertices in the same G_i cannot diminish the distance between a_0 and a_d so (2.1) remains a diametral arc. This shows: *In a d.c. graph all distance graphs (2.4)* are simplexes

$$G(A_i) = S(\alpha_i) = S(A_i).$$

Our next observation is:

One has for a diametral arc (2.1) in a d.c. graph

$$A_0 = a_0, \qquad A_d = a_d, \qquad \alpha_0 = \alpha_d = 1. \tag{2.7}$$

Proof: We need only examine A_d. If this set should contain two different vertices a_d and a_d', the edge (a_d, a_d') is in $G(A_d)$. We adjoin to G the new edge (a_d', a_{d-2}). This does not reduce the distance from a_0 to a_d for a diametral arc must pass from a_0 to A_{d-2}, and if it should continue in the new edge the last section would consist of the two edges

$$(a_{d-2}, a_d'), \qquad (a_d', a_d).$$

Next we add to G a new edge E_i' connecting A_{i-1} and A_i. Again there can be no reduction of the diametral arc (2.1), so in d.c. graph each vertex in A_{i-1} is connected by an edge to each vertex in A_i. This shows that each set

$$B_i = A_i + A_{i+1}, \qquad i = 0, 1, \ldots, d-1, \tag{2.8}$$

defines a simplex in G

$$S(B_i) = S(\beta_i), \qquad | B_i | = \beta_i = \alpha_i + \alpha_{i+1} . \tag{2.9}$$

A graph

$$G = \Sigma S(B_i), \qquad V = \Sigma B_i , \qquad i = 0, 1,\dots d - 1 \tag{2.10}$$

which is the sum of simplexes we call a *simplex chain* when

$$B_i \cap B_j = \phi, \tag{2.11}$$

when i and j are not consecutive, while we now define A_i through

$$B_{i-1} \cap B_i = A_i \neq \phi. \tag{2.12}$$

Such a chain is *tight* when each vertex in B_i belongs to either B_{i-1} or B_{i+1}. We then have the disjoint decomposition (2.8) for B_i provided we put for the *end sets*

$$A_o = B_o - B_o \cap B_1 , \qquad A_d = B_{d-1} - B_{d-1} \cap B_{d-2} . \tag{2.13}$$

We can now state the result:

THEOREM 2.1. *A graph G with diameter d is critical if and only if it is a tight simplex chain (2.10) in which the end sets (2.13) are single vertices a_o and a_d .*

PROOF: We have verified that a d.c. graph has these properties. In a simplex chain all diametral arcs must connect the vertices a_0 and a_d. Then any added edge must connect non-consecutive sets A_i and A_j and the diameter is reduced.

Theorem 2.1 holds also for infinite graphs with finite diameters.

3. THE NUMBER OF EDGES

From the form of a d.c. graph as determined by Theorem 2.1 we see that the number of its edges is

$$\nu_e = \sum_{i=0}^{d-1} (\alpha_i \cdot \alpha_{i+1} + \tfrac{1}{2}\alpha_{i+1} \cdot (\alpha_{i+1} - 1)). \tag{3.1}$$

In the special case of a d.c. graph $G_2(a_o , a_2)$ having diameter $d = 2$ one finds from (2.3)

$$\alpha_o = \alpha_2 = 1, \qquad \alpha_1 = n - 2$$

and the number of edges is

$$\nu_e = \tfrac{1}{2}(n+1)(n-2). \tag{3.2}$$

For a graph $G_3(a_0, a_3)$ with diameter $d = 3$

$$\alpha_0 = \alpha_3 = 1, \qquad \alpha_1 + \alpha_2 = n - 2$$

and the number of edges is found to be by (3.1)

$$\nu_e = \tfrac{1}{2}(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 1) = \tfrac{1}{2}(n-1)(n-2). \tag{3.3}$$

For larger values of d the number of edges in a d.c. graph depends upon the choice of the α_i. For a given $d \geqslant 4$ we shall determine the d.c. graphs with a maximal number of edges (max. d.c. graphs).

Suppose $\alpha_{i-1} \geqslant 2$ and α_i are consecutive numbers in (2.3). We replace them by

$$\alpha_{i-1} - 1, \qquad \alpha_i + 1$$

so that (2.3) remains valid. The terms in (3.1) involving α_{i-1} and α_i are

$$\alpha_{i-2} \cdot \alpha_{i-1} + \tfrac{1}{2}\alpha_{i-1}(\alpha_{i-1} - 1) + \alpha_{i-1} \cdot \alpha_i + \tfrac{1}{2}\alpha_i(\alpha_i - 1) + \alpha_i \cdot \alpha_{i+1}$$

and these are changed into

$$\alpha_{i-2}(\alpha_{i-1} - 1) + \tfrac{1}{2}(\alpha_{i-1} - 1)(\alpha_{i-1} - 2) + (\alpha_{i-1} - 1)(\alpha_i + 1)$$
$$+ \tfrac{1}{2}(\alpha_i + 1)\alpha_i + (\alpha_i + 1)\alpha_{i+1}.$$

When the first expression is subtracted from the second the difference is

$$\Delta = \alpha_{i+1} - \alpha_{i-2}.$$

Thus if $\alpha_{i+1} > \alpha_{i-2}$ the change yields a d.c. graph with the same diameter and a larger number of edges. Hence if G is a max. d.c. graph it follows that $\alpha_{i-1} = 1$ when $\alpha_{i+1} > \alpha_{i-2}$. If $\alpha_{i-2} = 1$ one must have either $\alpha_{i+1} = 1$ or $\alpha_{i-1} = 1$.

The numbers α_i in (2.3) fall into sequences of two kinds: sequences with $\alpha = 1$ separated by sequences with $\alpha > 1$. The observation just made show that the latter sequences can consist only of one or two terms. Suppose that

$$\alpha_{i-1} = \alpha_{i+1} = 1, \qquad \alpha_i > 1. \tag{3.4}$$

According to (3.1) the number of edges in G in the section from a_{i-1} to a_{i+1} is then

$$\tfrac{1}{2}\alpha_i(\alpha_i + 3).$$

Suppose there is another section (3.4)

$$\alpha_{j-1} = \alpha_{j+1} = 1, \qquad \alpha_j > 1.$$

The total number of edges in the two sections is then

$$\tfrac{1}{2}\alpha_i(\alpha_i + 3) + \tfrac{1}{2}\alpha_j(\alpha_j + 3).$$

If α_i and α_j are replaced by $\alpha_i - 1$ and $\alpha_j + 1$ it is readily seen that the number of edges in G is increased provided $\alpha_j \geqslant \alpha_i$. We conclude that in a max. d.c. graph there can be at most one section (3.4).

Next take a section

$$\alpha_{i-2} = \alpha_{i+1} = 1, \qquad \alpha_{i-1} > 1, \qquad \alpha_i > 1. \tag{3.5}$$

It contains

$$\tfrac{1}{2}(\alpha_{i-1} + \alpha_i)(\alpha_{i-1} + \alpha_i + 1)$$

edges between a_{i-2} and a_{i+1}. By a similar argument as above one sees that, if there is a second section (3.5), the number of edges can be increased. Thus in a max. d.c. graph there is at most one section (3.5). Finally, if there is one section of type (3.4) and another of type (3.5), an increase in the number of edges is possible.

It follows that in a max. d.c. graph one has $\alpha_i = 1$ except for a single or for a pair of consecutive indices. In case there is a section of type (3.4) one sees from (2.3) that $\alpha_i = n - d$ and the number of edges is

$$\nu_e = d - 2 + \tfrac{1}{2}(n - d)(n - d + 3).$$

When there is a pair α_{i-1} and α_i as in (3.5) one finds the same expression. After rewriting it slightly we arrive at:

THEOREM 3.1. *In an arbitrary connected graph G with n vertices and diameter d the number of edges has the upper bound*

$$\nu_e \leqslant d + \tfrac{1}{2}(n - d - 1)(n - d + 4). \tag{3.6}$$

The bound is attained only for max. d.c. graphs. Leaving out the simplest case $d = 2$ these graphs can be described as follows; To a graph $G_3(a_0, a_3)$ of diameter $d = 3$ one attaches graph arcs at a_0 and a_3 having a total length $d - 3$. A special graph of this type is

$$G_2(a_0, a_2) = S_{n-d+2} - E, \qquad E = (a_0, a_2)$$

with an arc of length $d - 2$ attached at a_0.

4. Multiply Connected Graphs

We say that a graph is *κ-vertex connected* when it cannot be separated by fewer than κ vertices. Theorem 2.1 yields immediately:

THEOREM 4.1. *A κ-vertex connected graph of diameter d is diameter critical if and only if it has the form given in Theorem 2.1 with*

$$\alpha_i \geqslant \kappa, \qquad i = 1, 2, \ldots, d-1.$$

When $d = 2$ we have

$$n - 2 = \alpha_1 \geqslant \kappa$$

and the number of edges is given by (3.2) as before. For $d = 3$ the conditions become

$$\alpha_1 \geqslant \kappa, \alpha_2 \geqslant \kappa; \qquad \alpha_1 + \alpha_2 = n - 2 \geqslant 2\kappa$$

and the number of edges is given in (3.3).

We assume $d \geqslant 4$ and determine all d.c. κ-connected graphs with maximal number of edges. We notice first that in such a graph

$$\alpha_1 = \alpha_{d-1} = \kappa, \qquad \alpha_0 = \alpha_d = 1. \tag{4.2}$$

It suffices to show $\alpha_1 = \kappa$; this follows from the fact that if $\alpha_1 > \kappa$ one can replace α_1 and α_2 by $\alpha_1 - 1$ and $\alpha_2 + 1$ to obtain a new graph with more edges. By an argument similar to that used in Section 3 one proves:

THEOREM 4.2. *A κ-vertex connected diameter critical graph with $d \geqslant 4$ and a maximal number of edges has the following characteristic properties*:

1. *The conditions (4.2) are fulfilled.*

2. *For all other values one has $\alpha = \kappa$ except for a single $\alpha_i \geqslant \kappa$ or a consecutive pair with $\alpha_{i-1} \geqslant \kappa$, $\alpha_i \geqslant \kappa$.*

In either of the two cases one finds that the number of edges is the same and has the expression given below in (4.3). This leads us to the generalization of Theorem 3.1.

THEOREM 4.3. *Let G be an arbitrary κ-vertex connected graph. Then for the number of edges one has the upper bound for $d \leqslant 4$*

$$\nu_e = \tfrac{1}{2}d \cdot \kappa(3\kappa - 1) - 5\kappa^2 + 3\kappa + \tfrac{1}{2}(n - 2 - (d - 2)\kappa)(n - 3 - (d - 6)\kappa). \tag{4.3}$$

For $\kappa = 1$ one finds the expression (3.6). For a graph without separating vertices $\kappa = 2$ so in this case the bound becomes

$$5d - 14 + \tfrac{1}{2}(n + 2 - 2d)(n + 9 - 2d).$$

Reference

1. O. Ore, *Theory of Graphs*, (Colloquium Publ. 38), American Mathematical Society, Providence, R. I., 1962.