On Néron models of moduli spaces of theta characteristics

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Abstract
Let \( f : C \rightarrow B \) be a smoothing of a stable curve \( C \) and \( S_f \) be the moduli space of theta characteristics on the smooth fibers of \( f \). We describe the Néron model \( N(S_f) \), in terms of combinatorial invariants of the dual graph of \( C \). Furthermore, we provide a modular description of \( N(S_f) \) and we construct an immersion \( \psi_f : N(S_f) \rightarrow J_{E^w} \), where \( J_{E^w} \) is a suitable relative compactified Jacobian. We show that \( \psi_f \) factors through the locus of \( J_{E^w} \) parametrizing locally free rank-1 sheaves.

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1. Introduction

1.1. Étale models

Let \( C \) be a projective scheme of dimension 1 over an algebraically closed field of characteristic zero, or, for short, a curve. Let \( f : C \rightarrow B \) be a general smoothing of \( C \), i.e. a family of curves over a smooth and connected curve \( B \), where \( C \) is non-singular and where \( C = f^{-1}(0) \) for some \( 0 \in B \) and \( C^* = f^{-1}(B - 0) \) is smooth over \( B - 0 \). Let \( S_f^* \) be the moduli scheme of theta characteristics on the fibers of \( f|_{C^*} \), an étale scheme over \( B - 0 \). It makes sense to ask: is it possible to give a description of the maximal étale \( B \)-model of \( S_f^* \), via combinatorial invariants of \( C \)? A goal of this paper is to give a positive answer to this question, when \( C \) is a stable curve without non-trivial automorphisms.

This distinguished \( B \)-model is necessarily the Néron model \( N(S_f^*) \) of \( S_f^* \) over \( B \). More generally, the Néron model provides a smooth and separated \( B \)-model of a scheme defined over the field of fractions of \( B \). The Néron model is canonically determined by a universal property, known as the Néron mapping property. Recall that the theory of Néron models has been introduced in [N] for abelian
varieties and it became apparent in [R] their connection with the Picard functor. They have been employed in arithmetic and geometry and recently also in the moduli theory of curves (see [C2,Ch,B]).

The posed question has been recently considered in [Ch]. There, it is shown a necessary and sufficient condition for the existence of a finite Néron model of the moduli space of $r$-torsion line bundles on the fibers of $f|_{C^*}$, via combinatorial invariants of the semistable reduction of $C$. More generally, one can consider the Néron model $N(\operatorname{Pic}^d C^*)$, where $\operatorname{Pic}^d C^*$ is the degree-$d$ relative Jacobian of $f|_{C^*}$. Assume that $C$ is a stable curve of genus $g$ and $d$ an integer such that $(d-g+1, 2g-2) = 1$. Let $\mathcal{P}_{d,g}$ be the universal Picard variety over $\mathcal{M}_g$ constructed in [C1]. In [C2, Theorem 6.1], it is shown that $N(\operatorname{Pic}^d C^*) = B \times \mathcal{M}_g \mathcal{P}_{d,g}$, where $\mathcal{P}_{d,g}$ is the representable stack version of the open subset of $\mathcal{P}_{d,g}$ parametrizing equivalence classes of balanced line bundles on stable curves. However, a theta characteristic of a curve of genus $g$ has degree $g - 1$, then [C2, Theorem 6.1] does not hold in this case. To find a geometric description of $\mathcal{N}(S_f^*)$, we will need to consider different compactified Jacobians.

1.2. The main result

Fix a smoothing $f : C \to B$ of a stable curve $C$ and let $d = g - 1$. There are plenty of degree-$d$ relative compactified Jacobian. In [A], it is shown that the ones constructed in [C1,OS,S] are all isomorphic. A different degree-$d$ relative compactified Jacobian was constructed in [AK] for a family of integral curves and more generally in [E] for a family of reduced and connected curves. This compactified Jacobian is denoted by $J_f^\sigma$. We establish a relationship between $J_f^\sigma$ and the Néron model $\mathcal{N}(S_f^*)$. A comparison result between $J_f^\sigma$ and Néron models of Picard schemes is contained in [B]. However, the fact that we are considering the subfunctor of theta characteristics, allows us to find a rather explicit geometric description of $\mathcal{N}(S_f^*)$.

Since we work over an algebraically closed field of characteristic zero, we can apply the techniques and results of [CCC]. There, for a given line bundle $\mathcal{G}$ of $C$, the authors constructed a scheme $\overline{\mathcal{S}}_f(\mathcal{G})$, finite over $B$, compactifying the moduli space of pairs $(C_b, L_b)$. $C_b$ a fiber of $f$ and $L_b$ a square root $\mathcal{G}|_{C_b}$. The objects employed in this construction are limit square roots, that is certain line bundles supported on nodal curves obtained by blowing-up the curves of the family, i.e. curves obtained by replacing nodal singularities by rational curves, called exceptional curves. There is a distinguished combinatorial invariant attached to a blow-up $X$ of a curve, which is the graph $\Sigma_X$ whose vertices are the connected components of the residual in $X$ of the union of the exceptional components and whose edges are the exceptional components of $X$. We will describe $\mathcal{N}(S_f^*)$ via combinatorial properties of the graph $\Sigma_X$, as follows.

In Section 3.2 we introduce and classify the set $\mathcal{A}_f(C_0)$ of $f$-admissible twisters of $C$ with respect to $C_0$, where $C_0$ is an irreducible component of $C$. The set $\mathcal{A}_f(C_0)$ is a subset of the set of line bundles on $C$ that are limits of trivial bundles on the smooth fibers of $f$. Our main result, contained in Theorems 2.2 and 3.9, is:

**Theorem 1.1.** Let $f : C \to B$ be a general smoothing of a stable curve $C$ of genus $g \geq 3$ with $\operatorname{Aut}(C) = \{\text{id}\}$. Let $\nu : \overline{\mathcal{S}}_f^\nu(\omega_f) \to \overline{\mathcal{S}}_f(\omega_f)$ be the normalization map. Then the following properties are equivalent for every $\xi \in \overline{\mathcal{S}}_f(\omega_f)$:

(i) $\overline{\mathcal{S}}_f(\omega_f) \to B$ is étale at a point of $\nu^{-1}(\xi)$;

(ii) $\overline{\mathcal{S}}_f(\omega_f) \to B$ is étale at any point of $\nu^{-1}(\xi)$;

(iii) if $\xi$ is supported on the blow-up $X$ of $C$, then $\Sigma_X$ is bipartite.

Furthermore, for every irreducible component $C_0 \subset C$ we have:

$$\mathcal{N}(S_f^*) \simeq \bigcup_{T \in \mathcal{A}_f(C_0)} \mathcal{S}_f(\omega_f \otimes T)$$
where $S_f(\omega_f \otimes T)$ is the open subscheme of $\overline{S_f}(\omega_f \otimes T)$ parametrizing limit square roots supported on stable curves and where $\sim$ denotes the gluing along the generic fiber of $S_f(\omega_f \otimes T) \to B$.

Let $E$ be the polarization $E = \mathcal{O}_C$ and let $(J^\sigma_E)^{\text{free}}$ be the open subspace of $J^\sigma_E$ parametrizing locally free sheaves. Then there exists an immersion:

$$\psi_f : N(S^*_f) \hookrightarrow (J^\sigma_E)^{\text{free}}.$$ 

The idea of comparing moduli spaces of roots of line bundles and compactified Jacobians already appears in [CCC] and [F]. Using Theorem 1.1, we are able to recover the combinatorial result of [Ch], which classifies the curves for which $N(S^*_f)$ is finite over $B$ in term of their dual graph (see Proposition 2.4).

1.3. Notation and terminology

A curve is a connected, projective, reduced scheme of dimension 1 over an algebraically closed field of characteristic zero. A stable (semistable) curve $C$ is a nodal curve such that every smooth rational component meets the rest of the curve in at least 3 points (2 points). The genus of $C$ is $g_C = h^1(C, \omega_C)$, where $\omega_C$ is the dualizing sheaf of $C$. If $Z \subset C$ is a subcurve, the residual in $C$ of $Z$ is $Z_C := C \setminus Z$.

A family of curves is a proper and flat morphism $f : C \to B$ whose fibers are curves. If $b \in B$, we denote by $C_b = f^{-1}(b)$. A smoothing of a curve $C$ is a family $f : C \to B$, where $B$ is a smooth, connected, affine curve with a distinguished point $0 \in B$ such that $C^* := f^{-1}(B - 0)$ is smooth over $B - 0$ and $C = f^{-1}(0)$. A general smoothing is a smoothing with $C$ smooth.

A nodal curve $X$ is obtained by blowing-up a nodal curve $C$ at a subset $\Delta$ of nodes of $C$, if there is a morphism $\pi : X \to C$ such that, for every $p_i \in \Delta$, $\pi^{-1}(p_i) = E_i \cong \mathbb{P}^1$ and $\pi : X - \bigcup_i E_i \to C - \Delta$ is an isomorphism. For every $p_i \in \Delta$, we call $E_i$ an exceptional component. A family of curves $X' \to B$ is a family of blow-ups of a family $C \to B$ if there exists a $B$-morphism $\pi : X' \to C$ such that $\pi|_{X_b} : X_b \to C_b$ is obtained by blowing-up $C_b$, for every $b \in B$.

Let $I$ be a coherent sheaf on a curve $C$. We say that $I$ is torsion-free if its associated points are generic points of $C$. We say that $I$ is of rank 1 if $I$ is invertible on a dense open subset of $C$. We say that $I$ is simple if $\text{End}(I) = k$. Each line bundle on $C$ is torsion-free of rank 1 and simple. If $I$ is torsion-free of rank 1, we call $\deg(I) := \chi(I) - \chi(\mathcal{O}_C)$ the degree of $I$.

Denote by $\text{Aut}(C)$ the group of automorphism of a curve $C$. If $\Gamma$ is a graph with an orientation, then $\delta : \mathbb{C}^0(\Gamma, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{C}^1(\mathbb{Z}/2\mathbb{Z})$ denotes the coboundary operator. A graph $\Gamma$ is bipartite if there is a partition of its vertices into two sets $A$ and $B$ such that each edge of $\Gamma$ has a vertex in $A$ and the other vertex in $B$. Equivalently, $\Gamma$ is bipartite if each cycle of $\Gamma$ has an even number of edges.

2. Néron models of moduli spaces of square roots

2.1. Review of moduli spaces of limit square roots

Let $C$ be a nodal curve and $G \in \text{Pic}(C)$ of even degree. Consider a tern $(X, L, \alpha)$, where $\pi : X \to C$ is a blow-up of $C$, $L \in \text{Pic} X$ and $\alpha$ is a homomorphism $\alpha : L^\otimes 2 \to \pi^*G$. Then $(X, L, \alpha)$ is a limit square root of $(C, G)$ if:

(i) the restriction of $L$ to every exceptional component has degree 1;
(ii) the map $\alpha$ is an isomorphism at the points of $X$ not belonging to an exceptional component;
(iii) for every exceptional component $E$ such that $E \cap E^c = \{p, q\}$ the orders of vanishing of $\alpha$ at $p$ and $q$ add up to 2.

The curve $X$ is called the support of the limit square root. If $C \to B$ is a family of stable curves and $G \in \text{Pic} C$ has even relative degree, then a limit square root of $(C, G)$ is a tern $(X, L, \alpha)$, where
\( \pi : \mathcal{X} \to C \) is a family of blow-ups, \( \mathcal{L} \in \text{Pic} \mathcal{X} \) and \( \alpha \) is a homomorphism \( \alpha : \mathcal{L} \otimes 2 \to \pi^* \mathcal{G} \) such that 
\((X_b, \mathcal{L}|_{X_b}, \alpha|_{X_b})\) is a limit square root of \((C_b, \mathcal{G}|_{C_b})\), for every \( b \in B \).

If \( X \) is obtained by blowing-up the curve \( C \), set \( \bar{X} := X - \bigcup_{E \in \mathcal{E}(X)} \bar{E} \), where \( \mathcal{E}(X) \) is the set of exceptional components of \( X \).

**Remark 2.1.** There exists a notion of isomorphism of limit square roots. By [Co, Lemma 2.1], two limit square roots \((X, L, \alpha)\) and \((X, L', \alpha')\) are isomorphic if and only if \( L|_{\bar{X}} \simeq L'|_{\bar{X}} \).

Let \( f : C \to B \) be a family of nodal curves over a quasi-projective scheme \( B \) and \( \mathcal{G} \in \text{Pic}(C) \) of even relative degree. Let \( \tilde{S}_f(\mathcal{G}) \) be the contravariant functor from the category of locally Noetherian \( B \)-schemes to sets, defined on \( T \) by:

\[
\tilde{S}_f(\mathcal{G})(T) := \{ \text{limit square roots of } q^*\mathcal{G} \}/ \sim
\]

where \( q : C \times_B T \to C \) is the first projection and \( \sim \) means isomorphism of limit square roots. There exists a quasi-projective scheme \( \tilde{S}_f(\mathcal{G}) \), finite over \( B \), which coarsely represents \( \tilde{S}_f(\mathcal{G}) \). For more details, we refer to [CCC, Theorem 2.4.1]. Abusing notation, we will often denote by \( \xi \) both the isomorphism class of a limit square root and the point of \( \tilde{S}_f(\mathcal{G}) \) parametrizing this equivalence class.

Let \( C \) be a nodal curve and \( \mathcal{G} \in \text{Pic}(C) \) of even degree. Denote by \( \overline{\text{SC}}_e(G) \) the zero-dimensional scheme \( \overline{S}_f_c(G) \), where \( f_c : C \to \text{Spec}(k) \) is the structure morphism of \( C \). In particular, \( \overline{S}_e(G) \) is in bijection with the isomorphism classes of limit square roots of \((C, G)\). If \( f : C \to B \) is a family of curves and \( \mathcal{G} \in \text{Pic}(C) \), then the fiber of \( \tilde{S}_f(\mathcal{G}) \to B \) over \( b \in B \) is \( \overline{S}_c(G|_{C_b}) \). If \( f : C \to B \) is a smoothing of a stable curve \( C \) with distinguished point \( 0 \in B \) and \( \mathcal{G} \) is a line bundle on \( C \) of even relative degree, let \( C^* := f^{-1}(B - 0) \) and \( \mathcal{G}^* := \mathcal{G}|_{C^*} \) and denote \( S(C^*) := \overline{S}_f|_{C^*}(\mathcal{G}^*) \). Moreover, denote by \( S_f(\mathcal{G}) \) the open subscheme of \( \tilde{S}_f(\mathcal{G}) \) parametrizing limit square roots supported on stable curves.

Let \( X \) be obtained by blowing-up \( C \). Let \( \Sigma_X \) be the graph whose vertices (resp. edges) corresponds to the connected components of \( X \) (resp. to the exceptional component of \( X \), where an edge connects two vertices if the corresponding exceptional component intersects the corresponding connected components. By [CCC, 4.1], the multiplicity of \( \overline{S}_e(G) \) in \( \xi \) is \( 2h^0(S_X) \), if \((X, L, \alpha)\) is a representative of \( \xi \). If \( C \) is a stable curve, denote by \( \Gamma_C \) the usual dual graph of \( C \), whose edges (resp. vertices) corresponds to the nodes (resp. to the irreducible components) of \( C \). Let \( \Gamma_X \) the subgraph of \( \Gamma_C \) whose edges corresponds to the nodes of \( C \) which are not blown-up to get \( X \). As observed in [CCC], the graph \( \Sigma_X \) is obtained from \( \Gamma_C \) by contracting the edges contained in \( \Gamma_X \).

### 2.2. A combinatorial result on the Néron model of \( S(G^*) \)

Let \( B \) be a connected Dedekind scheme with field of fractions \( K \). Let \( X_K \) be a smooth and separated \( K \)-scheme of finite type. A Néron model of \( X_K \) is a \( B \)-scheme \( N(X_K) \), which is a smooth, separated and finite type \( B \)-model of \( X_K \) and satisfying the following universal property, well-known as Néron mapping property: for every smooth \( B \)-scheme \( Y \) and \( K \)-morphism \( \phi_K : Y_K \to X_K \), there exists a unique extension of \( \phi_K \) to a \( B \)-morphism \( \phi : Y \to N(X_K) \). If a Néron model exists, it is canonically determined, up to a unique isomorphism, by the Néron mapping property.

**Theorem 2.2.** Let \( f : C \to B \) be a general smoothing of a stable curve \( C \) of genus \( g \geq 3 \) with \( \text{Aut}(C) = \{ \text{id} \} \). Consider the moduli space \( \overline{S}_f(\mathcal{G}) \), where \( \mathcal{G} \in \text{Pic}(C) \) is of even relative degree, and its normalization \( \nu : \overline{S}_f(\mathcal{G}) \to \overline{S}_f(\mathcal{G}) \). Then the Néron model of \( S(G^*) \) is isomorphic to the étale locus of \( \overline{S}_f(\mathcal{G}) \to B \) and the following properties are equivalent for every \( \xi \in \overline{S}_f(\mathcal{G}) \):

(i) \( \overline{S}_f(\mathcal{G}) \to B \) is étale at a point of \( \nu^{-1}(\xi) \);
(ii) \( \overline{S}_f(\mathcal{G}) \to B \) is étale at any point of \( \nu^{-1}(\xi) \);
(iii) if \( X \) is the support of a representative of \( \xi \), then \( \Sigma_X \) is bipartite.
Proof. Let $\gamma_f : B \to \overline{M}_g$ be the moduli morphism, where $\overline{M}_g$ is the moduli space of Deligne–Mumford stable curves. Since $C$ is smooth and $\text{Aut}(C) = \{id\}$, the image of $\gamma_f$ is smooth at $\gamma_f(0)$. Up to shrink $B$ to an open (analytic) subset containing 0, we can assume $B \subset \text{Def}(C)$, where $\text{Def}(C)$ is the base of the universal deformation of $C$. Let $(X, L, \alpha)$ be a representative of some $\xi \in \overline{S}_C(G|C)$. Assume that $X$ is obtained by blowing-up the nodes $n_1, \ldots, n_m$ of $C$. Let $t_j$ be the coordinate of $\text{Def}(C)$ such that \( t_j = 0 \) is the locus where the node $n_j$ persists, for every $j = 1, \ldots, m$. Using the fact that $C$ is smooth and the implicit function theorem, we can describe $B$ as:

$$\{ (t_1, t_1 h_2(t_1), t_1 h_3(t_1), \ldots, t_1 h_{3g-3}(t_1) ) \}$$

where $h_i$ is an analytic function such that $h_i(0) \in \mathbb{C}^*$, for $i = 2, \ldots, m$. Consider the morphism $\rho : \text{Def}(C) \to \text{Def}(C)$ given by:

$$\rho : (t_1, \ldots, t_m, t_{m+1}, \ldots, t_{3g-3}) \mapsto (t_1^2, \ldots, t_m^2, t_{m+1}, \ldots, t_{3g-3}).$$

Pick $U_\xi = \rho^{-1}(B)$. Fix an orientation on the graph $\Sigma_X$ and let $e_1, \ldots, e_m$ be the edges of $\Sigma_X$, corresponding to the exceptional components of $X$. Consider the coboundary operator $\delta : C^0(\Sigma_X, \mathbb{Z}/2\mathbb{Z}) \to C^1(\Sigma_X, \mathbb{Z}/2\mathbb{Z})$. By [CCC, Lemmas 2.3.2 and 3.3.1], the moduli space $\overline{S}_f(G)$ is $U_\xi / \text{Im} (\delta)$, locally analytically at $\xi$. Here, an element $\theta = \sum_{i=1}^m \epsilon_i \cdot e_i \in C^1(\Sigma_X, \mathbb{Z}/2\mathbb{Z})$, where $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ for $i = 1, \ldots, m$, acts on $U_\xi$ via:

$$(t_1, \ldots, t_m, t_{m+1}, \ldots, t_{3g-3}) \mapsto (\epsilon_1 t_1, \ldots, \epsilon_m t_m, t_{m+1}, \ldots, t_{3g-3}).$$

Furthermore, $\rho|_{U_\xi}$ factors through a morphism $\mu : U_\xi / \text{Im}(\delta) \to B$, giving locally the finite morphism $\overline{S}_f(G) \to B$ described in Section 2.1.

The tangent cone of $U_\xi$ at the origin is:

$$T_0(U_\xi) = \{ t_1^2 - h_2(0)t_1^2 = 0, \ldots, t_m^2 - h_m(0)t_1^2 = 0, t_{m+1} = 0, \ldots, t_{3g-3} = 0 \}.$$ 

Hence $U_\xi$ has $2^{m-1}$ distinct branches intersecting transversally. Consider the automorphisms $\theta^-$ of $\text{Def}(C)$ defined as:

$$\theta^- : (t_1, \ldots, t_m, t_{m+1}, \ldots, t_{3g-3}) \mapsto (-t_1, \ldots, -t_m, t_{m+1}, \ldots, t_{3g-3}).$$

Notice that $\theta^-$ commutes with $\rho$ and acts over $U_\xi$ preserving the irreducible components of $T_0(U_\xi)$ and hence also the branches of $U_\xi$. We see that $\rho|_{U_\xi}$ is a cover of $B$ of degree $2^m$ and, for every branch $U'_\xi \subset U_\xi$, we have that $\rho|_{U'_\xi}$ is a degree-2 cover of $B$ with involution $\theta^-|_{U'_\xi}$.

Notice that $\theta^- \in \text{Im}(\delta)$ if and only if $\Sigma_X$ is bipartite.

We show (i) $\Rightarrow$ (iii). Assume that $\overline{S}_f(G) \to B$ is étale at a point of $\nu^{-1}(\xi)$. Consider the finite morphism $\mu : U_\xi / \text{Im}(\delta) \to B$, giving locally the morphism $\overline{S}_f(G) \to B$. Then, for at least one branch $U'_\xi \subset U_\xi$, the restriction $\mu|_{U'_\xi} / \text{Im}(\delta)$ is a bijection. Hence $\theta^-|_{U'_\xi} = \overline{\theta}|_{U'_\xi}$, for some $\overline{\theta} \in \text{Im}(\delta)$, otherwise $\mu|_{U'_\xi} / \text{Im}(\delta)$ would have degree 2. Since $\theta^-$ is the only non-trivial automorphism of $C^1(\Sigma_X, \mathbb{Z}/2\mathbb{Z})$ preserving $U'_\xi$, then $\theta^- = \overline{\theta} \in \text{Im}(\delta)$ and hence $\Sigma_X$ is bipartite.

We show (iii) $\Rightarrow$ (ii). If $\Sigma_X$ is bipartite, then $\theta^- \in \text{Im}(\delta)$ and hence $\mu|_{U'_\xi} / \text{Im}(\delta)$ is a bijection, for every branch $U'_\xi \subset U_\xi$. In particular, $S_f(G) \to B$ is étale at every points of $\nu^{-1}(\xi)$. The implication (ii) $\Rightarrow$ (i) is trivial.

To prove the first statement, by [BLR, Proposition 1.2.4] we can assume without loss of generality that $B = \text{Spec} R$, where $R$ is a discrete valuation ring. By [BLR, Corollary 6.5.4], the Néron model
N(S(\mathcal{G}^+)) of S(\mathcal{G}^+) exists. Let \overline{\mathcal{S}_f} (\mathcal{G}^\text{et}) be the étale locus of \overline{\mathcal{S}_f} (\mathcal{G}) \to B. Now, N(S(\mathcal{G}^+)) is étale over B and it is a birational model of \overline{\mathcal{S}_f} (\mathcal{G}). Then we have an immersion N(S(\mathcal{G}^+)) \hookrightarrow \overline{\mathcal{S}_f} (\mathcal{G}^\text{et}) and, by the Néron mapping property, a reverse immersion holds as well. □

**Lemma 2.3.** Let C be a stable curve and \Gamma_C its dual graph. Let X be a blow-up of C and \Gamma_X the subgraph of \Gamma_C associated to \Gamma_X as explained in Section 2.1. Then X is the support of a representative of some \xi \in \overline{\mathcal{S}_C}(\omega_C) if and only if \Gamma_X can be written as a possibly empty union of cycles of \Gamma_C whose mutual intersections contains no edge of \Gamma_C.

**Proof.** See [CC, Section 1.3, p. 6]. □

**Proposition 2.4.** Let f : C \to B be a general smoothing of a stable curve C of genus g \geq 3 with Aut(C) = \{id\}. Then N(S(\omega^+_f)) is finite over B if and only if for every pair (\Gamma_1, \Gamma_2) of cycles of \Gamma_C the intersection \Gamma_1 \cap \Gamma_2 contains an even number of edges of \Gamma_C.

**Proof.** Assume that the condition of the statement holds. If \Gamma is a cycle of \Gamma_C, then, applying the condition of the statement to the pair (\Gamma, \Gamma), we see that \Gamma has an even number of edges. In particular \Gamma_C is bipartite. Pick \xi \in \overline{\mathcal{S}_C}(\omega_C) and let X be the support of any representative of \xi. If X is obtained by blowing-up X at the whole set of its nodes, then \Sigma_X = \Gamma_C and \Sigma_X is bipartite. Otherwise, \Sigma_X is obtained by contracting \Gamma_X. Combining Lemma 2.3 and the condition of the statement, we have that the cycles of \Sigma_X X admit a finite number of edges, and then \Sigma_X is bipartite. Then \Sigma_X is bipartite in any case. By Theorem 2.2 we have that N(S(\omega^+_f)) \cong \overline{\mathcal{S}_f} (\omega_f), then N(S(\omega^+_f)) is finite over B.

Conversely, assume that the condition of the statement does not hold. By Theorem 2.2, it suffices to show that \overline{\mathcal{S}_f} is not étale over B. We have two cases. In the first case, there exists a cycle of \Gamma_C with an odd number of edges, i.e. \Gamma_C is not bipartite. By Lemma 2.3, there exists a \xi \in \overline{\mathcal{S}_C}(\omega_C) with a representative supported on the curve X obtained by blowing-up C at the whole set of its nodes and \Sigma_X = \Gamma_C. By Theorem 2.2, \overline{\mathcal{S}_f} is not étale at \xi. In the second case, \Gamma_C is bipartite and there are two different cycles \Gamma_1 and \Gamma_2 such that \Gamma_1 \cap \Gamma_2 is an odd number of edges of \Gamma_C. Consider the graph \Sigma obtained by contracting \Gamma_C at the edges of \Gamma_1. In \Sigma, the cycle obtained by contracting \Gamma_2 at the edges of \Gamma_1 has an odd number of edges, then \Sigma is not bipartite. Let X_{\Gamma_1} be obtained by blowing-up C at the nodes whose corresponding edges are not contained in \Gamma_1. By Lemma 2.3 there is a \xi \in \overline{\mathcal{S}_C}(\omega_C) with a representative supported on X_{\Gamma_1} and \Sigma_{\Gamma_1} = \Sigma. Hence \Sigma_{\Gamma_1} is not bipartite and \overline{\mathcal{S}_f} is not étale at \xi. □

3. Néron models of S(\omega^+_f) within J_\mathcal{E}

3.1. The compactified Jacobian J_\mathcal{E}^\text{d}

Let f : C \to B be a family of curves. Then f admits enough sections through the B-smooth locus of C if there are sections \sigma_1, \ldots, \sigma_n : B \to C of f such that:

(i) \sigma_i factors through the B-smooth locus of C for i = 1, \ldots, n;
(ii) for every b \in B, every irreducible component of C_b contains \sigma_i(b) for some i = 1, \ldots, n.

Let f : C \to B be a family of curves, where B is a locally Noetherian scheme. Assume that f admits enough sections through the B-smooth locus of C. Let J_\mathcal{E} be the contravariant functor from the category of locally Noetherian B-schemes to sets, associating to T the set of equivalence classes of B-flat, coherent sheaves \mathcal{I} on C \times_B T/T whose fibers over B are degree d, simple, rank-1, torsion-free sheaves. Here, \mathcal{I}_1 and \mathcal{I}_2 are equivalent if there is an invertible sheaf M on T such that \mathcal{I}_1 \cong \mathcal{I}_2 \otimes p^* M, for p : C \times_B T \to T the projection. In [E], it is shown that J_\mathcal{E} is finely represented by a scheme J_d. Furthermore, one can consider distinguished subschemes of J_d as follows. Fix an integer d.
A polarization on \( C \) is a vector bundle \( \mathcal{E} \) on \( C \) of rank \( r > 0 \) and relative degree \( r(g - 1 - d) \). We will denote by \( \mathcal{E} \) the canonical polarization on \( C \):

\[
\mathcal{E} = \begin{cases} 
\omega_f^{\otimes (g-1-d)} \oplus \mathcal{O}_C^{\otimes (2g-3)}, & d \neq g - 1, \\
\mathcal{O}_C, & d = g - 1,
\end{cases} \tag{3.1}
\]

where \( \omega_f \) is the relative dualizing sheaf of the family \( f \).

Let \( I \) be a simple, torsion free, rank-1 sheaf of degree \( d \) on a curve \( C \). Then \( I \) is semistable with respect to a polarization \( \mathcal{E} \) of rank \( r \), if for every non-empty, proper subcurve \( Z \subseteq C \),

\[
\chi(I_Z) \geq \frac{-\deg E|_Z}{r}, \tag{3.2}
\]

where \( I_Z \) is the maximum torsion-free quotient of \( I|_Z \). Furthermore, \( I \) is stable if (3.2) is strict for every \( Z \). Let \( W \) (resp. \( p \)) be a component of \( C \) (resp. a non-singular point of \( C \)). Then \( I \) is \( W \)-quasistable (resp. \( p \)-quasistable) with respect to a polarization \( \mathcal{E} \) if \( I \) is semistable with respect to \( \mathcal{E} \) and (3.2) is strict for every \( Z \) such that \( W \subseteq Z \) (resp. for every \( Z \) such that \( p \in Z \)).

Fix a section \( \sigma : B \to C \) through the \( B \)-smooth locus of \( f \). A simple, torsion free, rank-1 sheaf \( \mathcal{I} \) on \( C \times_B T/T \) is semistable (resp. stable, resp. \( \sigma \)-quasistable) with respect to a polarization \( \mathcal{E} \), if \( \mathcal{I}|_{C_b} \) is semistable (resp. stable, resp. \( \sigma(b) \)-quasistable) with respect to \( \mathcal{E}|_{C_b} \), for every \( b \in B \). Consider the subspace \( J^p_f \) of \( J_d \) parametrizing sheaves \( \sigma \)-semistable with respect to the canonical polarization \( \mathcal{E} \) defined in (3.1). By [E, Theorem A], \( J^p_f \) is proper over \( B \). Notice that \( J^p_f \) finely represents the subfunctor \( J^p_f \) of \( J_d \) of the sheaves which are \( \sigma \)-quasistable with respect to \( \mathcal{E} \).

**Lemma 3.1.** Let \( C \) be a stable curve of genus \( g \geq 3 \) and let \( M \) be a line bundle on \( C \) of degree \( d \). Then:

(i) \( M \) is semistable (resp. stable) with respect to the canonical polarization if and only if for every non-empty, proper subcurve \( Z \subseteq C \):

\[
\left| \deg M|_Z - \frac{d}{2g - 2} \deg \omega_C|_Z \right| \leq \frac{\#(Z \cap Z^C)}{2} \tag{3.3}
\]

(resp. the strict inequality holds in (3.3)).

(ii) \( M \) is \( W \)-quasistable with respect to the canonical polarization \( \mathcal{E} \) if and only if (3.3) is satisfied and:

\[
\deg M|_Z - \frac{d}{2g - 2} \deg \omega_C|_Z > -\frac{\#(Z \cap Z^C)}{2},
\]

for every non-empty, proper subcurve \( Z \subseteq C \) such that \( W \subseteq Z \).

**Proof.** Since \( M \in \text{Pic}(C) \), \( M \) is semistable (resp. stable) with respect to the canonical polarization if and only if for each non-empty, proper subcurve \( Z \subseteq C \):

\[
\chi(M|_Z) \geq \frac{(-\deg E|_Z)}{\text{rank}(E)} \tag{3.4}
\]

(resp. (3.4) is strict for each \( Z \)). We have \( \chi(M|_Z) = \deg(M|_Z) + 1 - g_Z \) and \( \deg E|_Z = (g - 1 - d) \deg \omega_C|_Z \). Thus \( M \) is semistable (resp. stable) if and only if for each non-empty, proper subcurve \( Z \subseteq C \):
we have deg
\[\deg(M|_Z) \geq g_Z - 1 - \frac{(g - 1 - d) \deg \omega_C|_Z}{2g - 2}\]
\[= \frac{d(\deg \omega_C|_Z)}{2g - 2} + g_Z - 1 - \frac{\deg \omega_C|_Z}{2} = \frac{d(\deg \omega_C|_Z)}{2g - 2} - \frac{(Z \cap Z^c)}{2}\] (3.5)
(resp. if and only if (3.5) is strict for each \(Z\)). If \(M\) is semistable (resp. stable), we can apply the inequality (3.5) to \(Z^c\), and we get:
\[\deg(M|_Z) \leq \frac{d(\deg \omega_C|_Z)}{2g - 2} + \frac{(Z \cap Z^c)}{2}\] (3.6)
(resp. we get that (3.6) is strict). Then \(M\) is semistable (resp. stable) if and only (3.5) holds (resp. the strict inequality holds in (3.3)), for each non-empty, proper subcurve \(Z \subseteq C\). The item (ii) is similar.

3.2. Admissible twisters

Let \(f : C \to B\) be a smoothing of a semistable curves \(C\). Recall that an \(f\)-twister of \(C\), or simply a twister of \(C\), is a line bundle \(T\) on \(C\) such that \(T \cong \mathcal{O}_C(D)|_C\), where \(D\) is a Cartier divisor of \(C\) with support contained in \(C\).

**Definition 3.2.** Let \(C\) be a stable curve and \(T\) a twister of \(C\). We say that a line bundle \(M \in \text{Pic}\, C\) is a \(T\)-spin curve if \(M^\otimes 2 \cong \omega_C \otimes T\). If \(C_0\) is an irreducible component of \(C\), a twister \(T\) of \(C\) is admissible with respect to \(C_0\) if the set of \(T\)-spin curves is non-empty and every \(T\)-spin curve is \(C_0\)-quasistable with respect to the canonical polarization \(\mathcal{O}_C\).

Recall that \(T\)-spin curves have been used in [P] to study degenerations of theta characteristics to non-stable curves.

**Lemma 3.3.** Let \(f : C \to B\) be a general smoothing of a stable curve \(C\), where \(B\) is the spectrum of a discrete valuation ring, and let \(T\) be an \(f\)-twister of \(C\). Then the following properties are equivalent:

(i) \(T\) is admissible with respect to \(C_0\).

(ii) The set of \(T\)-spin curves is non-empty and there is an integer \(r_T \geq 0\) and a unique partition of \(C\) into non-empty subcurves \(Z_0, \ldots, Z_{r_T}\) such that:
   (a) for every connected component \(Z_h\) of \(Z\) we have \(Z_{h} \cap Z_{h-1} \neq \emptyset\), for every \(h = 1, \ldots, r_T\);
   (b) \(C_0 \subset Z_0\) and \(Z_0\) is connected;
   (c) \(T \cong \mathcal{O}_C(D)|_C\), where \(D = \sum_{i=1}^{r_T} i \cdot Z_i\) and:

\[T \otimes \mathcal{O}_{Z_h} \cong \begin{cases} 
\mathcal{O}_{Z_h}(\sum_{p \in Z_h \cap Z_{h+1}} p) & \text{if } h = 0, \\
\mathcal{O}_{Z_h}(\sum_{p \in Z_h \cap Z_{h+1}} (p - q)) & \text{if } h = 1, \ldots, r_T - 1, \\
\mathcal{O}_{Z_h}(\sum_{p \in Z_h \cap Z_{h+1}} p) & \text{if } h = r_T.
\end{cases}\] (3.7)

**Proof.** Assume that (ii) holds and let \(L\) be a \(T\)-spin curve. For every non-empty subcurve \(Z \subseteq C\) we have \(\deg_g T = \sum_{p \in Z \cap Z^c} m_p\), for some \(m_p \in \{-1, 0, 1\}\) and hence \(\deg_g T \leq \#(Z \cap Z^c)\). Thus \(L\) is semistable by Lemma 3.1. Let \(Z\) be a non-empty, proper subcurve \(Z \subseteq C\). Set \(T|_Z \cong \mathcal{O}_Z(\sum_{p \in Z \cap Z^c} m_p p)_p\), for \(m_p \in \mathbb{Z}\). Then \(\deg T|_Z \geq -\#(Z \cap Z^c)\). We need to prove that, if \(C_0 \subseteq Z\), then \(\deg T|_Z \geq -\#(Z \cap Z^c)\). Assume by contradiction that \(\deg T|_Z = -\#(Z \cap Z^c)\). Since \(C_0 \subseteq Z_0\), we have \(Z \cap Z_0 \neq \emptyset\). We claim that \(Z_0 \subseteq Z\). In fact, assume that \(Z_0 \not\subseteq Z\). Since \(Z_0\) is connected, there is an irreducible component \(Y_0\) such that \(Y_0 \subseteq Z_0 - Z\) and \(Y_0 \cap Z \neq \emptyset\). Consider a point \(p_0 \in Y_0 \cap Z\).
We have \( p_0 \in Z \cap Z^c \) and \( m_{p_0} = 0 \), then \( \deg T|_Z > -\#(Z \cap Z^c) \), which is a contradiction. Hence \( Z_0 \subseteq Z \). If \( Z_0 \not\subseteq Z_1 \), then either we have an irreducible component \( Y_1 \) such that \( Y_1 \subseteq Z_1 - Z \) and \( Y_1 \cap Z \neq \emptyset \), or there exists a connected component \( W_1 \) of \( Z_1 \) such that \( Z \subseteq W_1 \). In the first case, arguing as before for \( Y_0 \), we get a contradiction. In the second case, the condition (a) implies that \( \emptyset \neq W_1 \cap Z_0 \subseteq W_1 \cap Z \). Consider \( p_1 \in W_1 \cap Z \). Then by construction \( p_1 \in Z \cap Z^c \) and \( m_{p_1} = 1 \), and hence \( \deg T|_Z > -\#(Z \cap Z^c) \), which is a contradiction. Then \( Z_1 \subseteq Z \). Iterating, we get an integer \( h \in \{0, \ldots, r_T - 1\} \) such that \( \bigcup_{h=0}^{h} Z_h \) is a connected component of \( Z \). But \( m_p = 1 \) for each \( p \in Z_h \cap Z^c \), thus \( \deg T|_{Z_h} = (\#(Z_h \cap Z^c)) \) and hence \( \deg T|_Z > -\#(Z \cap Z^c) \), which is a contradiction.

Assume that (i) holds. There is a semistable \( T \)-spin curve and by Lemma 3.1, for each non-empty subcurve \( Z \subseteq C \), we have \( |\text{deg}_Z T| \leq \#(Z \cap Z^c) \). Let \( C_1 \cdots C_y \) be the irreducible components of \( C \). Let \( T \cong O_C(D)|_C \) for a divisor \( D = \sum_{1 \leq i \leq y} a_i C_i, a_i \in \mathbb{Z} \). Since \( O_C(nC)|_C \cong O_C \) for every \( n \in \mathbb{Z} \), we can assume without loss of generality that \( \min_{1 \leq i \leq y} a_i = 0 \). Set:

\[
Z_h := \bigcup_{a_i = h} C_i \quad \text{for every } h = 0, \ldots, r_T,
\]

where \( r_T := \max_{1 \leq i \leq y} a_i \). We prove that \( r_T \) and \( Z_0, \ldots, Z_{r_T} \) satisfy (ii). If \( Z_0 = C \) we are done. Otherwise:

\[
T \otimes O_{Z_0} \simeq O_{Z_0} \left( \sum_{p \in Z_0 \cap Z^c_0} m_p p \right).
\]

for some \( 0 < m_p \in \mathbb{Z} \). Now, \( |\text{deg}_{Z_0} T| \leq \#(Z_0 \cap Z^c_0) \), hence:

\[
\#(Z_0 \cap Z^c_0) \geq |\text{deg}_{Z_0} T| = \sum_{p \in Z_0 \cap Z^c_0} m_p \geq \#(Z_0 \cap Z^c_0).
\]

Thus \( m_p = 1 \), for every \( p \in Z_0 \cap Z^c_0 \). In particular, we have:

\[
T \otimes O_{Z_0^c} \simeq O_{Z_0^c} \left( - \sum_{p \in Z_0 \cap Z^c_0} p \right)
\]

and hence \( Z_1 \supseteq \{ C_i \cap (Z_0 \cap Z_0^c) \neq \emptyset, C_i \subseteq Z_0^c \} \). Notice that \( Z_1 \neq \emptyset \) and \( Z_0 \cap Z_h \) if and only if \( h = 1 \).

Assume that \( C \neq Z_0 \cup Z_1 \). For every \( p \in (Z_1 \cap Z_1^c) - Z_0 \), there exists \( 0 < m_p \in \mathbb{Z} \) such that:

\[
T \otimes O_{Z_1} \simeq O_{Z_1} \left( \sum_{p \in (Z_1 \cap Z_1^c) - Z_0} m_p p - \sum_{q \in Z_0 \cap Z_1} q \right).
\]

Arguing as before for the subcurve \( Z_0 \cup Z_1 \), we get \( m_p = 1 \), for \( p \in (Z_1 \cap Z_1^c) - Z_0 \). Hence:

\[
Z_2 \supseteq \{ C_i \cap Z_1 \neq \emptyset, C_i \subseteq (Z_0 \cup Z_1)^c \}.
\]

Then \( Z_2 \neq \emptyset \) and \( Z_1 \cap Z_h \neq \emptyset \) if and only if \( |h - 1| \leq 1 \). Iterating, we get that \( Z_h \neq \emptyset \) for \( h = 0, \ldots, r_T \) and \( Z_{h_1} \cap Z_{h_2} \neq \emptyset \) if and only if \( |h_1 - h_2| \leq 1 \). Notice that (c) and (3.7) follow by construction.

We show (b). If \( T \) is trivial, we have nothing to prove. By (3.7) we have \( \deg T|_{Z_0} = -\#(Z_0^c \cap Z_0) \).

If \( C_0 \subseteq Z_0 \), we get a contradiction, being \( C_0 \subseteq Z_0^c \) and \( T \) admissible. Then \( C_0 \subseteq Z_0 \). Assume that \( Z_0 \) is not connected and let \( Z_0^c \) be a connected component of \( Z_0 \) such that \( C_0 \not\subseteq Z_0^c \). Then \( C_0 \subseteq (Z_0^c)^c \) and \( \deg T|_{Z_0^c} = -\#((Z_0^c)^c \cap Z_0^c) \), again a contradiction.
We show (a). Assume that there exists a connected component $Z_h^E$ of $Z_h$ such that $Z_h^E \cap Z_{h-1} = \emptyset$, for some $h = 1, \ldots, r_T$. Then $C_0 \subseteq (Z_h^E)^c$ and $\deg T|_{(Z_h^E)^c} = -\#((Z_h^E)^c \cap Z_h^E)$, a contradiction.

Notice that the partition $Z_0, \ldots, Z_h$ of $C$ is the unique satisfying (ii). \hfill $\Box$

**Definition 3.4.** Keep the notations of Lemma 3.3. We call the partition $Z_0, \ldots, Z_{r_T}$ of $C$ the partition of $C$ induced by $T$. We denote by $\text{Ad}_f(C_0)$ the set of the admissible $f$-twisters $T$ of $C$ with respect to $C_0$. We say that a node $p$ of $C$ is $T$-twisted if $p \in Z_{i-1} \cap Z_i$, for some $i = 1, \ldots, r_T$.

**Remark 3.5.** Let $T$ and $\bar{T}$ be two admissible $f$-twisters of $C$ with respect to $C_0$ and let $Z_0, \ldots, Z_{r_T}$ and $\bar{Z}_0, \ldots, \bar{Z}_{r_T}$ be the partitions of $C$ induced respectively by $T$ and $\bar{T}$. Let $S$ (resp. $\bar{S}$) be the set of $T$-twisted nodes (resp. $\bar{T}$-twisted nodes). If $S = \bar{S}$ and $Z_0 = \bar{Z}_0$, then $T = \bar{T}$. Indeed, the connected components of the two partitions are the same, because they are obtained by taking the desingularization of $C$ at the nodes of $S$. Since $Z_0 = \bar{Z}_0$, we have $T = \bar{T}$ by condition (a) of Lemma 3.3.

**Definition 3.6.** Let $f : C \to B$ be a general smoothing of a nodal curve $C$. Let $X$ be obtained by blowing-up $C$ at a set $\Delta$ of nodes of $C$. Let $\pi : X \to C$ be the blow-up morphism. Consider the smoothing $f' : \tilde{X} \to B'$ of $X$, where $B'$ is the degree-2 covering of $B$, totally ramified over $0 \in B$, and $\tilde{X}$ is the blow-up of $C \times_B B$ at $\Delta$. Fix $M \in \text{Pic} C$ and $L \in \text{Pic} X$. We say that $L$ and $M$ are $f$-related if there exists an $f'$-twister $T$ of $X$ such that $L \simeq \pi^* M \otimes T$.

**Lemma 3.7.** Let $f : C \to B$ be a general smoothing of a stable curve $C$ of genus $g \geq 3$, where $B$ is the spectrum of a discrete valuation ring. Let $T$ be an admissible $f$-twister of $C$. Let $Z_0, \ldots, Z_{r_T}$ be the partition of $C$ induced by $T$. Assume that $M$ is a $T$-spin curve and that a representative $(X, L, \alpha)$ of some $\xi \in \Sigma_C(\omega_C)$ fullfills the following properties:

(i) $X$ is obtained by blowing-up $C$ at the $T$-twisted nodes;
(ii) for every $h = 0, \ldots, r_T$, the restriction of $L$ to $Z_h$ is:

$$L \otimes \mathcal{O}_{Z_h} \simeq \begin{cases} 
M \otimes \mathcal{O}_{Z_h}(-\sum_{p \in Z_h \cap Z_{h+1}} p) & \text{if } h = 0, \ldots, r_T - 1, \\
M \otimes \mathcal{O}_{Z_h} & \text{if } h = r_T.
\end{cases}$$

Then there exists a representative $(X, L_M, \alpha_M)$ of $\xi$ such that $M$ and $L_M$ are $f$-related.

**Proof.** Let $B' \to B$ be the degree 2 cover of $B$, totally ramified over $0 \in B$. Let $\tilde{X}$ be obtained by blowing-up $C \times_B B'$ at the set of the $T$-twisted nodes of $C$. Then the projection $f' : \tilde{X} \to B'$ is a smoothing of the fiber $X = (f')^{-1}(0)$ and $X$ is obtained by blowing-up $C$ at the set of $T$-twisted nodes. Let $\pi : X \to C$ be the induced blow-up morphism. Notice that $\tilde{X}$, the residual in $X$ of the union of the exceptional component of $X$, is the disjoint union of $Z_0, \ldots, Z_{r_T}$. Furthermore, $\tilde{X}$ is smooth at every node lying on an exceptional component of $X$ and has a singularity of type $A_1$ at the remaining nodes. Let $E_h$ be the set of exceptional components of $X$ intersecting $Z_{h-1}$ and $Z_h$, for each $h = 1, \ldots, r_T$. Consider the Cartier divisor of $\tilde{X}$:

$$D_M = -\sum_{h=1}^{r_T} \left( h \cdot Z_h + h \cdot \sum_{E \in E_h} E \right).$$

Pick the $f'$-twister $T_M = \mathcal{O}_{\tilde{X}}(D_M) \otimes \mathcal{O}_X$ of $X$. Set $L_M := \pi^* M \otimes T_M \in \text{Pic} X$. By construction, $L_M$ and $M$ are $f$-related. We are done if we show that we can construct a representative $(X, L_M, \alpha_M)$ of $\xi$.

First of all, we define $\alpha_M$ as follows. By construction, $L_M|_E \simeq \mathcal{O}_E(1)$ for every exceptional component $E$ and by condition (ii) we get $L_M|_{Z_h} = L|_{Z_h}$ for every $h = 0, \ldots, r_T$. By definition, $M^\otimes 2 \simeq \omega_C \otimes T$
and, by formula (3.7) of Lemma 3.3:

\[(\omega_C \otimes T)|_{Z_h} \simeq \begin{cases} 
\omega_{Z_h}(\sum_{p \in Z_h \cap Z_{h+1}} 2p) & \text{if } h = 0, \ldots, r_T - 1, \\
\omega_{Z_h} & \text{if } h = r_T. 
\end{cases} \]

Thus, \((L_M|_{Z_h})^\otimes \simeq \omega_{Z_h}\), for every \(h = 0, \ldots, r_T\). Let \(\alpha_M : (L_M)^\otimes \to \pi^*(\omega_X)\) be the homomorphism which agrees on each \(Z_h\) with:

\[\alpha_h : (L_M|_{Z_h})^\otimes \simeq \omega_{Z_h} \simeq \pi^*(\omega_C) \otimes \mathcal{O}_{Z_h} \left(- \sum_{p \in Z_h \cap Z_h'} p \right) \to \pi^*(\omega_C) \otimes \mathcal{O}_{Z_h}\]

and which is zero on the exceptional components of \(X\). Now, \(\bar{X} = \bigcup_{i=0}^{r_T} Z_i\), then \(L_M|_{\bar{X}} \simeq L|_{\bar{X}}\) and \((X, L_M, \alpha_M)\) is a representative of \(\xi\) by Remark 2.1. \(\square\)

Let \(f : C \to B\) be a general smoothing of a stable curve \(C\). For any \(f\)-twister \(T\) of \(C\) consider the moduli space:

\[\overline{S}_f(\omega_f \otimes T) \to B\]

whose fiber over \(b \in B\) parametrizes limit square roots of \(\omega_f \otimes T \otimes \mathcal{O}_{C_b}\). These moduli spaces are isomorphic away from the special fiber. Hence they have the same normalization, which, in the notations of Theorem 2.2, we write as:

\[\nu_T : \overline{S}_f^T(\omega_f) \to \overline{S}_f(\omega_f \otimes T).\]

Let \(S_f(\omega_f \otimes T)\) be the open subscheme of \(\overline{S}_f(\omega_f \otimes T)\) parametrizing limit square root supported on stable curves. Notice that \(S_f(\omega_f \otimes T)\) is étale over \(B\), by [CCC, 4.1]. In particular, there is an immersion \(S_f(\omega_f \otimes T) \hookrightarrow \overline{S}_f^T(\omega_f)\).

**Remark 3.8.** Let \(f : C \to B\) be a smoothing of a nodal curve \(C\) and let \(\mathcal{G} \in \text{Pic}(C)\). Let \(L \in \text{Pic}(C)\) be endowed with an isomorphism \(t_0 : L^\otimes \to \mathcal{G}|_C\). By [CCC, Remark 3.0.6.], up to shrinking \(B\) to a complex neighborhood of 0, there exists a line bundle \(\mathcal{L} \in \text{Pic}(C)\) extending \(L\) and an isomorphism \(\iota : \mathcal{L}^\otimes \to \mathcal{G}\) extending \(t_0\). Moreover, if \((\mathcal{L}', \iota')\) is another extension of \((L, t_0)\), then there is an isomorphism \(\chi : \mathcal{L} \to \mathcal{L}'\), restricting to the identity, and with \(\iota = \iota' \circ \chi^2\).

**Theorem 3.9.** Let \(f : C \to B\) be a general smoothing of a stable curve \(C\) of genus \(g \geq 3\) with \(\text{Aut}(C) = \{id\}\). Let \(C_0\) be an irreducible component of \(C\). Then:

\[N(S(\omega_f^*)) \simeq \bigcup_{T \in \text{Ad}_f(C_0)} S_f(\omega_f \otimes T) \sim,\]

where \(\sim\) denotes the gluing along the generic fiber of \(S_f(\omega_f^* \otimes T) \to B\).

Assume that \(f\) admits enough sections through the \(B\)-smooth locus of \(f\) and let \(\sigma\) be a section of \(f\) through the \(B\)-smooth locus of \(C\) such that \(\sigma(0) \in C_0\). Fix the canonical polarization \(\mathcal{E} = \mathcal{O}_C\) on \(C\). If \(J_E^\sigma\) is the open subscheme of \(J_E^\sigma\) parametrizing locally free sheaves, then there exists an immersion:

\[\psi_f : N(S(\omega_f^*)) \hookrightarrow (J_E^\sigma)^{\text{free}}.\]
Proof. We can assume without loss of generality that $B$ is the spectrum of a discrete valuation ring. Recall that $S_f(\omega_f \otimes T) \hookrightarrow \overline{S_f}^T(\omega_f)$, for every twister $T$. By Theorem 2.2, it suffices to show the equivalence of the following properties, for every $\xi \in \overline{S_f}^T(\omega_f)$:

(i) $\overline{S_f}^T(\omega_f) \to B$ is étale at $\xi' \in v^{-1}(\xi)$;

(ii) there exists a unique $T \in \text{Ad}_f(C_0)$ such that $\overline{S_f}^T(\omega_f)$ and $S_f(\omega_f \otimes T)$ are isomorphic, locally at $\xi' \in v^{-1}(\xi)$.

We show (i) $\Rightarrow$ (ii). Let $(X, L, \alpha)$ be any representative of $\xi$. We show that there exists a $T$ satisfying (ii). If $X = C$, then it suffices to set $T = \mathcal{O}_C \in \text{Ad}_f(C_0)$. Assume that $X \neq C$ and let $\pi : X \to C$ be the blow-up map and $\tilde{x}_0, \ldots, \tilde{x}_c$ be the connected components of $\tilde{x}$, corresponding to the vertices of $\Sigma_X$. Let $v_i$ be the vertex of $\Sigma_X$ corresponding to $\tilde{x}_i$. By Theorem 2.2, the graph $\Sigma_X$ is bipartite. Assume that $C_0 \subset \tilde{x}_0$ and set $A_0 := \{v_0\}$. For every $i \geq 1$, define inductively the set $A_i$ as the set of vertices $v$ of $\Sigma_X$ such that there exists an edge containing $v$ and a vertex of $A_{i-1}$. Let $A_0, \ldots, A_r$ be the non-empty sets defined in this way. Abusing notation, we can see $\tilde{x}_i$ as a subcurve of $C$. Consider the divisor $D = \sum_{i=0}^r \sum_{v_j \in A_i} i \cdot \tilde{x}_j$ of $C$. Set $T := \mathcal{O}_C(D)|_C$. Notice that $Z_0 = \tilde{x}_0$ and $T$ satisfies the conditions of Lemma 3.6 (ii), then $T \in \text{Ad}_f(C_0)$. Let $Z_0, \ldots, Z_{r_T}$ be the partition of $C$ induced by $T$. Being $\Sigma_X$ bipartite, each edge of $\Sigma_X$ has a vertex in $A_{i-1}$ and the other vertex in $A_i$, for some $i = 1, \ldots, r$. In particular, $X$ is obtained by blowing-up $C$ at the $T$-twisted nodes. Consider the subset $\mathcal{M}$ of $S_f(\omega_f \otimes T)$ defined as the set of $T$-spin curve $M \in \text{Pic}C$ satisfying for every $h = 0, \ldots, r_T$:

$$M \otimes O_{Z_h} \simeq \begin{cases} L \otimes O_{Z_h}(\sum_{p \in Z_h \cap Z_{h+1}} p) & \text{if } h = 0, \ldots, r_T - 1, \\ L \otimes O_{Z_h} & \text{if } h = r_T. \end{cases}$$

Notice that $S_f(\omega_f \otimes T) \to B$ is étale at each $M \in \mathcal{M}$. Then $S_f(\omega_f \otimes T)$ and $\overline{S_f}^T(\omega_f)$ are isomorphic, locally at each $M \in \mathcal{M}$. Our goal is to show that $\overline{S_f}^T(\omega_f)$ and $S_f(\omega_f \otimes T)$ are isomorphic, locally at every $\xi' \in v^{-1}(\xi)$. It is enough to show that $\mathcal{M} = v^{-1}(\xi)$. By Lemma 3.7, for every $M \in \mathcal{M}$, there is a representative $(X, L, M, \alpha_M)$ of $\xi$ such that $L_M$ and $M$ are $f$-related. Keep the notations of Definition 3.6. Since $L_M$ and $M$ are $f$-related, it follows from Remark 3.8 that $L_M$ and $\pi^*M$ are limits of the same family of theta characteristics on the family $f': X \to X$. Thus, $M \in v^{-1}(\xi)$ and hence $\mathcal{M} \subset v^{-1}(\xi)$. Now, the ramification index of $\psi : S_f(\omega_f) \to B$ at $\xi$ is $2^b_1(\Sigma_X)$, then $|v^{-1}(\xi)| \leq 2^b_1(\Sigma_X)$ and, by construction, $|\mathcal{M}| = 2^{b_1(\Sigma_X)}$. This implies $\mathcal{M} = v^{-1}(\xi)$.

We claim that $T$ is uniquely determined within $\text{Ad}_f(C_0)$, i.e. if $\overline{S_f}^T(\omega_f)$ and $S_f(\omega_f \otimes \tilde{T})$ are isomorphic, locally at $\xi' \in v^{-1}(\xi)$ for some $T \in \text{Ad}_f(C_0)$, then $\tilde{T} = T$. Indeed, in this case, there exists a $\tilde{T}$-spin curve $\tilde{M}$ such that $\tilde{M} \in v^{-1}(\xi)$. We claim that $X$ is obtained by blowing-up $C$ at the $\tilde{T}$-twisted nodes. Otherwise, let $\tilde{X}$ be obtained by blowing-up $C$ at the $\tilde{T}$-twisted nodes, with $\tilde{X} \neq X$. By Lemma 3.7, there exists $\tilde{\xi} \in \tilde{S}_C(\omega_C)$, with a representative $(\tilde{X}, \tilde{L}, \tilde{a})$, where $\tilde{L}$ is $f$-related to $M$. Arguing as before, we get $\tilde{M} \in v^{-1}(\tilde{\xi})$ and hence $\xi = \tilde{\xi}$, contradicting Remark 2.1. Now, let $\tilde{Z}_0, \ldots, \tilde{Z}_{r_T}$ be the partition of $C$ induced by $\tilde{T}$. Since $X$ is obtained by blowing-up $C$ at the $\tilde{T}$-twisted nodes, the set of $\tilde{T}$-twisted nodes coincides with the set of $\tilde{T}$-twisted nodes. Then $\tilde{Z}_0 \cap Z_0$ are $\tilde{T}$-twisted nodes. Being $\emptyset \neq C_0 \subset \tilde{Z}_0 \cap Z_0$ and $Z_0$ connected, we have $Z_0 \subset Z_0$. Arguing similarly we get $Z_0 \subset Z_0$ and hence $\tilde{Z}_0 = Z_0$. Then $\tilde{T} = T$, by Remark 3.5. The implication (ii) $\Rightarrow$ (i) is trivial.

Now we prove the second part. First of all, we show the existence of a morphism $S_f(\omega_f \otimes T) \to J^\omega_f$, for every $T \in \text{Ad}_f(C_0)$. In fact, let $S_f(\omega_f \otimes T)$ be the subfunctor of the functor $\overline{S_f}^T(\omega_f \otimes T)$ defined in (2.1), associating to a locally Noetherian $B$-scheme $T$ the set of isomorphism classes of limit square roots of $\omega_f$ supported on $C \times T$, for $f : C \times T \to T$ the first projection. By definition of admissible twister, we have a transformation of functors:

$$S_f(\omega_f \otimes T) \to J^\omega_f \xrightarrow{\sim} \text{Hom}(-, J^\omega_f).$$
Now, $S_f(\omega_f \otimes T)$ coarsely represents $S_f(\omega_f \otimes T)$. Therefore, we get a morphism $S_f(\omega_f \otimes T) \to J^\sigma_E$. By the first part of the theorem, we have:

$$N(S(\omega^*_f)) \cong \bigcup_{T \in \text{Ad}_f(C_0)} S_f(\omega_f \otimes T)$$

hence we get a morphism $\psi_f : N(S(\omega^*_f)) \to J^\sigma_E$, which is injective because the line bundles parametrized by the points of $N(S(\omega^*_f))$ over $0 \in B$ are non-isomorphic $T$-spin curves. Now, $\psi_f : N(S(\omega^*_f)) \to \text{Im} \psi_f$ is an injective $B$-morphism and $N(S(\omega^*_f))$ is $B$-smooth. Then $\text{Im} \psi_f$ is $B$-smooth and $\psi_f$ is an immersion. By construction $\text{Im} \psi_f \subset (J^\sigma_E)^{\text{free}}$.

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References