Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 3782–3804
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Path transferability of graphs

Ryuzo Torii

Department of Mathematics, School of Education, Waseda University, Nishi-Waseda 1-6-1, Shin'juku-ku, Tokyo 169-8050, Japan

Received 9 May 2007; received in revised form 30 June 2007; accepted 13 July 2007
Available online 30 August 2007

Abstract

Path transferability of a graph is a notion that arises from the movement of a path along the graph, the behavior of the path seems as a train on a railroad. In this paper, we introduce two graph notions, transferability and reversibility, and study their properties.

Keywords: Path transferability; Path reversibility

1. Introduction

The graphs discussed here are finite, simple and connected. We follow [3] for all basic notation and terminology. A *path* consists of distinct vertices $v_0, v_1, \ldots, v_n$ and edges $v_0v_1, v_1v_2, \ldots, v_{n-1}v_n$. When the direction of the path P needs to be emphasized, it is denoted $\langle P \rangle$(we distinguish between $\langle v_0v_1 \ldots v_{n-1}v_n \rangle$ and $\langle v_nv_{n-1} \ldots v_1v_0 \rangle$). If there is no danger of confusion, we use the same notation P instead of $\langle P \rangle$. We denote the reverse path of P by P^{-1}. The number of edges in a path P is called its *length* and is denoted by $\|P\|$. A path of length n is called an *n-path*. The set of all directed n-paths in a graph G is denoted by $\mathbb{P}_n(G)$. The last (resp. first) vertex of a path P in its direction is called the *head* (resp. *tail*) of P and is denoted by $h(P)$(resp. $t(P)$); for $P = \langle v_0v_1 \ldots v_{n-1}v_n \rangle$, we set $h(P) = v_n$ and $t(P) = v_0$. The set of all inner vertices of P, (i.e., the vertices that are neither the head nor the tail) is denoted by $\mathrm{Inn}(P)$ (Fig. 1).

This paper focuses on the movement of a path along a graph: let P be an n-path. We assume that $h(P)$ has a neighboring vertex v which does not belong to $\mathrm{Inn}(P)$. Then we have a new n-path P' by deleting the vertex $t(P)$ from P and adding v to P as its new head, (it seems that P takes one step and reaches the next position P'). We say that P *can transfer* (*or move*) *to* P' *by a step* and denote it by $P \xrightarrow{v} P'$ (or briefly $P \to P'$, or sometimes $P \to v$). If there is a sequence $P_0 \xrightarrow{x_1} P_1 \xrightarrow{x_2} \cdots \xrightarrow{x_m} P_m$, we shortly denote it by $P_0 \xrightarrow{x_1} \xrightarrow{x_2} \cdots \xrightarrow{x_m} P_m$. If there is a sequence of paths $P \to \cdots \to Q$ for two paths P and Q, then we say that P *can transfer* (*or move*) *to* Q, and denote it by $P \dashrightarrow Q$. The following is basic and important.

Proposition 1. *Let P, Q be distinct n-paths in G. If $P \dashrightarrow Q$, then $Q^{-1} \dashrightarrow P^{-1}$.*

In this paper, we regard a path as a "train" that moves along a graph. The main question we study is whether a path can transfer to everywhere on the graph by several steps.

E-mail address: torii@toki.waseda.jp.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.07.072

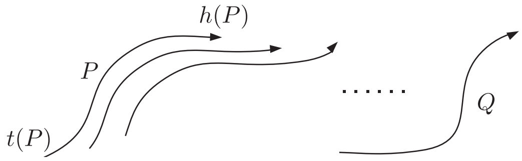

Fig. 1.

Definition 1. A graph G is called *n-path-transferable* or *n-transferable* if $\mathbb{P}_n(G) \neq \emptyset$ and if any two n-paths in G can transfer from one to another by finite number of steps, that is, $P \dashrightarrow Q$ holds for any pair of directed n-paths $P, Q \in \mathbb{P}_n(G)$.

Definition 2. An n-path P in a graph is called *reversible* if P can transfer to P^{-1}, and a graph G is called *n-path-reversible* or *n-reversible* if $\mathbb{P}_n(G) \neq \emptyset$ and if any directed n-path in G is reversible.

Remark 1. We define any graph to be 0-transferable and 0-reversible.

Remark 2. In a graph with minimum degree at least two, except cycle graphs, 1- or 2-paths can transfer from one to another. Conversely, we need at least two cycles to reverse a 1- or 2-path. Hence, the following statements are equivalent:

(1) A graph G is k-transferable ($k = 1, 2$).
(2) A graph G is k-reversible (k=1, 2).
(3) G is a graph with minimum degree $\geqslant 2$ which has at least two cycles.

Remark 3. Let $P = \langle v_0 v_1 \dots v_n \rangle$ be an n-path with $n \geqslant 1$. If P is reversible, then P can take at least one step, that is, there is a vertex v and a path Q that satisfies $P \overset{v}{\to} Q$. Furthermore, if P is reversible, then we have the following sequence of n-paths:

$$
\begin{aligned}
P \dashrightarrow & \langle \dots\dots\dots\dots v_n \rangle \\
\overset{v_{n-1}}{\to} & \langle \dots\dots\dots v_n v_{n-1} \rangle \\
\overset{v_{n-2}}{\to} & \langle \dots\dots v_n v_{n-1} v_{n-2} \rangle \\
& \vdots \\
\overset{v_0}{\to} & \langle v_n \dots\dots v_1 v_0 \rangle = P^{-1}.
\end{aligned}
$$

The longer a path is, the more difficult it is to move. The next theorem gives us an assurance for this fact. The proof will be shown later.

Theorem 2. *If a graph G is n-reversible, then G is $(n-1)$-reversible.*

The maximum number n for which G is n-reversible is called the *reversibility* of G and is denoted by $\tau(G)$. By definition, if G is n-transferable, then G is n-reversible. However, we will show that there is no difference between them.

Main theorem. *Let n be a non-negative integer, G a finite simple connected graph. The graph G is n-transferable if and only if G is n-reversible.*

The maximum number n for which G is n-transferable is called the *transferability* of G. By the main theorem, we use the same notation $\tau(G)$ for transferability and reversibility. The transferability of complete graphs can be obtained from this theorem.

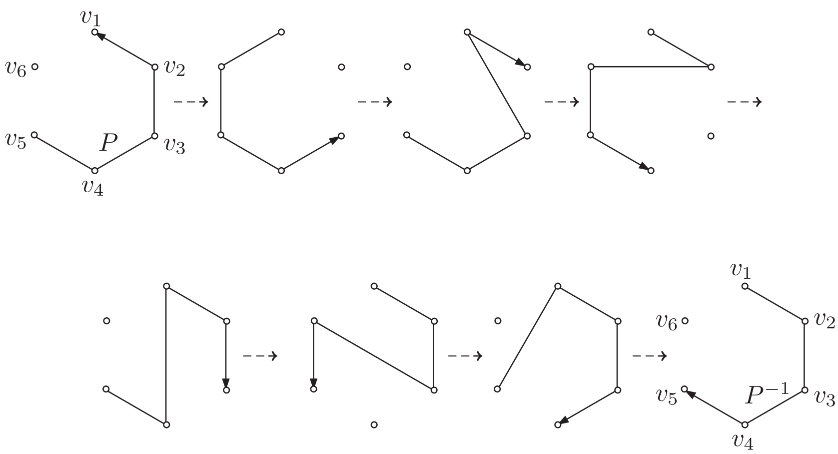

Fig. 2. A process of $P \dashrightarrow P^{-1}$ in K_6.

Theorem 3. *Let K_n be an n-vertex complete graph. For $n = 1, 2, 3$, $\tau(K_n) = 0$, and for $n \geqslant 4$, $\tau(K_n) = n - 2$.*

Proof. It is easy to see that the assertion holds for $n = 1, 2, 3$. We assume that $n \geqslant 4$. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n and $P = \langle v_{n-1} v_{n-2} \ldots v_2 v_1 \rangle$ an $(n-2)$-path in the graph. It is sufficient to show that $P \dashrightarrow P^{-1}$. We have the following sequence:

$$
\begin{aligned}
P &\stackrel{v_n}{\to} \stackrel{v_{n-1}}{\to} \cdots \stackrel{v_3}{\to} \langle v_1 v_n v_{n-1} v_{n-2} \ldots v_4 v_3 \rangle \\
&\stackrel{v_1}{\to} \stackrel{v_2}{\to} \langle v_{n-1} v_{n-2} \ldots v_4 v_3 v_1 v_2 \rangle \\
&\stackrel{v_n}{\to} \stackrel{v_{n-1}}{\to} \cdots \stackrel{v_4}{\to} \langle v_1 v_2 v_n v_{n-1} v_{n-2} \ldots v_4 \rangle \\
&\stackrel{v_1}{\to} \stackrel{v_2}{\to} \stackrel{v_3}{\to} \langle v_{n-1} v_{n-2} \ldots v_4 v_1 v_2 v_3 \rangle \\
&\vdots \\
&\stackrel{v_n}{\to} \stackrel{v_{n-1}}{\to} \langle v_1 v_2 \ldots v_{n-3} v_n v_{n-1} \rangle \\
&\stackrel{v_1}{\to} \stackrel{v_2}{\to} \cdots \stackrel{v_{n-3}}{\to} \stackrel{v_{n-2}}{\to} \langle v_{n-1} v_1 v_2 \ldots v_{n-3} v_{n-2} \rangle \\
&\stackrel{v_{n-1}}{\to} \langle v_1 v_2 v_3 \ldots v_{n-2} v_{n-1} \rangle = P^{-1},
\end{aligned}
$$

so the assertion holds (Fig. 2). □

Recently, the author found that the following papers are in some sense related to the study of this paper. However, we do not use the notions and results of these papers.

In [1], Broersma and Hoede introduce "path graph", which is a generalization of line graph. A digraph version of path graph is studied in [2]. By using their notation, we can redefine that G is n-transferable if and only if the digraph $\mathscr{D} = (\mathscr{V}, \mathscr{E})$ is non-empty, strongly connected, here $\mathscr{V} = \{P \mid P \in \mathbb{P}_n(G)\}$ and $\mathscr{E} = \{(P, Q) \mid P \to Q;\ P, Q \in \mathbb{P}_n(G)\}$.

On the other hand, in [4,5], Robertson et al. proposed the following for an approach to "linkless embedding conjecture" suggested by Sachs; let G be a graph, H, H' subgraphs of G, each is a hexad or a pentad (here hexad implies a subdivision of $K_{3,3}$, pentad a subdivision of K_5). If G is 4-connected, then there is a sequence $H = H_1, \ldots, H_n = H'$ such that each is a hexad or a pentad and that each differs only a "little" from the preceding one.

We study properties of n-reversible graphs in this paper, and almost all of this paper is devoted to the proof of the main theorem.

2. Proof of Theorem 2

Lemma 4. *Let G be an n-reversible graph, and P an n-path in G. Then P can arrive at any vertex in G, that is, for any vertex v there is an n-path Q such that $P \dashrightarrow Q$, $h(Q) = v$.*

Proof. Let $P = \langle v_0 v_1 \dots v_n \rangle$ be an n-path in G. Since P is reversible, there is a sequence of n-paths

$$\begin{aligned} P &\dashrightarrow \langle \dots\dots\dots\dots v_n \rangle \\ &\stackrel{v_{n-1}}{\to} \langle \dots\dots\dots v_n v_{n-1} \rangle \\ &\stackrel{v_{n-2}}{\to} \langle \dots\dots v_n v_{n-1} v_{n-2} \rangle \\ &\ \ \vdots \\ &\stackrel{v_0}{\to} \langle v_n \dots\dots v_1 v_0 \rangle = P^{-1}. \end{aligned}$$

Thus P can arrive at all vertices $v_0, v_1, \dots, v_n$ of $V(P)$ itself. Let U be the set of all vertices at which P can arrive. The set U is not empty.

We assume that $U \neq V(G)$, and let w be one of the vertices in $V(G) - U$. Since G is connected, there is a path between U and w. We denote it by $L = w w_1 \dots w_{l-1} w_l$, and choose the length of L as short as possible. By the choice of L, only w_l is the vertex of L that belongs to U, i.e., $w, w_1, \dots, w_{l-1} \notin U$, $w_l \in U$.

When P arrive at w_l, let the n-path be Q. We can move the path Q toward w_{l-1} by a step; otherwise the reason that Q cannot move to w_{l-1} is that w_{l-1} is one of the inner vertices of Q, however, P must have arrived at w_{l-1} before arriving at the position of Q, and this contradicts the definition of U. Therefore, Q can move to w_{l-1} and then P can arrive at w_{l-1}, this contradicts $w_{l-1} \notin U$. Thus $U = V(G)$ as desired. $\square$

Lemma 5. *Let G be an n-reversible graph and P an $(n-1)$-path in G. If P is contained in some n-path, then P is reversible.*

Proof. Let $Q = \langle v_0 v_1 \dots v_n \rangle$ be an n-path which includes an $(n-1)$-path $P = \langle v_1 \dots v_n \rangle$ as a subpath. The other case $t(P) = t(Q)$ is similar, so we omit it. Since Q is reversible, there is a sequence of n-paths;

$$\begin{aligned} Q \stackrel{w_1}{\to} \cdots \stackrel{w_k}{\to} \stackrel{v_n}{\to} Q_0 &= \langle \dots\dots\dots\dots v_n \rangle \\ \stackrel{v_{n-1}}{\to} Q_1 &= \langle \dots\dots\dots v_n v_{n-1} \rangle \\ &\ \ \vdots \\ \stackrel{v_0}{\to} Q_n &= \langle v_n \dots\dots v_1 v_0 \rangle = Q^{-1}. \end{aligned}$$

For this sequence, P can also take the same steps keeping with Q's steps (it seems that a "train" Q conveys its "freight" P):

$$\begin{aligned} P \stackrel{w_1}{\to} \cdots \stackrel{w_k}{\to} \stackrel{v_n}{\to} &\langle \dots\dots\dots\dots v_n \rangle \subseteq Q_0 \\ \stackrel{v_{n-1}}{\to} &\langle \dots\dots\dots v_n v_{n-1} \rangle \subseteq Q_1 \\ &\ \ \vdots \\ \stackrel{v_1}{\to} &\langle v_n \dots\dots v_2 v_1 \rangle = P^{-1} \subseteq Q^{-1}. \end{aligned}$$

Thus P is reversible. $\square$

Proof of Theorem 2. Let G be an n-reversible graph and $P = \langle v_1 v_2 \dots v_n \rangle$ an $(n-1)$-path in G. We will show that P is contained in some n-path, and then P is reversible by Lemma 5. By Lemma 4, there is an n-path that arrives at v_1, and we denote it by $Q_1 = \langle \dots\dots\dots w v_1 \rangle$.

Case 1: We assume that w is not in $V(P)$.

In this case, the n-path $P^+ = \langle w v_1 v_2 \dots v_n \rangle$ has P as its subpath.

Case 2: We assume that $w = v_2$.

In this case, the path Q_1 has the form $Q_1 = \langle \ldots\ldots v_2 v_1 \rangle$. Since Q_1 is reversible, there is a path Q_2 such that

$$Q_1 \dashrightarrow \xrightarrow{v_1} \xrightarrow{v_2} Q_2 = \langle \ldots\ldots\ldots v_1 v_2 \rangle.$$

We move the path Q_2 along the path P as close to v_n as possible, and let the resulting path be Q_k, that is,

$$\begin{aligned} Q_1 \dashrightarrow Q_2 &= \langle \ldots\ldots\ldots\ldots v_1 v_2 \rangle \\ &\vdots \\ \xrightarrow{v_k} Q_k &= \langle \ldots\ldots v_1 v_2 \ldots v_k \rangle. \end{aligned}$$

If $k = n$, then $P \subset Q_k$ as desired. We thus assume that $k < n$. The reason that Q_k cannot take a step to v_{k+1} is that v_{k+1} is an inner vertex of Q_k. Hence, Q_k has the form

$$Q_k = \langle \ldots\ldots u_2 u_1 v_{k+1} w_1 \ldots w_l v_1 v_2 \ldots v_k \rangle.$$

Here, we consider the following n-path instead of Q_k,

$$Q'_k = \langle \ldots\ldots u_2 u_1 v_{k+1} v_k \ldots v_2 v_1 w_l \ldots w_1 \rangle.$$

Since Q'_k is reversible, there is a sequence of n-paths

$$\begin{aligned} Q'_k &\dashrightarrow \langle \ldots\ldots w_1 \ldots w_l v_1 \rangle \\ &\xrightarrow{v_2} \langle \ldots\ldots\ldots v_1 v_2 \rangle \\ &\vdots \\ &\xrightarrow{v_k} \langle \ldots v_1 v_2 \ldots v_k \rangle \\ &\xrightarrow{v_{k+1}} \langle \ldots v_1 v_2 \ldots v_k v_{k+1} \rangle =: Q_{k+1}. \end{aligned}$$

The last n-path Q_{k+1} contains more edges of P than Q_k. Repeating the argument above, we finally find an n-path that fully contains P.

Case 3: We assume that w is a vertex of $V(P) - v_2$.

In this case, we can find an n-path that fully contains P in the same way as in Case 2, and P is reversible. $\square$

3. Proof of main theorem

Let G be an n-reversible graph and P, Q two n-paths in G that satisfies $P \to Q$. We set $t(P) = u$ and $h(Q) = v$. As long as we treat n-reversible graphs, $Q \dashrightarrow P$ also holds. We regard these steps as a *back step* of Q, and denote it by $Q \overset{u}{\leftarrow} P$ (or briefly $Q \leftarrow P$). We notice that the notations $P \to Q$ and $Q \leftarrow P$ are not the same in meaning. In fact, these two imply $P \overset{v}{\to} Q$ and $Q \overset{u}{\leftarrow} P$, respectively.

Let $R = \langle x v_1 \ldots v_n \rangle$, $S = \langle y v_1 \ldots v_n \rangle$ be two n-paths in G. In Proposition 6, we will show that $R \dashrightarrow S$. Such a move will be called a *tail flip* of R, and will be denoted $R \overset{y}{>} S$ (or briefly $R > S$). *Head flip* is similarly introduced and is denoted by $<$.

Proposition 6. *Let G be an n-reversible graph and $P = \langle x v_1 \ldots v_n \rangle$, $Q = \langle y v_1 \ldots v_n \rangle$ two n-paths in G. Then $P \dashrightarrow Q$.*

Proof. Let P and Q be as above. Since P is reversible, there is a vertex $z \notin \{v_1, v_2, \ldots, v_n\}$ (it may be x or y) to which P can transfer by a step. Then $P \overset{z}{\to} \overset{y}{\leftarrow} Q$, and therefore $P \dashrightarrow Q$. $\square$

Let $P = \langle v_0 v_1 v_2 \ldots v_{n-2} v_{n-1} v_n \rangle$ and $Q = \langle v_n v_1 v_2 \ldots v_{n-2} v_{n-1} v_0 \rangle$ be two n-paths in a graph G. To prove the main theorem, we will show that $P \dashrightarrow Q$ if G is n-reversible. We call this the *cross flip* of P and denote it by $P \propto Q$. To prove this, we will prepare several lemmas and propositions.

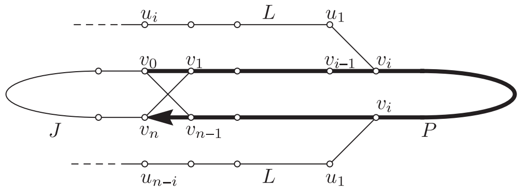

Fig. 3.

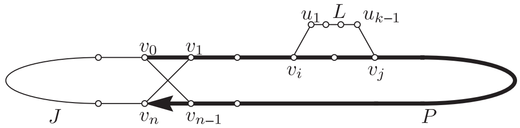

Fig. 4.

Lemma 7. *Let G be an n-reversible graph and $P = \langle v_0 v_1 v_2 \dots v_{n-2} v_{n-1} v_n \rangle$, $Q = \langle v_n v_1 v_2 \dots v_{n-2} v_{n-1} v_0 \rangle$ two n-paths in G. We assume that there is a path L such that $t(L) = v_i$, $V(L) \cap V(P) = v_i$ for some i, $1 \leqslant i \leqslant n-1$. We further assume that there is another path J such that $t(J) = v_0$, $h(J) = v_n$, $V(J) \cap (V(P) \cup \mathrm{Inn}(L)) = \{v_0, v_n\}$. If $\|L\| \geqslant i$ or $\|L\| \geqslant n - i$, then $P \dashrightarrow Q$ (Fig. 3).*

Proof. Let $k = \|L\|$. We assume that $k \geqslant i$ (the other case $k \geqslant n - i$ is similar). Let $L = v_i u_1 \dots u_i \dots u_k$ and $J = v_0 w_1 \dots w_l v_n$. Then we have the following sequence of n-paths:

$$
\begin{aligned}
P &\overset{w_l}{\rightarrow} \overset{w_{l-1}}{\rightarrow} \cdots \overset{w_1}{\rightarrow} \overset{v_0}{\rightarrow} \overset{v_1}{\rightarrow} \cdots \overset{v_{i-l-1}}{\rightarrow} \langle v_i v_{i+1} \dots v_{n-1} v_n w_l \dots w_1 v_0 v_1 \dots v_{i-l-1} \rangle \\
&\overset{u_1}{\leftarrow} \cdots \overset{u_i}{\leftarrow} \langle u_i \dots u_1 v_i v_{i+1} \dots v_{n-1} v_n \rangle \\
&\overset{v_0}{\lessdot} \langle u_i \dots u_1 v_i v_{i+1} \dots v_{n-1} v_0 \rangle \\
&\overset{w_1}{\rightarrow} \cdots \overset{w_l}{\rightarrow} \overset{v_n}{\rightarrow} \overset{v_1}{\rightarrow} \overset{v_2}{\rightarrow} \cdots \overset{v_{n-1}}{\rightarrow} \overset{v_0}{\rightarrow} Q,
\end{aligned}
$$

and therefore $P \dashrightarrow Q$. $\square$

Lemma 8. *Let G be an n-reversible graph and P, Q as in Lemma 7. We assume that there is a path L such that $t(L) = v_i$, $h(L) = v_j$, $V(L) \cap V(P) = \{v_i, v_j\}$ for some i, j, $0 \leqslant i < j \leqslant n$. We further assume that there is another path J such that $t(J) = v_0$, $h(J) = v_n$, $V(J) \cap (V(P) \cup \mathrm{Inn}(L)) = \{v_0, v_n\}$. If $\|L\| > j - i$, then $P \dashrightarrow Q$ (Fig. 4).*

Proof. Let $k = \|L\|$. We assume that $k > j - i$. Let $L = v_i u_1 \dots u_{k-1} v_j$, and $J = v_0 w_1 \dots w_l v_n$. Then

$$
\begin{aligned}
P &\overset{w_l}{\rightarrow} \cdots \overset{w_1}{\rightarrow} \overset{v_0}{\rightarrow} \overset{v_1}{\rightarrow} \cdots \overset{v_i}{\rightarrow} \overset{u_1}{\rightarrow} \cdots \overset{u_{k-1}}{\rightarrow} \overset{v_j}{\rightarrow} \overset{v_{j+1}}{\rightarrow} \cdots \overset{v_{n-1}}{\rightarrow} \overset{v_0}{\rightarrow} \\
&\overset{w_1}{\rightarrow} \cdots \overset{w_l}{\rightarrow} \overset{v_n}{\rightarrow} \overset{v_1}{\rightarrow} \cdots \overset{v_i}{\rightarrow} \overset{v_{i+1}}{\rightarrow} \cdots \overset{v_j}{\rightarrow} \cdots \overset{v_{n-1}}{\rightarrow} \overset{v_0}{\rightarrow} Q,
\end{aligned}
$$

and therefore $P \dashrightarrow Q$. $\square$

Theorem 9. *Let G be an n-reversible graph and P, Q as in Lemma 7. If $v_0 v_n \in E(G)$, then $P \dashrightarrow Q$.*

Proof. We set $V = V(P)$, $W = V(G) - V(P)$. Then we have $W \neq \emptyset$; otherwise P has only one orbit $P \overset{v_0}{\rightarrow} \overset{v_1}{\rightarrow} \cdots \overset{v_n}{\rightarrow} P \overset{v_0}{\rightarrow} \overset{v_1}{\rightarrow} \cdots$, and therefore, cannot be reversible.

We first show that any vertex in W is connected to v_0 by a path whose vertices except v_0 are in W: let u be a vertex in W. Since G is connected, there is a path between u and V. Extending this path as long as possible in W, we set the path $L = v_i u_1 u_2 \dots u_k$. If $k \geqslant i$, then the assertion holds by Lemma 7, so we assume that $k < i$. We consider an n-path

$$R_1 = \langle v_{i-k} \dots v_2 v_1 v_n v_{n-1} \dots v_i u_1 u_2 \dots u_k \rangle.$$

Since R_1 is reversible and L cannot be extended in W, the head vertex u_k is adjacent to one of the vertices $v_{i-k}, v_{i-k+1}, \dots, v_{i-1}, v_0$. If u_k is adjacent to one of $v_{i-k+1}, \dots, v_{i-2}, v_{i-1}$, then the assertion holds by Lemma 8, so we assume that u_k is adjacent to v_0. Since u is an arbitrary vertex in W and it lies on L, we conclude that any vertex in W is connected to v_0 by a path.

Let $L = v_0 w_1 w_2 \dots w_l$ and $W_0 = \{w_1, \dots, w_l\} \subseteq W$. We choose the length of L as long as possible. It is easy to see that $P \dashrightarrow Q$ if $l \geqslant n-1$, so we assume that $1 \leqslant l < n-1$.

Case 1: $l \geqslant 2$. We consider an n-path

$$R_2 = \langle v_l v_{l+1} \dots v_{n-1} v_0 w_1 \dots w_l \rangle.$$

If w_l is adjacent to one of $v_1, v_2, \dots, v_{l-1}, v_l$, then the assertion holds by Lemma 7, so we assume that w_l is adjacent to none of $v_1, v_2, \dots, v_l$. Since R_2 is reversible and L is a longest path, w_l is adjacent to v_n. We set

$$R_3 = \langle v_{l+1} v_{l+2} \dots v_{n-1} v_n v_0 w_1 \dots w_l \rangle.$$

The vertex w_l is adjacent to v_{l+1} since w_l is adjacent to none of $v_1, v_2, \dots, v_l$. We further consider the next step of the following n-path:

$$R_4 = \langle v_{n-1} v_{n-2} \dots v_{l+1} w_l v_n v_0 w_1 \dots w_{l-1} \rangle.$$

If w_{l-1} is adjacent to one of $v_1, v_2, \dots, v_{l-1}, v_{n-1}$, then the assertion holds by Lemma 7. And if w_{l-1} is adjacent to v_l, then the assertion holds by Lemma 8. We thus assume that w_{l-1} is adjacent to some vertex in $W - W_0$, say w. We set

$$R_5 = \langle v_{n-2} \dots v_{l+1} w_l v_n v_0 w_1 \dots w_{l-1} w \rangle.$$

If w is adjacent to one of $v_1, v_2, \dots, v_l, v_{n-2}, v_{n-1}$, then the assertion holds by Lemma 7. Otherwise, w is adjacent to some vertex in $W - W_0$ since R_5 is reversible, however, this contradicts the maximality of the length of L.

Case 2: $l = 1$. All vertices in W are adjacent to v_0 because any vertex in W is connected to v_0 by a path without crossing V. Let $W = \{w_1, \dots, w_m\}$. We notice that the vertices in W are pairwise non-adjacent. We will define n-paths $S_1, S_2, \dots$ inductively.

$$S_1 = \langle v_2 v_3 \dots v_n v_0 w_1 \rangle,$$
$$S_i = \langle v_{2i} v_{2i+1} \dots v_n v_0 v_1 \dots v_{2i-2} w_1 \rangle.$$

If w_1 is adjacent to v_{2i-1}, then the assertion holds by Lemma 7 or 8. We thus assume that w_1 is not adjacent to v_{2i-1} and that w_1 is adjacent to v_{2i}. And then we set the next path

$$S_{i+1} = \langle v_{2i+2} v_{2i+3} \dots v_n v_0 v_1 \dots v_{2i} w_1 \rangle.$$

While we set the paths $S_1, S_2, \dots$, we also obtain that $w_1 v_2, w_1 v_4, \dots \in E(G)$. The sequence must end by $w_1 v_n$; otherwise, if it ends by $w_1 v_{n-1}$, then the assertion holds by Lemma 7. Particularly n is even. We deduce a similar fact for the other vertices of W:

$$w_j v_0, w_j v_2, \dots, w_j v_{n-2}, w_j v_n \in E(G),$$
$$w_j v_1, w_j v_3, \dots, w_j v_{n-3}, w_j v_{n-1} \notin E(G),$$

for each j, $1 \leqslant j \leqslant m$. Let $U_1 = \{v_0, v_2, \dots, v_{n-2}, v_n\}$, $U_2 = \{v_1, v_3, \dots, v_{n-3}, v_{n-1}\}$. Each vertex in U_1 is adjacent to each vertex in W, and there are no edges between U_2 and W. To decide the relation between U_1 and U_2, we set

$$P_{2t-1} := \langle v_0 v_1 \dots v_{2t-2} w_1 v_{2t} \dots v_{n-1} v_n \rangle,$$
$$Q_{2t-1} := \langle v_n v_1 \dots v_{2t-2} w_1 v_{2t} \dots v_{n-1} v_0 \rangle,$$

and

$$V_{2t-1} := \{v_0, v_1, \dots, v_{2t-2}\} \cup \{w_1\} \cup \{v_{2t}, \dots, v_{n-1}, v_n\},$$
$$W_{2t-1} := \{v_{2t-1}\} \cup \{w_2, \dots, w_m\},$$

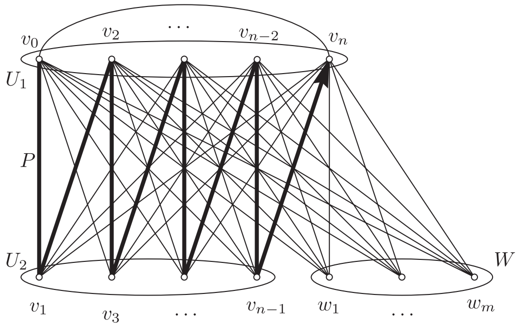

Fig. 5.

for each t, $1 \leqslant t \leqslant n/2$. Only $(2t-1)$th vertices of P_{2t-1} and Q_{2t-1} differ from the vertices of P and Q, respectively. For these paths, we can find that

$$P \overset{v_0}{\to} \overset{v_1}{\to} \cdots \overset{v_{2t-2}}{\to} \overset{w_1}{\to} \overset{v_{2t}}{\to} \cdots \overset{v_{n-1}}{\to} \overset{v_n}{\to} P_{2t-1},$$
$$Q_{2t-1} \overset{v_n}{\to} \overset{v_1}{\to} \cdots \overset{v_{2t}}{\to} \overset{v_{2t+1}}{\to} \overset{v_{2t+2}}{\to} \cdots \overset{v_{n-1}}{\to} \overset{v_n}{\to} Q,$$

thus $P_{2t-1} \dashrightarrow Q_{2t-1}$ implies $P \dashrightarrow Q$. We apply the same method to the vertex v_{2t-1}, and deduce that v_{2t-1} is adjacent to the vertices in V_{2t-1} alternatively, that is,

$$v_{2t-1}v_0, v_{2t-1}v_2, \dots, v_{2t-1}v_{n-2}, v_{2t-1}v_n \in E(G),$$
$$v_{2t-1}v_1, v_{2t-1}v_3, \dots, v_{2t-1}v_{n-3}, v_{2t-1}v_{n-1} \notin E(G).$$

The index t varies for $1 \leqslant t \leqslant n/2$. We therefore deduce that the vertices in U_1 and the vertices in U_2 are mutually adjacent and that the vertices in U_2 are pairwise non-adjacent.

We assume that there is an edge in U_1 other than v_0v_n. Then we can find an n-path whose head and tail are in W and which passes through all vertices of U_1. However, this path cannot take even one step, and this fact contradicts the reversibility of G. We therefore deduce that U_1 has only one edge v_0v_n, and then G is a complete bipartite graph $K_{n/2+1,n/2+m}$ with an additional edge v_0v_n, whose partition sets are U_1 and $U_2 \cup W$ (see Fig. 5). If $n \geqslant 4$, this graph cannot be reversible: in fact, no matter how P takes any steps, the order of v_0, v_2, v_4 cannot be changed, so P is not reversible. If $n = 2$, it is easy to see that the graph is 2-transferable. As a consequence, we complete the proof. □

Let $P = \langle v_0v_1v_2 \dots v_n \rangle$, $Q = \langle v_1v_0v_2 \dots v_n \rangle$ be two n-paths in an n-reversible graph. In Proposition 10, we will show that $P \dashrightarrow Q$. Such a move will be called the *Δ-tail flip* of P, and will be denoted $P \rhd Q$. *Δ-head flip* is similarly introduced and is denoted by $\lhd$.

Proposition 10. *Let G be an n-reversible graph and P, Q as above. Then $P \dashrightarrow Q$.*

Proof. If v_n is adjacent to some vertex $z \notin V(P)$, then $P \overset{z}{\to} \overset{v_0}{>} \overset{v_1}{\leftarrow} Q$. We thus assume that v_n is adjacent to none of the vertices out of $V(P)$. Since P and Q are reversible, v_n is adjacent to v_0 and v_1. We can find that

$$\begin{aligned} P &\overset{v_0}{\to} \langle v_1v_2 \dots v_nv_0 \rangle \\ &\dashrightarrow \langle v_0v_2 \dots v_nv_1 \rangle \quad \text{(by Theorem 9)} \\ &\overset{v_1}{\leftarrow} \langle v_1v_0v_2 \dots v_n \rangle = Q, \end{aligned}$$

and therefore $P \dashrightarrow Q$. □

Lemma 11. *Let G be an n-reversible graph and P, Q and L as in Lemma 7. If $v_0v_{n-2}, v_nv_{n-2} \in E(G)$ and $\|L\| \geqslant i$, or if $v_0v_2, v_nv_2 \in E(G)$ and $\|L\| \geqslant n-i$, then $P \dashrightarrow Q$ (Fig. 6).*

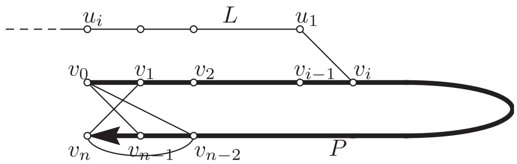

Fig. 6.

Proof. Let $k = \|L\|$. We assume that $v_0v_{n-2}, v_nv_{n-2} \in E(G)$ and $k \geqslant i$ (the other case is similar). We set $L = v_iu_1 \dots u_i \dots u_k$. Then we have the following sequence of n-paths:

$$\begin{aligned}
&P \lhd \langle v_0v_1 \dots v_{n-3}v_{n-2}v_nv_{n-1}\rangle \\
&\quad \overset{v_0}{\to} \overset{v_1}{\to} \cdots \overset{v_{i-1}}{\to} \langle v_iv_{i+1} \dots v_{n-3}v_{n-2}v_nv_{n-1}v_0v_1 \dots v_{i-2}v_{i-1}\rangle \\
&\quad \overset{u_1}{\leftarrow} \overset{u_2}{\leftarrow} \cdots \overset{u_i}{\leftarrow} \langle u_i \dots u_2u_1v_iv_{i+1} \dots v_{n-3}v_{n-2}v_nv_{n-1}\rangle \\
&\quad \lhd \overset{v_0}{\lessdot} \lhd \langle u_i \dots u_2u_1v_iv_{i+1} \dots v_{n-3}v_{n-2}v_0v_{n-1}\rangle \\
&\quad \overset{v_n}{\to} \overset{v_1}{\to} \overset{v_2}{\to} \cdots \overset{v_{i-1}}{\to} \overset{v_i}{\to} \cdots \overset{v_{n-2}}{\to} \overset{v_0}{\to} \overset{v_{n-1}}{\to} \langle v_nv_1v_2 \dots v_{n-3}v_{n-2}v_0v_{n-1}\rangle \\
&\quad \lhd Q,
\end{aligned}$$

and therefore $P \dashrightarrow Q$. $\square$

We can also deduce the following as in Lemma 8. The proof is similar.

Lemma 12. *Let G be an n-reversible graph and P, Q and L as in Lemma 8. If $\|L\| \geqslant j-i$, and if $v_0v_{n-2}, v_nv_{n-2} \in E(G)$ or $v_0v_2, v_nv_2 \in E(G)$, then $P \dashrightarrow Q$.*

Let $P = \langle v_0v_1v_2 \dots v_n\rangle$ be a path in a graph and $\hat{P} = \langle w_l \dots w_1v_0v_1v_2 \dots v_n\rangle$ a longest path with $h(P) = h(\hat{P})$, $P \subseteq \hat{P}$. Then the subpath $w_l \dots w_1v_0$ is called a *rut* of P, and the length l is denoted by $r(P)$.

Theorem 13. *Let G be an n-reversible graph and P, Q as in Lemma 7. If $r(P) \geqslant 2$ or $r(Q) \geqslant 2$, then $P \dashrightarrow Q$.*

Proof. Let V, W be as in the proof of Theorem 9. The case $v_0v_n \in E(G)$ is already treated in Theorem 9, so we assume that $v_0v_n \notin E(G)$. Without loss of generality, we may assume that $r(P) \geqslant r(Q)$, $r(P) \geqslant 2$. We set $l = r(P)$ and denote one of the ruts of P by $L = w_l \dots w_1v_0$. Let $W_0 = \{w_1, w_2, \dots, w_l\} \subseteq W$. By the choice of L, w_l is not adjacent to any vertices in $W - W_0$. If $l \geqslant n-1$, then it is easy to see that $P \dashrightarrow Q$, we thus assume that $l < n-1$. We further assume that the cross flip of a path is allowed if a rut of the path has length $> l$.

Here, we consider the two cases whether $w_lv_n \in E(G)$ or not.

Case 1: $w_lv_n \in E(G)$. In this case, we further consider several cases for the neighbors of v_2 and v_{n-2}.

Case 1.1: We assume that v_2 has a neighbor, say w, in $W - W_0$. We set

$$\begin{aligned}
R_1 &= \langle v_1v_nv_{n-1}v_{n-2} \dots v_3v_2w\rangle, \\
R_1' &= \langle v_1v_0v_{n-1}v_{n-2} \dots v_3v_2w\rangle.
\end{aligned}$$

If w has a neighbor in W, or if w is adjacent to v_1, then the assertion holds by Lemma 7. Hence we assume that w has no neighbors in W, and then w must be adjacent to v_0 and v_n since R_1 and R_1' are reversible. We set

$$R_2 = \langle v_{l+2} \dots v_{n-1}v_nwv_0w_1 \dots w_l\rangle.$$

If w_l is adjacent to one of $v_1, v_2, \dots, v_{l-1}, v_l$, then the assertion holds by Lemma 7, so we assume that w_l is adjacent to none of $v_1, v_2, \dots, v_l$. Since R_2 is reversible and L is a longest path, w_l is adjacent to v_{l+1} or v_{l+2}. We first assume that $w_lv_{l+1} \in E(G)$. Then we can consider the following n-path:

$$R_3 = \langle v_{n-2}v_{n-3} \dots v_{l+1}w_lv_nwv_0w_1 \dots w_{l-1}\rangle.$$

If w_{l-1} is adjacent to one of $v_1, v_2, \ldots, v_l, v_{n-2}, v_{n-1}$, then the assertion holds by Lemmas 7 or 8, we thus assume that w_{l-1} is adjacent to some vertex in $W - W_0$, say w'. We set

$$R_4 = \langle v_{n-3} \ldots v_{l+1} w_l v_n w v_0 w_1 \ldots w_{l-1} w' \rangle.$$

If w' is adjacent to one of $v_1, v_2, \ldots, v_l, v_{n-3}, v_{n-2}, v_{n-1}$, then the assertion holds by Lemma 7 or 8. Otherwise, w' is adjacent to another vertex in $W - W_0$ since R_4 is reversible, however, this contradicts the maximality of the length of L.

The assertion also holds for the other case $w_l v_{l+2} \in E(G)$ in a similar way.

As a consequence, we deduce that v_2 has no neighbors in $W - W_0$. We similarly deduce that v_{n-2} has no neighbors in $W - W_0$.

Case 1.2: We assume that $l \geqslant 3$ and v_2 is adjacent to one of the vertices in $W_0 - \{w_1, w_l\}$. Let w_i, $1 < i < l$, be such a vertex. Then

$$\begin{aligned} P &\overset{w_l}{\rightarrow} \overset{w_i}{\gtrdot} \overset{w_{i+1}}{\leftarrow} \langle w_{i+1} w_i v_2 \ldots v_{n-2} v_{n-1} v_n \rangle \\ &\overset{v_0}{\lessdot} \overset{w_1}{\rightarrow} \langle w_i v_2 \ldots v_{n-2} v_{n-1} v_0 w_1 \rangle \\ &\overset{v_1}{\gtrdot} \overset{v_n}{\leftarrow} Q. \end{aligned}$$

Therefore v_2, as well as v_{n-2}, is adjacent to none of the vertices in $W_0 - \{w_1, w_l\}$.

Case 1.3: We assume that v_2 is adjacent both to w_1 and to w_l. Then

$$\begin{aligned} P &\overset{w_l}{\rightarrow} \overset{w_1}{\gtrdot} \overset{w_2}{\leftarrow} \langle w_2 w_1 v_2 \ldots v_{n-2} v_{n-1} v_n \rangle \\ &\overset{v_0}{\lessdot} \overset{v_1}{\rightarrow} \langle w_1 v_2 \ldots v_{n-2} v_{n-1} v_0 v_1 \rangle \\ &\overset{w_l}{\gtrdot} \overset{w_1}{\lessdot} \overset{v_1}{\gtrdot} \overset{v_n}{\leftarrow} Q. \end{aligned}$$

We thus conclude that v_2, as well as v_{n-2}, is not adjacent both to w_1 and to w_l.

Case 1.4: We assume that v_2 is adjacent neither to w_1 nor to w_l. We first consider the following n-paths:

$$S_1 = \langle v_0 v_1 v_n v_{n-1} v_{n-2} \ldots v_3 v_2 \rangle,$$
$$S_1' = \langle v_n v_1 v_0 v_{n-1} v_{n-2} \ldots v_3 v_2 \rangle.$$

Since v_2 has no neighbors in W, v_2 is adjacent to v_0 and v_n. If v_{n-2} is adjacent to w_1 or w_l, then the assertion holds by Lemma 11, so we assume that v_{n-2} is adjacent to none of the vertices in W. We next consider the following n-paths:

$$S_2 = \langle v_n v_{n-1} v_0 v_1 v_2 v_3 \ldots v_{n-3} v_{n-2} \rangle,$$
$$S_2' = \langle v_0 v_{n-1} v_n v_1 v_2 v_3 \ldots v_{n-3} v_{n-2} \rangle.$$

Since v_{n-2} has no neighbors in W, v_{n-2} is adjacent to v_n and v_0. Here, we set

$$S_3 = \langle v_{l+1} v_{l+2} \ldots v_{n-2} v_n v_{n-1} v_0 w_1 \ldots w_l \rangle.$$

If w_l is adjacent to one of $v_1, v_2, \ldots, v_{l-1}, v_l$, then the assertion holds by Lemma 11, so we assume that w_l is adjacent to none of $v_1, v_2, \ldots, v_l$. Since S_3 is reversible and L is a longest path, w_l is adjacent to v_{l+1}. We further consider the following n-path:

$$S_4 = \langle v_{n-2} \ldots v_{l+1} w_l v_n v_{n-1} v_0 w_1 \ldots w_{l-1} \rangle.$$

If w_{l-1} is adjacent to one of $v_1, v_2, \ldots, v_{l-1}, v_{n-2}$, then the assertion holds by Lemma 11, and if w_{l-1} is adjacent to v_l, then the assertion holds by Lemma 12. We thus assume that w_{l-1} is adjacent to some vertex, say w, in $W - W_0$. We set

$$S_5 = \langle v_{n-3} \ldots v_{l+1} w_l v_n v_{n-1} v_0 w_1 \ldots w_{l-1} w \rangle.$$

If w is adjacent to one of $v_1, v_2, \ldots, v_l, v_{n-3}, v_{n-2}$, then the assertion holds by Lemma 11. Otherwise, w is adjacent to some vertex in $W - W_0$ since S_5 is reversible, however, this contradicts the maximality of the length of L. Therefore v_2, as well as v_{n-2}, is adjacent either to w_1 or to w_l.

Case 1.5: Finally, from what has been discussed above, we conclude that

(A1) v_2 is adjacent to precisely one vertex in W, which is either w_1 or w_l.
(B1) v_1 is not adjacent to any vertex in W, particularly $v_1w_1, v_1w_l \notin E(G)$.

The vertex v_{n-2} is also adjacent either to w_1 or to w_l. By symmetry, it is sufficient to consider the following two cases:

Case 1.5.1: We assume that $v_2w_l \in E(G), v_{n-2}w_l \in E(G)$. In this case, we first show that $P \dashrightarrow Q$ if $v_3w_{l-1} \in E(G)$; if $l \geqslant 5$ and $v_3w_{l-1} \in E(G)$, then

$$\begin{aligned}
&P \overset{w_1}{\leftarrow} \langle w_1v_0v_1v_2v_3 \dots v_{n-3}v_{n-2}v_{n-1}\rangle \\
&\quad \overset{w_l}{\lessdot} \overset{v_n}{\rightarrow} \overset{v_{n-1}}{\rightarrow} \overset{v_0}{\rightarrow} \langle v_2v_3 \dots v_{n-3}v_{n-2}w_lv_nv_{n-1}v_0\rangle \\
&\quad \overset{w_{l-1}}{\gtrdot} \overset{w_{l-2}}{\leftarrow} \overset{w_{l-3}}{\leftarrow} \overset{w_{l-4}}{\leftarrow} \langle w_{l-4}w_{l-3}w_{l-2}w_{l-1}v_3 \dots v_{n-3}v_{n-2}w_l\rangle \\
&\quad \overset{v_{n-1}}{\lessdot} \overset{v_0}{\rightarrow} \overset{w_1}{\rightarrow} \overset{w_2}{\rightarrow} \langle w_{l-1}v_3 \dots v_{n-3}v_{n-2}v_{n-1}v_0w_1w_2\rangle \\
&\quad \overset{v_2}{\gtrdot} \overset{v_1}{\leftarrow} \overset{v_n}{\leftarrow} Q.
\end{aligned}$$

For the other cases, $l = 4, 3, 2$, we can find that

$$P \overset{w_1}{\leftarrow} \overset{w_4}{\lessdot} \overset{v_n}{\rightarrow} \overset{v_{n-1}}{\rightarrow} \overset{v_0}{\rightarrow} \overset{w_3}{\gtrdot} \overset{w_2}{\leftarrow} \overset{w_1}{\leftarrow} \overset{v_0}{\leftarrow} \overset{v_{n-1}}{\lessdot} \overset{v_0}{\rightarrow} \overset{w_1}{\rightarrow} \overset{w_2}{\rightarrow} \overset{v_2}{\gtrdot} \overset{v_1}{\leftarrow} \overset{v_n}{\leftarrow} Q,$$

$$P \overset{w_1}{\leftarrow} \overset{w_3}{\lessdot} \overset{v_n}{\rightarrow} \overset{v_{n-1}}{\rightarrow} \overset{v_0}{\rightarrow} \overset{w_2}{\gtrdot} \overset{w_1}{\leftarrow} \overset{v_0}{\leftarrow} \overset{v_1}{\leftarrow} \overset{v_{n-1}}{\lessdot} \overset{v_n}{\rightarrow} \overset{v_1}{\rightarrow} \overset{w_3}{\gtrdot} \overset{v_2}{\leftarrow} \overset{v_0}{\lessdot} \overset{v_1}{\rightarrow} \overset{v_n}{\rightarrow} \overset{v_2}{\gtrdot} \overset{w_3}{\leftarrow} \overset{w_1}{\lessdot} \overset{v_1}{\gtrdot} \overset{v_n}{\leftarrow} Q,$$

$$P \overset{w_1}{\leftarrow} \overset{w_2}{\lessdot} \overset{v_n}{\rightarrow} \overset{v_{n-1}}{\rightarrow} \overset{v_0}{\rightarrow} \overset{w_1}{\gtrdot} \overset{v_0}{\leftarrow} \overset{v_1}{\leftarrow} \overset{v_2}{\leftarrow} \overset{v_{n-1}}{\lessdot} \overset{v_n}{\rightarrow} \overset{v_1}{\rightarrow} \overset{w_2}{\gtrdot} \overset{v_2}{\leftarrow} \overset{v_0}{\lessdot} \overset{v_1}{\rightarrow} \overset{v_n}{\rightarrow} \overset{v_2}{\gtrdot} \overset{w_2}{\leftarrow} \overset{w_1}{\lessdot} \overset{v_1}{\gtrdot} \overset{v_n}{\leftarrow} Q,$$

respectively. Therefore, if $v_3w_{l-1} \in E(G)$, then $P \dashrightarrow Q$. We thus conclude that $v_3w_{l-1} \notin E(G)$.

Here, we set $P' = \langle v_1v_2v_3 \dots v_{n-2}v_{n-1}v_nw_l\rangle$, $Q' = \langle w_lv_2v_3 \dots v_{n-2}v_{n-1}v_nv_1\rangle$, $V' = \{v_1, v_2, \dots, v_n, w_l\}$, $W' = V(G) - V'$ and $W'_0 = \{v_0, w_1, w_2, \dots, w_{l-1}\} \subseteq W'$. We notice that $P \overset{w_l}{\rightarrow} P'$, and that $Q' \overset{w_{l-1}}{\leftarrow} \overset{v_0}{\lessdot} \overset{w_1}{\rightarrow} \overset{v_1}{\gtrdot} \overset{v_n}{\leftarrow} Q$. Hence, $P' \dashrightarrow Q'$ implies $P \dashrightarrow Q$, and the same assertion as $(A1)$, $(B1)$ holds for P' and Q'. That is,

(A1)$'$ v_3 is adjacent to precisely one vertex in W', which is either v_0 or w_{l-1}.
(B1)$'$ v_2 is not adjacent to any vertex in W', particularly $v_2w_1 \notin E(G)$.

Since $v_3w_{l-1} \notin E(G)$, v_3 is adjacent to v_0.

We set $P'' = \langle v_0v_3v_4 \dots v_{n-2}v_{n-1}v_nv_1v_2\rangle$, $Q'' = \langle v_2v_3v_4 \dots v_{n-2}v_{n-1}v_nv_1v_0\rangle$, $V'' = V = \{v_0, \dots, v_n\}$, $W'' = W$. We notice that $P \overset{w_l}{\rightarrow} \overset{w_{l-1}}{\rightarrow} \overset{v_0}{\gtrdot} \overset{w_1}{\leftarrow} \overset{v_1}{\lessdot} \overset{v_2}{\rightarrow} P''$. If $v_4w_1 \in E(G)$, then

$$\begin{aligned}
&P'' \overset{w_l}{\rightarrow} \overset{w_1}{\gtrdot} \overset{w_2}{\leftarrow} \langle w_2w_1v_4 \dots v_{n-2}v_{n-1}v_nv_1v_2\rangle \\
&\quad [\overset{w_3}{\leftarrow} \overset{w_4}{\leftarrow}; l \geqslant 4][\overset{w_3}{\leftarrow} \overset{v_2}{\leftarrow}; l = 3][\overset{v_2}{\leftarrow} \overset{v_1}{\leftarrow}; l = 2] \\
&\quad \overset{v_0}{\lessdot} \overset{v_1}{\rightarrow} \overset{v_n}{\rightarrow} \overset{w_l}{\rightarrow} \overset{v_3}{\gtrdot} \overset{v_2}{\leftarrow} \overset{w_l}{\leftarrow} \overset{w_1}{\lessdot} \overset{v_1}{\gtrdot} \overset{v_n}{\leftarrow} Q,
\end{aligned}$$

and therefore $P \dashrightarrow Q$. We thus assume that $v_4w_1 \notin E(G)$. On the other hand, we have that $Q'' \overset{w_l}{\leftarrow} \overset{w_{l-1}}{\leftarrow} \overset{v_0}{\lessdot} \overset{w_1}{\rightarrow} \overset{v_1}{\gtrdot} \overset{v_n}{\leftarrow} Q$, hence $P'' \dashrightarrow Q''$ implies $P \dashrightarrow Q$. As a consequence, the same assertion as $(A1)$, $(B1)$ holds for P'' and Q'':

(A2) v_4 is adjacent to precisely one vertex in W, which is either w_1 or w_l.
(B2) v_3 is not adjacent to any vertex in W, particularly $v_3w_1 \notin E(G)$.

Since $v_4w_1 \notin E(G)$, v_4 is adjacent to w_l. We observe that P'' is obtained from P by shifting the vertices of V other than v_0 by two steps (see Fig. 7). Iterating in this way, we obtain that:

(A) none of $v_2, v_4, \dots, v_{n-2}, v_n$ is adjacent to w_1;
(B) none of $v_1, v_3, \dots, v_{n-3}, v_{n-1}$ is adjacent to w_1.

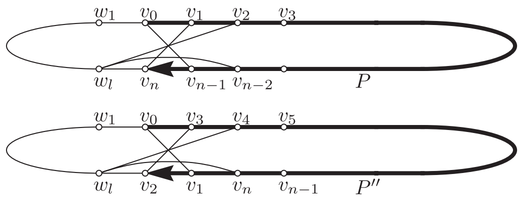

Fig. 7.

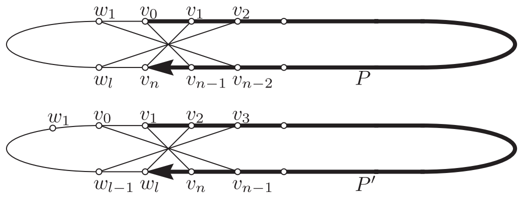

Fig. 8.

Particularly, n is even and w_1 is adjacent to none of the vertices in $V - v_0$. We consider an n-path

$$T = \langle v_{l+2}v_{l+3}\dots v_{n-2}v_{n-1}v_0v_1v_nw_l\dots w_2w_1\rangle.$$

The vertex w_1 is adjacent to some vertex in $W - W_0$ since w_1 is not adjacent to any of $v_2, \dots, v_{l+1}, v_{l+2}$, however, this contradicts the maximality of L.

Case 1.5.2: We assume that $v_2w_l \in E(G)$, $v_{n-2}w_1 \in E(G)$. In this case, we set $P' = \langle v_1v_2v_3\dots v_{n-2}v_{n-1}v_nw_l\rangle$, $Q' = \langle w_lv_2v_3\dots v_{n-2}v_{n-1}v_nv_1\rangle$ as in Case 1.5.1. We can deduce that $P \dashrightarrow Q$ if $v_0v_3 \in E(G)$, thus assume that $v_0v_3 \notin E(G)$. Since $P' \dashrightarrow Q'$ implies $P \dashrightarrow Q$, the same assertion as $(A1)$, $(B1)$ holds for P' and Q'. Here, let V', W' be as in Case 1.5.1.

(A1)′ v_3 is adjacent to precisely one vertex in W', which is either v_0 or w_{l-1}.
(B1)′ v_2 is not adjacent to any vertex in W', particularly $v_2w_1 \notin E(G)$ (Fig. 8).

Since $v_3v_0 \notin E(G)$, v_3 is adjacent to w_{l-1}. It is easy to see that $P' \dashrightarrow Q'$ if v_n is adjacent to some vertex in $W - w_l$, so we assume that v_n is adjacent to none of the vertices in $W - w_l$. Then P can only move to P'. We similarly deduce that P' can only move to w_{l-1} because the same form appears for P' and Q'. Iterating in this way, we conclude that P has only one orbit $P \xrightarrow{w_l} P' \xrightarrow{w_{l-1}} \cdots \xrightarrow{w_1} \xrightarrow{v_0} \xrightarrow{v_1} \cdots \xrightarrow{v_n} P \xrightarrow{w_l} \cdots$, and this contradicts the reversibility of P.

Case 2: $w_lv_n \notin E(G)$. We consider the next step of the following n-path:

$$X_1 = \langle v_{n-l}v_{n-l-1}\dots v_2v_1v_0w_1w_2\dots w_l\rangle.$$

If w_l is adjacent to v_{n-1}, then $P \overset{w_l}{\lessdot} \overset{v_n}{\gtrdot} \overset{v_0}{\lessdot} Q$, so we assume that $w_lv_{n-1} \notin E(G)$. Since $w_lv_n \notin E(G)$ and L is a longest path, w_l is adjacent to one of the vertices $v_{n-l}, v_{n-l+1}, \dots, v_{n-2}$. Let v_j be such a vertex. Then we deduce that

$$\begin{aligned}
P &\overset{w_1}{\leftarrow} \overset{w_2}{\leftarrow} \cdots \overset{w_{n-j}}{\leftarrow} \langle w_{n-j}\dots w_1v_0v_1v_2\dots v_{n-l}\dots v_{j-1}v_j\rangle \\
&\overset{w_l}{\rightarrow} \overset{w_{l-1}}{\rightarrow} \cdots \overset{w_{l-(n-j)+1}}{\rightarrow} \langle v_0v_1v_2\dots v_{j-1}v_jw_lw_{l-1}\dots w_{l-(n-j)+1}\rangle \\
&\overset{v_n}{\gtrdot} \overset{v_{n-1}}{\leftarrow} \overset{v_0}{\leftarrow} \overset{w_1}{\leftarrow} \overset{w_2}{\leftarrow} \cdots \overset{w_{n-j-2}}{\leftarrow} \langle w_{n-j-2}\dots w_2w_1v_0v_{n-1}v_nv_1v_2\dots v_{j-1}v_j\rangle \\
&\overset{v_{j+1}}{\rightarrow} \overset{v_{j+2}}{\rightarrow} \cdots \overset{v_{n-2}}{\rightarrow} \langle v_0v_{n-1}v_nv_1v_2\dots v_{n-3}v_{n-2}\rangle =: Y.
\end{aligned}$$

Let the last n-path be Y. If $v_{n-2}v_0 \in E(G)$, then $Y \xrightarrow{v_0} \xrightarrow{v_{n-1}} \lhd Q$, so we assume that $v_{n-2}v_0 \notin E(G)$. Since Y and P are reversible, each of v_{n-2}, v_n is adjacent to vertices in W. If there are two different vertices $x, x' \in W$ such that

$v_{n-2}x,\, v_nx' \in E(G)$, then $Y \overset{x}{\rightarrow} \overset{x'}{\gtrdot} \overset{v_{n-1}}{\lessdot} \overset{v_0}{\rightarrow} Q$, and therefore $P \dashrightarrow Q$. We hence assume that v_{n-2} and v_n have only one neighbor in W, say w.

On the other hand, we consider the next step of the following n-path:

$$X_1' = \langle v_l v_{l+1} \ldots v_{n-2} v_{n-1} v_0 w_1 w_2 \ldots w_l \rangle.$$

If w_l is adjacent to v_1, then $P \overset{w_l}{\gtrdot} \overset{v_0}{\lessdot} \overset{v_n}{\gtrdot} Q$, so we assume that $w_l v_1 \notin E(G)$. Since $w_l v_n \notin E(G)$ and L is a longest path, w_l is adjacent to one of $v_2, v_3, \ldots, v_{l-1}, v_l$. Let $v_{j'}$ be such a vertex. We deduce that

$$\begin{aligned}
Q &\overset{w_1}{\rightarrow} \overset{w_2}{\rightarrow} \cdots \overset{w_{j'}}{\rightarrow} \langle v_{j'} v_{j'+1} \ldots v_l \ldots v_{n-2} v_{n-1} v_0 w_1 w_2 \ldots w_{j'} \rangle \\
&\overset{w_l}{\leftarrow} \overset{w_{l-1}}{\leftarrow} \cdots \overset{w_{l-j'+1}}{\leftarrow} \langle w_{l-j'+1} \ldots w_{l-1} w_l v_{j'} v_{j'+1} \ldots v_{n-2} v_{n-1} v_0 \rangle \\
&\overset{v_n}{\lessdot} \overset{v_1}{\rightarrow} \overset{v_0}{\rightarrow} \overset{w_1}{\rightarrow} \cdots \overset{w_{j'-2}}{\rightarrow} \langle v_{j'} v_{j'+1} \ldots v_{n-2} v_{n-1} v_n v_1 v_0 w_1 w_2 \ldots w_{j'-2} \rangle \\
&\overset{v_{j'-1}}{\leftarrow} \cdots \overset{v_2}{\leftarrow} \langle v_2 v_3 \ldots v_{n-2} v_{n-1} v_n v_1 v_0 \rangle =: Y'.
\end{aligned}$$

Let the last n-path be Y'. If $v_2 v_0 \in E(G)$, then $Y' \overset{v_0}{\leftarrow} \overset{v_1}{\leftarrow} \triangleright P$, and therefore $P \dashrightarrow Q$. We hence assume that $v_2 v_0 \notin E(G)$. Since $(Y')^{-1}$ is reversible, v_2 is adjacent to some vertex in W. If v_2 is adjacent to a vertex in $W - w$, say x'', then $Y' \overset{x''}{\leftarrow} \overset{w}{\lessdot} \overset{v_1}{\gtrdot} \overset{v_0}{\leftarrow} P$, and therefore $P \dashrightarrow Q$. We thus assume that v_2 is adjacent to none of the vertices in $W - w$, and that $v_2 w \in E(G)$. As a consequence, we deduce that v_2, v_{n-2} and v_n have only one vertex w as their common neighbors in W.

Let $P' = \langle v_1 v_2 v_3 \ldots v_{n-2} v_{n-1} v_n w \rangle$, $Q' = \langle w v_2 v_3 \ldots v_{n-2} v_{n-1} v_n v_1 \rangle$. We notice that $P \overset{w}{\rightarrow} P'$, and that $Q' \overset{v_0}{\rightarrow} Y' \dashrightarrow Q$. Hence, $P' \dashrightarrow Q'$ implies $P \dashrightarrow Q$.

If $w \notin W_0$, then P' has a rut of length more than l, and by assumption, $P' \dashrightarrow Q'$. Hence, we assume that $w \in W_0$. If $w = w_1$, then $P \dashrightarrow Q$ by Lemma 7 (in fact, two paths $v_j w_l \ldots w_1$ and $v_0 w v_n$ play the roles of L and J in the lemma), and the case $w = w_l$ has already been treated in Case 1.5.1, so we assume that $w = w_k$, $1 < k < l$ (Fig. 9). We consider the next step of the following n-path:

$$X_2 = \langle v_1 v_2 w_k v_n v_{n-1} \ldots v_3 \rangle.$$

If v_3 is adjacent to v_1, then $P' \triangleright \overset{w_k}{\leftarrow} \overset{v_0}{\lessdot} \overset{w_1}{\rightarrow} \triangleright \overset{v_n}{\leftarrow} Q$, so we assume that $v_3 v_1 \notin E(G)$.

We first show that v_3 is adjacent to none of the vertices in $W - W_0$; otherwise v_3 is adjacent to some vertex in $W - W_0$, say y, then we consider the following n-path:

$$X_3 = \langle v_2 w_k v_n v_{n-1} \ldots v_3 y \rangle.$$

Since X_3 is reversible, y is adjacent to one of the vertices v_0, v_1, v_2, or to some vertex in $W - w_k$, however, then we can deduce that $P \dashrightarrow Q$ or $P' \dashrightarrow Q'$ by Lemma 7 or 8. Therefore, v_3 is adjacent to none of the vertices in $W - W_0$.

We next show that $v_3 v_0 \in E(G)$; otherwise, we assume that $v_3 v_0 \notin E(G)$. Since X_2 is reversible, v_3 is adjacent to some vertex in $W - w_k$, say z. On the other hand, we consider the following n-path:

$$X_4 = \langle v_{l-k-1} \ldots v_{n-3} v_{n-2} w_k w_{k+1} \ldots w_l \rangle.$$

If w_l is adjacent to v_n or v_{n-1}, or if w_l is adjacent to one of $v_1, v_2, \ldots, v_{l-k-1}$ for $l - k - 1 \geqslant 1$, then $P \dashrightarrow Q$ or $P' \dashrightarrow Q'$ by Lemma 7 or 8. We hence deduce that w_l is adjacent to one of the vertices $w_1, w_2, \ldots, w_{k-1}, v_0$ since X_4 is reversible. Then we can find a vertex $z' \in \{w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_l, v_0\}$ which satisfies $zz' \in E(G)$ and

$$\begin{aligned}
Y' &\overset{z}{\gtrdot} \overset{z'}{\leftarrow} \overset{w_{l-1}}{\lessdot} \langle z' z v_3 v_4 \ldots v_{n-1} v_n w_k \rangle \\
&\overset{v_2}{\rightarrow} \overset{w_{k-1}}{\lessdot} \Big(\text{or } \overset{w_{k+1}}{\lessdot}\Big) \overset{v_2}{\gtrdot} \overset{v_1}{\leftarrow} \overset{v_0}{\leftarrow} P,
\end{aligned}$$

and thus $P \dashrightarrow Q$. We hence conclude that $v_3 v_0 \in E(G)$.

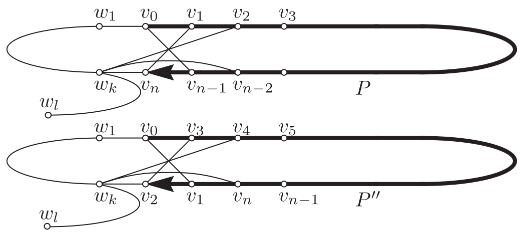

Fig. 9.

Let P'' and Q'' be two n-paths as in Case 1.5.1. We can similarly deduce that $P \dashrightarrow P''$, $Q'' \dashrightarrow Q$. By above consideration, we observe that P'' is obtained from P by shifting the vertices of V other than v_0 by two steps as in Case 1.5.1. Continuing in this way, we can conclude that the assertion holds in this case.

As a consequence, we establish this theorem. $\square$

Lemma 14. *Let G be an n-reversible graph and P, Q as in Lemma 7. We assume that $v_{i+1}v_{i+3} \in E(G)$ for some index i, $0 \leqslant i \leqslant n-4$. If there are two vertices $x, y \notin V(P)$ with $xv_0, xv_n, yv_i, yv_{i+2} \in E(G)$, then $P \dashrightarrow Q$.*

Proof. We have the following sequence of n-paths:

$$\begin{aligned}
P &\xrightarrow{x}\xrightarrow{v_0}\xrightarrow{v_1}\cdots\xrightarrow{v_i}\langle v_{i+2}v_{i+3}\ldots v_{n-2}v_{n-1}v_nxv_0v_1v_2\ldots v_{i-1}v_i\rangle\\
&\xrightarrow{y}\xrightarrow{v_{i+2}}\xrightarrow{v_{i+1}}\xrightarrow{v_{i+3}}\langle v_{i+6}\ldots v_{n-2}v_{n-1}v_nxv_0v_1v_2\ldots v_{i-1}v_iyv_{i+2}v_{i+1}v_{i+3}\rangle\\
&\xrightarrow{v_{i+4}}\cdots\xrightarrow{v_{n-1}}\langle v_0v_1v_2\ldots v_{i-1}v_iyv_{i+2}v_{i+1}v_{i+3}v_{i+4}\ldots v_{n-2}v_{n-1}\rangle\\
&\xrightarrow{v_0}\xrightarrow{x}\xrightarrow{v_n}\langle v_3\ldots v_{i-1}v_iyv_{i+2}v_{i+1}v_{i+3}v_{i+4}\ldots v_{n-2}v_{n-1}v_0xv_n\rangle\\
&\xrightarrow{v_1}\xrightarrow{v_2}\cdots\xrightarrow{v_i}\langle v_{i+1}v_{i+3}v_{i+4}\ldots v_{n-2}v_{n-1}v_0xv_nv_1v_2\ldots v_i\rangle\\
&\xrightarrow{v_{i+1}}\xrightarrow{v_{i+2}}\cdots\xrightarrow{v_{n-1}}\xrightarrow{v_0}Q. \qquad \square
\end{aligned}$$

Lemma 15. *Let G be an n-reversible graph and P, Q as in Lemma 7. We assume that $v_{i+1}v_{i+4} \in E(G)$ for some index i, $0 \leqslant i \leqslant n-5$. If there are two vertices $x, y \notin V(P)$ with $xv_0, xv_n, yv_i, yv_{i+3} \in E(G)$, then $P \dashrightarrow Q$ (Fig. 10).*

Proof. The proof is similar to that of Lemma 14. $\square$

Lemma 16. *Let G be an n-reversible graph and P, Q as in Lemma 7. We assume that $v_{i+1}v_{i+5}, v_{i+2}v_{i+6} \in E(G)$ for some index i, $0 \leqslant i \leqslant n-7$. If there are two vertices $x, y \notin V(P)$ with $xv_0, xv_n, yv_i, yv_{i+3} \in E(G)$, then $P \dashrightarrow Q$.*

Proof. We can find that

$$\begin{aligned}
P &\xrightarrow{x}\xrightarrow{v_0}\xrightarrow{v_1}\cdots\xrightarrow{v_i}\langle v_{i+2}v_{i+3}\ldots v_{n-2}v_{n-1}v_nxv_0v_1v_2\ldots v_{i-1}v_i\rangle\\
&\xrightarrow{y}\xrightarrow{v_{i+3}}\xrightarrow{v_{i+4}}\xrightarrow{v_{i+5}}\xrightarrow{v_{i+1}}\xrightarrow{v_{i+2}}\xrightarrow{v_{i+6}}\langle v_{i+9}\ldots v_{n-2}v_{n-1}v_nxv_0v_1v_2\ldots v_{i-1}v_iyv_{i+3}v_{i+4}v_{i+5}v_{i+1}v_{i+2}v_{i+6}\rangle\\
&\xrightarrow{v_{i+7}}\cdots\xrightarrow{v_{n-1}}\langle v_0v_1v_2\ldots v_{i-1}v_iyv_{i+3}v_{i+4}v_{i+5}v_{i+1}v_{i+2}v_{i+6}\ldots v_{n-2}v_{n-1}\rangle\\
&\xrightarrow{v_0}\xrightarrow{x}\xrightarrow{v_n}\langle v_3\ldots v_{i-1}v_iyv_{i+3}v_{i+4}v_{i+5}v_{i+1}v_{i+2}v_{i+6}\ldots v_{n-2}v_{n-1}v_0xv_n\rangle\\
&\xrightarrow{v_1}\xrightarrow{v_2}\cdots\xrightarrow{v_i}\langle v_{i+4}v_{i+5}v_{i+1}v_{i+2}v_{i+6}\ldots v_{n-2}v_{n-1}v_0xv_nv_1v_2\ldots v_i\rangle\\
&\xrightarrow{y}\xrightarrow{v_{i+3}}\xrightarrow{v_{i+4}}\xrightarrow{v_{i+5}}\langle v_{i+6}\ldots v_{n-2}v_{n-1}v_0xv_nv_1v_2\ldots v_iyv_{i+3}v_{i+4}v_{i+5}\rangle\\
&\xrightarrow{v_{i+6}}\cdots\xrightarrow{v_{n-1}}\xrightarrow{v_0}\xrightarrow{x}\xrightarrow{v_n}\xrightarrow{v_1}\xrightarrow{v_2}\cdots\xrightarrow{v_i}\xrightarrow{v_{i+1}}\cdots\xrightarrow{v_{n-1}}\xrightarrow{v_0}Q. \qquad \square
\end{aligned}$$

Lemma 17. *Let G be an n-reversible graph and P, Q as in Lemma 7. If there are three vertices $x, y, z \notin V(P)$ with $xv_0, xv_n, yv_i, yv_{i+3}, zv_{i+1}, zv_{i+4} \in E(G)$ for some index i, $1 \leqslant i \leqslant n-5$,* then $P \dashrightarrow Q$.

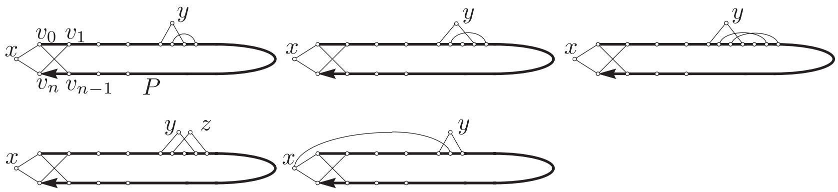

Fig. 10. The configurations of Lemmas 14–18.

Proof. We have the following sequence:

$$
\begin{aligned}
P &\xrightarrow{x}\xrightarrow{v_0}\xrightarrow{v_1}\cdots\xrightarrow{v_i}\langle v_{i+2}v_{i+3}\dots v_{n-2}v_{n-1}v_nxv_0v_1v_2\dots v_{i-1}v_i\rangle\\
&\xrightarrow{y}\xrightarrow{v_{i+3}}\xrightarrow{v_{i+2}}\xrightarrow{v_{i+1}}\xrightarrow{z}\langle v_{i+7}\dots v_{n-2}v_{n-1}v_nxv_0v_1v_2\dots v_{i-1}v_iyv_{i+3}v_{i+2}v_{i+1}z\rangle\\
&\xrightarrow{v_{i+4}}\cdots\xrightarrow{v_{n-1}}\langle v_1v_2\dots v_{i-1}v_iyv_{i+3}v_{i+2}v_{i+1}zv_{i+4}v_{i+5}\dots v_{n-2}v_{n-1}\rangle\\
&\xrightarrow{v_0}\xrightarrow{x}\xrightarrow{v_n}\langle v_4\dots v_{i-1}v_iyv_{i+3}v_{i+2}v_{i+1}zv_{i+4}v_{i+5}\dots v_{n-2}v_{n-1}v_0xv_n\rangle\\
&\xrightarrow{v_1}\cdots\xrightarrow{v_i}\xrightarrow{v_{i+1}}\xrightarrow{v_{i+2}}\cdots\xrightarrow{v_{n-1}}\xrightarrow{v_0}Q. \qquad \square
\end{aligned}
$$

Lemma 18. *Let G be an n-reversible graph and P, Q as in Lemma 7. If there are two vertices $x, y \notin V(P)$ with $xv_0, xv_n, xv_{i+1}, yv_i, yv_{i+2} \in E(G)$ for some index i, $1 \leqslant i \leqslant n-3$,* then $P \dashrightarrow Q$.

Proof. We can find that

$$
\begin{aligned}
P &\xrightarrow{x}\xrightarrow{v_0}\xrightarrow{v_1}\cdots\xrightarrow{v_i}\langle v_{i+2}v_{i+3}\dots v_{n-2}v_{n-1}v_nxv_0v_1v_2\dots v_{i-1}v_i\rangle\\
&\xrightarrow{y}\xrightarrow{v_{i+2}}\cdots\xrightarrow{v_{n-1}}\xrightarrow{v_n}\langle v_0v_1v_2\dots v_{i-1}v_iyv_{i+2}v_{i+3}\dots v_{n-2}v_{n-1}v_n\rangle = P'.
\end{aligned}
$$

Let the last n-path be P'. To compare P' with P, we observe that their $(i+1)$th vertices are different. Let Q' be the following n-path that has the same vertices as Q except the $(i+1)$th vertex:

$$Q' = \langle v_nv_1v_2\dots v_{i-1}v_iyv_{i+2}v_{i+3}\dots v_{n-2}v_{n-1}v_0\rangle.$$

We notice that $r(P') \geqslant 2$ (in fact, $\|v_{i+1}xv_0\| = 2$), therefore $P' \dashrightarrow Q'$ by Theorem 13. And then $Q' \xrightarrow{x}\xrightarrow{v_n}\xrightarrow{v_1}\cdots\xrightarrow{v_{i-1}}\xrightarrow{v_i}\xrightarrow{v_{i+1}}\cdots\xrightarrow{v_{n-1}}\xrightarrow{v_0} Q$, thus $P \dashrightarrow Q$. $\square$

Theorem 19. *Let G be an n-reversible graph and P, Q as in Lemma 7. If $r(P) = r(Q) = 1$ and $|V(G)| \geqslant n+3$, then $P \dashrightarrow Q$.*

Proof. We set $V = \{v_0, \dots, v_n\}$, $W = V(G) - V$, $W \neq \emptyset$. The case $v_0v_n \in E(G)$ is already treated in Theorem 9, so we assume that $v_0v_n \notin E(G)$. Since $v_0v_n \notin E(G)$ and P is reversible, v_n is adjacent to some vertex in W. Let the set of all vertices in W that are adjacent to v_n be $W_0 = \{w_1, w_2, \dots, w_m\}$. Here, we consider the following n-paths $R_1, \dots, R_m$:

$$R_i = \langle v_1v_2\dots v_{n-2}v_{n-1}v_nw_i\rangle.$$

The vertex w_i is adjacent to none of the vertices in W since $r(P) = 1$, and hence w_i is adjacent to v_0 or v_1. If w_i is adjacent to v_1, then $P \overset{w_i}{>} \overset{v_0}{<} \overset{v_n}{>} Q$, we thus assume that $w_iv_0 \in E(G)$ for each i, $1 \leqslant i \leqslant m$.

We will show that $W = W_0$; otherwise we assume that $W - W_0 \neq \emptyset$. Since G is connected and there are no edges between W_0 and $W - W_0$, there is at least one edge between V and $W - W_0$. Let v_ju_1 be such an edge, here $1 \leqslant j \leqslant n-1$, $u_1 \in W - W_0$. We consider an n-path

$$S = \langle v_{j-1}v_{j-2}\dots v_1v_0v_{n-1}v_{n-2}\dots v_{j+1}v_ju_1\rangle.$$

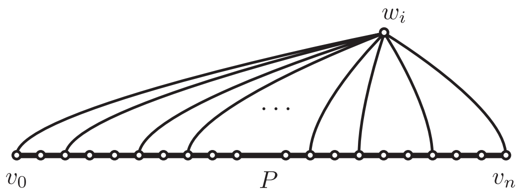

Fig. 11.

Since $u_1 \notin W_0$, u_1 is not adjacent to v_n. If u_1 is adjacent to v_{j-1}, then the assertion holds by Lemma 8. We hence assume that u_1 is adjacent to some vertex in $W - W_0$, say u_2. We consider the following n-path:

$$S' = \langle v_{j-2}v_{j-3}\ldots v_1v_0v_{n-1}v_{n-2}\ldots v_{j+1}v_ju_1u_2\rangle.$$

If u_2 is adjacent to v_{j-2} or v_{j-3}, then the assertion holds by Lemma 8, so we assume that u_2 is adjacent to some vertex in $W - W_0$, say u_3. Iterating in this way, we obtain a sequence of vertices $u_1, u_2, \ldots$, in $W - W_0$, however, when we have got the jth vertex u_j, the assertion will hold by Lemma 7. We therefore deduce that $W = W_0$.

We set $W = \{w_1, w_2, \ldots, w_m\}$. We notice that the vertices in W are pairwise non-adjacent, and that $m \geqslant 2$ by assumption. We consider an n-path

$$T = \langle v_3v_4\ldots v_{n-2}v_{n-1}v_nw_2v_0w_1\rangle.$$

Since T is reversible, w_1 is adjacent to v_2 or v_3. We first assume that $w_1v_3 \in E(G)$. Then we can define the next n-path

$$T' = \langle v_6v_7\ldots v_{n-2}v_{n-1}v_nw_2v_0v_1v_2v_3w_1\rangle.$$

Since T' is reversible, w_1 is adjacent to one of v_4, v_5, v_6. If $w_1v_4 \in E(G)$, then the assertion holds by Lemma 8, so we assume that $w_1v_5 \in E(G)$ or $w_1v_6 \in E(G)$. In this way, we can find that w_1 is adjacent to the vertices of P at intervals of two or three edges. Similarly, the vertices w_i, $1 \leqslant i \leqslant m$, are also adjacent to the vertices of P at intervals of two or three edges (Fig. 11).

Case 1: We assume that no vertex in W has a 3-interval; suppose that $w_iv_0, w_iv_2, \ldots, w_iv_{n-2}, w_iv_n \in E(G)$ for each i, $1 \leqslant i \leqslant m$.

Let $U_1 = \{v_0, v_2, \ldots, v_{n-2}, v_n\}$, $U_2 = \{v_1, v_3, \ldots, v_{n-3}, v_{n-1}\}$. Each vertex in U_1 is adjacent to each vertex in W, and there are no edges between U_2 and W. We set

$$P_{2t-1} := \langle v_0v_1\ldots v_{2t-2}w_1v_{2t}\ldots v_{n-1}v_n\rangle,$$
$$Q_{2t-1} := \langle v_nv_1\ldots v_{2t-2}w_1v_{2t}\ldots v_{n-1}v_0\rangle,$$

and

$$V_{2t-1} := \{v_0, v_1, \ldots, v_{2t-2}\} \cup \{w_1\} \cup \{v_{2t}, \ldots, v_{n-1}, v_n\},$$
$$W_{2t-1} := \{v_{2t-1}, w_2, \ldots, w_m\}.$$

For these two paths, we deduce that

$$P \overset{w_2}{\rightarrow} \overset{v_0}{\rightarrow} \overset{v_1}{\rightarrow} \cdots \overset{v_{2t-2}}{\rightarrow} \overset{w_1}{\rightarrow} \overset{v_{2t}}{\rightarrow} \cdots \overset{v_{n-1}}{\rightarrow} \overset{v_n}{\rightarrow} P_{2t-1},$$
$$Q_{2t-1} \overset{w_2}{\rightarrow} \overset{v_n}{\rightarrow} \overset{v_1}{\rightarrow} \cdots \overset{v_{2t}}{\rightarrow} \overset{v_{2t+1}}{\rightarrow} \overset{v_{2t+2}}{\rightarrow} \cdots \overset{v_{n-1}}{\rightarrow} \overset{v_n}{\rightarrow} Q,$$

thus $P_{2t-1} \dashrightarrow Q_{2t-1}$ implies $P \dashrightarrow Q$. We apply the same method to v_{2t-1} as above, and deduce that v_{2t-1} is adjacent to the vertices of P_{2t-1} at intervals of two or three edges. If v_{2t-1} has a 3-interval, then we deduce that $P_{2t-1} \dashrightarrow Q_{2t-1}$ by Lemma 18, so we assume that v_{2t-1} has no 3-intervals:

$$v_{2t-1}v_0, v_{2t-1}v_2, \ldots, v_{2t-1}v_{n-2}, v_{2t-1}v_n \in E(G),$$
$$v_{2t-1}v_1, v_{2t-1}v_3, \ldots, v_{2t-1}v_{n-3}, v_{2t-1}v_{n-1} \notin E(G).$$

The index t varies for $1 \leqslant t \leqslant n/2$, we therefore deduce that the vertices in U_1 and the vertices in U_2 are mutually adjacent and that the vertices in U_2 are pairwise non-adjacent. If there are two or more edges in U_1, then we can find an n-path

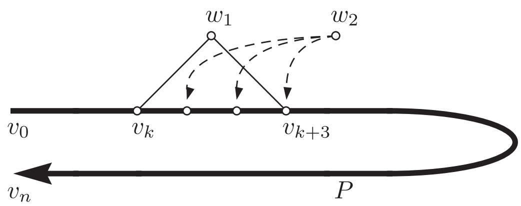

Fig. 12.

whose head and tail are in W and which passes through all vertices of U_1. However, this path cannot move, and this fact contradicts the reversibility of G. We therefore deduce that U_1 has at most one edge, and then G is either a complete bipartite graph $K_{n/2+1,n/2+m}$ with partition sets U_1 and $U_2 \cup W$, or a graph $K_{n/2+1,n/2+m}$ with an additional edge in U_1. However, we have already seen in the proof of Theorem 9 that these graphs are not n-reversible, a contradiction.

Case 2: We assume that some vertex in W has a 3-interval. Let the two vertices in P that make the interval be v_k and v_{k+3}, and choose the index k as small as possible. Without loss of generality, we assume that $w_1v_k, w_1v_{k+3} \in E(G)$. Here, we consider the neighbors of w_2; this vertex is adjacent to one of $v_{k+1}, v_{k+2}, v_{k+3}$ (Fig. 12).

Case 2.1: $w_2v_{k+1} \in E(G)$. In this case, w_2 is also adjacent to v_{k-2} or v_{k-1}. If $w_2v_{k-2} \in E(G)$, this contradicts the minimality of k. If $w_2v_{k-1} \in E(G)$, then the assertion holds by Lemma 18.

Case 2.2: $w_2v_{k+2} \in E(G)$. The vertex w_2 is adjacent to v_{k-1} or v_k. If $w_2v_{k-1} \in E(G)$, this contradicts the minimality of k. We hence assume that $w_2v_k \in E(G)$. On the other hand, w_2 is also adjacent to v_{k+4} or v_{k+5}. If $w_2v_{k+4} \in E(G)$, then the assertion holds by Lemma 18, we thus assume that $w_2v_{k+5} \in E(G)$. We consider the following n-path:

$$A = \langle v_{k+5} \dots v_{n-1}v_nw_1v_0v_1 \dots v_kw_2v_{k+2}v_{k+1}\rangle.$$

If v_{k+1} is adjacent to v_{k+3} or v_{k+4}, then the assertion holds by Lemma 14 or 15. The case that v_{k+1} is adjacent to some vertex in W is already treated in Case 2.1, therefore v_{k+1} must be adjacent to v_{k+5}.

Here, we will show that $k + 5 = n$; otherwise, if $k + 5 < n$, then we consider the following n-path:

$$B = \langle v_{k+3}v_{k+4}v_{k+5}v_{k+1}v_k \dots v_0w_1v_n \dots v_{k+7}v_{k+6}\rangle.$$

If v_{k+6} is adjacent to v_{k+2} or v_{k+3}, then the assertion holds by Lemma 16 or 15. Since B is reversible, v_{k+6} must be adjacent to some vertex in $W - \{w_1, w_2\}$, say w_3. The vertex w_3 is adjacent to v_{k+3} or v_{k+4}, and then the assertion holds by Lemma 17 or 18. Therefore, we conclude that $k + 5 = n$.

By considering the following n-path, we can also deduce that $v_kv_{k+4} \in E(G)$:

$$A' = \langle v_kv_{k-1} \dots v_1v_0w_2v_{k+5}w_1v_{k+3}v_{k+4}\rangle.$$

Furthermore, if $k > 0$, then the assertion holds in the same way as above by considering the following n-path:

$$B' = \langle v_{k+2}v_{k+1}v_kv_{k+4}v_{k+5} \dots v_nw_1v_0 \dots v_{k-2}v_{k-1}\rangle.$$

We therefore deduce that $k = 0$. As a consequence, vertices and edges of G are obtained:

$$V(G) \supseteq \{v_0, v_1, v_2, v_3, v_4, v_5\} \cup \{w_1, w_2\},$$
$$E(G) \supseteq \{v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_0v_4, v_1v_5, w_1v_0, w_1v_3, w_1v_5, w_2v_0, w_2v_2, w_2v_5\}.$$

And then, we have that $P \xrightarrow{w_2} \xrightarrow{v_0} \xrightarrow{v_1} \xrightarrow{v_2} \xrightarrow{v_3} \xrightarrow{w_1} \xrightarrow{v_5} \xrightarrow{w_2} \xrightarrow{v_0} \xrightarrow{v_4} \xrightarrow{v_3} \xrightarrow{w_1} \xrightarrow{v_5} \xrightarrow{v_1} \xrightarrow{v_2} \xrightarrow{w_2} \xrightarrow{v_0} \xrightarrow{w_1} \xrightarrow{v_5} \xrightarrow{v_1} \xrightarrow{v_2} \xrightarrow{v_3} \xrightarrow{v_4} \xrightarrow{v_0} Q$.

Case 2.3: $w_2v_{k+3} \in E(G)$. We further assume that $w_2v_k \in E(G)$ and that all vertices in W are also adjacent to v_k and v_{k+3} since the other cases are already treated.

We first assume that $|W| \geqslant 3$, and consider the following n-path:

$$C = \langle v_{k+5}v_{k+6} \dots v_{n-2}v_{n-1}v_nw_3v_0v_1v_2 \dots v_{k-1}v_kw_2v_{k+3}w_1\rangle.$$

If w_1 is adjacent to v_{k+4}, then the assertion holds by Lemma 8, so w_1 is adjacent to v_{k+5} since C is reversible. By considering the following n-path:

$$C' = \langle v_{k+7}v_{k+8} \dots v_{n-2}v_{n-1}v_nw_3v_0v_1v_2 \dots v_{k-1}v_kw_2v_{k+3}v_{k+4}v_{k+5}w_1\rangle,$$

we deduce that $w_1v_{k+7} \in E(G)$ in a similar way. Continuing in this way, we obtain that

$$w_1v_{k+5}, w_1v_{k+7}, \dots, w_1v_{n-2}, w_1v_n \in E(G).$$

A similar fact can be deduced for the other side of w_1 and for the other vertices of W, that is, each vertex in W is adjacent to the vertices $v_0, v_2, v_4, \dots, v_{k-2}, v_k, v_{k+3}, v_{k+5}, \dots, v_{n-2}, v_n$. We notice that k is even and n is odd.

Secondly, we assume that $|W| = 2$. We consider the following n-paths:

$$D_1 = \langle w_1v_{k+3}v_{k+2}\dots v_2v_1v_nv_{n-1}v_{n-2}\dots v_{k+5}v_{k+4}\rangle,$$
$$D_1' = \langle w_1v_{k+3}v_{k+2}\dots v_2v_1v_0v_{n-1}v_{n-2}\dots v_{k+5}v_{k+4}\rangle.$$

Since D_1 and D_1' are reversible and v_{k+4} is adjacent neither to w_1 nor to w_2, the vertex v_{k+4} is adjacent to v_0 and v_n. We consider the following n-paths:

$$D_2 = \langle v_{k+5}v_{k+6}\dots v_{n-2}v_{n-1}v_nv_{k+4}v_0v_1v_2\dots v_{k-1}v_kw_2v_{k+3}w_1\rangle,$$
$$D_2' = \langle v_{k+5}v_{k+6}\dots v_{n-2}v_{n-1}v_nv_{k+4}v_0v_1v_2\dots v_{k-1}v_kw_1v_{k+3}w_2\rangle.$$

Since D_2 and D_2' are reversible, w_1 and w_2 are adjacent to v_{k+5}. We set

$$D_3 = \langle w_1v_{k+5}v_{k+4}\dots v_2v_1v_nv_{n-1}v_{n-2}\dots v_{k+7}v_{k+6}\rangle,$$
$$D_3' = \langle w_1v_{k+5}v_{k+4}\dots v_2v_1v_0v_{n-1}v_{n-2}\dots v_{k+7}v_{k+6}\rangle.$$

Since D_3 and D_3' are reversible and v_{k+6} is adjacent neither to w_1 nor to w_2, the vertex v_{k+6} is adjacent to v_0 and v_n. Successively, the following n-paths are defined:

$$D_4 = \langle v_{k+7}v_{k+8}\dots v_{n-2}v_{n-1}v_nv_{k+6}v_0v_1v_2\dots v_{k-1}v_kw_2v_{k+3}v_{k+4}v_{k+5}w_1\rangle,$$
$$D_4' = \langle v_{k+7}v_{k+8}\dots v_{n-2}v_{n-1}v_nv_{k+6}v_0v_1v_2\dots v_{k-1}v_kw_1v_{k+3}v_{k+4}v_{k+5}w_2\rangle.$$

Since these paths are reversible, w_1 and w_2 are adjacent to v_{k+7}. Continuing in this way, we can obtain the sequence of edges $w_1v_{k+5}, w_2v_{k+5}, w_1v_{k+7}, w_2v_{k+7}, \dots$, alternatively. We can deduce a similar fact for the other sides of w_1 and w_2. As a consequence, we similarly deduce for the case $|W| = 2$ that each vertex in W is adjacent to the vertices $v_0, v_2, v_4, \dots, v_{k-2}, v_k, v_{k+3}, v_{k+5}, \dots, v_{n-2}, v_n$.

Here, we set

$$U_1 = \{v_0, v_2, \dots, v_{k-2}, v_k, v_{k+3}, v_{k+5}, \dots, v_{n-2}, v_n\},$$
$$U_2 = \{v_1, v_3, \dots, v_{k-1}, v_{k+1}, v_{k+2}, v_{k+4}, \dots, v_{n-3}, v_{n-1}\}.$$

Each vertex in U_1 is adjacent to each vertex in W, and there are no edges between U_2 and W. We consider the n-paths $P_1, P_3, \dots, P_{k-1}, P_{k+4}, P_{k+6}, \dots, P_{n-1}$, and $Q_1, Q_3, \dots, Q_{k-1}, Q_{k+4}, Q_{k+6}, \dots, Q_{n-1}$ as in Theorem 9; only the tth vertices of P_t and Q_t differ from the vertices of P and Q, respectively. We can deduce that $P \dashrightarrow P_t$ and $Q_t \dashrightarrow Q$ for each pair of two paths, so $P_t \dashrightarrow Q_t$ implies $P \dashrightarrow Q$.

To apply the same method as in Theorem 9, we deduce that each vertex in U_1 is adjacent to each vertex in U_2 and that U_2 has no edges other than $v_{k+1}v_{k+2}$. If there is an edge in U_1, then we can find an n-path whose head and tail are in W and which passes through all vertices of U_1. However, this path cannot move, and this fact contradicts the reversibility of G. We therefore deduce that U_1 has no edges. And then G is a complete bipartite graph $K_{(n+1)/2+1,(n+1)/2+m}$ with an additional edge $v_{k+1}v_{k+2}$. However, this graph is not n-reversible for $n \geqslant 5$, and is 3-transferable for $n = 3$. $\square$

Theorem 20. *Let G be an n-reversible graph and P, Q as in Lemma* 7. *If $r(P) = r(Q) = 1$ and $|V(G)| = n + 2$, then $P \dashrightarrow Q$.*

Proof. Let the vertex not in $V(P)$ be v_{n+1}. Then $V(G) = \{v_0, v_1, \dots, v_n, v_{n+1}\}$. For the sake of convenience, the index of the vertices in $V(G)$ can be extended to any integer; we regard two vertices v_i and v_j as the same vertex if i is congruent to j modulo $n + 2$. The case $v_0v_n \in E(G)$ is already treated in Theorem 9, so we assume that $v_0v_n \notin E(G)$. Since P and Q are reversible, both v_0 and v_n are adjacent to v_{n+1}. If $v_1v_{n+1} \in E(G)$ or $v_{n-1}v_{n+1} \in E(G)$, then $P \dashrightarrow Q$, we thus assume that $v_1v_{n+1}, v_{n-1}v_{n+1} \notin E(G)$.

If there are no edges between v_i and v_{i+2} for any i, $1 \leqslant i \leqslant n+1$, then the path P cannot stray out of the orbit $P \overset{v_{n+1}}{\to} \overset{v_0}{\to} \overset{v_1}{\to} \cdots \overset{v_n}{\to} P \overset{v_{n+1}}{\to} \cdots$, and this contradicts the reversibility of P. Hence, there is at least one edge between v_i and v_{i+2}. Let $v_t v_{t+2}$ be the edge that first appears in the sequence of the pairs, i.e., $v_t v_{t+2} \in E(G)$ and $v_{i-1}v_{i+1} \notin E(G)$ for $0 \leqslant i \leqslant t$. We define a sequence of n-paths $R_1, R_2, \ldots, R_t, R_{t+1}$ inductively as follows: We first set

$$R_1 = \langle v_0 v_1 v_n v_{n-1} v_{n-2} \ldots v_3 v_2 \rangle.$$

We suppose that the ith n-path R_i is already obtained, and denote it by the following:

$$R_i = \langle v_{i-1} v_i v_{i-3} v_{i-4} \ldots v_1 v_0 v_{n+1} v_n \ldots v_{i+1} \rangle.$$

For $1 \leqslant i \leqslant t$, v_{i+1} is adjacent to v_{i-2} since $v_{i-1}v_{i+1} \notin E(G)$. And then the next n-path can be defined.

$$R_{i+1} = \langle v_i v_{i+1} v_{i-2} v_{i-3} \ldots v_1 v_0 v_{n+1} v_n \ldots v_{i+2} \rangle.$$

While the paths are defined, the edges $v_2v_{n+1}, v_3v_0, \ldots, v_tv_{t-3}, v_{t+1}v_{t-2}$ are also obtained one after another.

We will show that the index t is even; otherwise the graph G has the edges $v_2v_{n+1}, v_3v_0, \ldots, v_tv_{t-3}, v_{t+1}v_{t-2}$, v_tv_{t+2}, and we have the following sequence:

$$\begin{aligned}
&P \overset{v_{n+1}}{\to} \overset{v_0}{\to} \overset{v_1}{\to} \cdots \overset{v_t}{\to} \langle v_{t+2} v_{t+3} \ldots v_{n-1} v_n v_{n+1} v_0 v_1 v_2 \ldots v_{t-2} v_{t-1} v_t \rangle \\
&\quad \overset{v_t}{\leftarrow} \overset{v_{t+1}}{\leftarrow} \langle v_{t+1} v_t v_{t+2} v_{t+3} \ldots v_{n-1} v_n v_{n+1} v_0 v_1 v_2 \ldots v_{t-2} \rangle \\
&\quad \overset{v_{t-2}}{\leftarrow} \overset{v_{t-1}}{\leftarrow} \overset{v_{t-4}}{\leftarrow} \overset{v_{t-3}}{\leftarrow} \cdots \overset{v_1}{\leftarrow} \overset{v_2}{\leftarrow} \overset{v_{n+1}}{\leftarrow} \overset{v_0}{\leftarrow} \langle v_0 v_{n+1} v_2 v_1 v_4 v_3 \ldots v_{t-3} v_{t-4} v_{t-1} v_{t-2} v_{t+1} v_t v_{t+2} v_{t+3} \ldots v_{n-2} v_{n-1} \rangle \\
&\quad \overset{v_0}{\to} \overset{v_{n+1}}{\to} \overset{v_n}{\to} \overset{v_1}{\to} \overset{v_2}{\to} \overset{v_3}{\to} \cdots \overset{v_{t-1}}{\to} \overset{v_t}{\to} \overset{v_{t+1}}{\to} \cdots \overset{v_{n-1}}{\to} \overset{v_0}{\to} Q.
\end{aligned}$$

we thus assume that t is even. We deduce a similar fact for the other side: let $v_{t'}v_{t'+2}$ be the edge that last appears in the sequence of the pairs of v_i and v_{i+2}, i.e., $v_{t'}v_{t'+2} \in E(G)$ and $v_iv_{i+2} \notin E(G)$ for $t' < i \leqslant n-1$. We can similarly deduce $P \dashrightarrow Q$ if $n-t'$ is odd, so we deduce that $n-t'$ is even.

We have known that v_1v_n, v_2v_{n+1}, $v_3v_0, \ldots, v_tv_{t-3}$, $v_{t+1}v_{t-2}$, $v_tv_{t+2} \in E(G)$ and that $v_{t'}v_{t'+2}$, $v_{t'+1}v_{t'+4}$, $v_{t'+2}v_{t'+5}, \ldots, v_{n-3}v_n$, $v_{n-2}v_{n+1}$, $v_{n-1}v_0 \in E(G)$.

We assume that $t+2 \leqslant t'$. Then we consider the following n-path:

$$S = \langle v_{t+1} v_t v_{t+2} v_{t+3} \ldots v_{t'-1} v_{t'} v_{t'+2} v_{t'+1} v_{t'+4} v_{t'+3} \ldots v_n v_{n-1} v_0 v_{n+1} v_2 v_1 v_4 v_3 \ldots v_{t-4} v_{t-5} v_{t-2} v_{t-1} \rangle.$$

The vertex v_{t-1} is adjacent to v_{t-3} or v_{t+1}, however, this contradicts $v_{i-1}v_{i+1} \notin E(G)$ for $0 \leqslant i \leqslant t$. We thus deduce that $t \leqslant t' < t+2$.

Case 1: We assume $t'=t$. There is only one edge between v_i and v_{i+2}, $0 \leqslant i \leqslant n+1$, which is the edge $v_tv_{t+2}=v_{t'}v_{t'+2}$. In this case, both n and t are even, and $v_{t+1}v_{t-2}, v_tv_{t-3}, \ldots, v_2v_{n+1}, v_1v_n, v_0v_{n-1}, v_{n+1}v_{n-2}, \ldots, v_{t+5}v_{t+2}, v_{t+4}v_{t+1}$, $v_tv_{t+2} \in E(G)$. If $v_{t+4}v_t \in E(G)$, we can define the following n-path:

$$X_1 = \langle v_{t-1} v_{t-2} \ldots v_{t+6} v_{t+5} v_{t+2} v_t v_{t+4} v_{t+1} \rangle,$$

however, this path cannot move since v_{t+1} is adjacent neither to v_{t-1} nor to v_{t+3}, a contradiction. We therefore deduce that $v_{t+4}v_t \notin E(G)$, and consider the following n-path:

$$X_2 = \langle v_{t+2} v_{t+5} v_{t+6} \ldots v_n v_{n+1} v_0 v_1 v_2 \ldots v_{t-1} v_t v_{t+1} v_{t+4} \rangle.$$

If $v_{t+6}v_{t+2} \notin E(G)$, then the path X_2 cannot stray out of the orbit $X_2 \overset{v_{t+3}}{\to} \overset{v_{t+2}}{\to} \overset{v_{t+5}}{\to} \overset{v_{t+6}}{\to} \cdots \overset{v_t}{\to} \overset{v_{t+1}}{\to} \overset{v_{t+4}}{\to} X_2 \overset{v_{t+3}}{\to} \ldots$, and this contradicts the reversibility of X_2. We thus deduce that $v_{t+6}v_{t+2} \in E(G)$. Then we can define the following n-path:

$$X_3 = \langle v_{t-1} v_{t-2} \ldots v_{t+8} v_{t+7} v_{t+4} v_{t+5} v_{t+6} v_{t+2} v_t v_{t+1} \rangle,$$

however, X_3 cannot move since v_{t+1} is adjacent neither to v_{t-1} nor to v_{t+3}, a contradiction.

Case 2: We assume that $t' = t+1$. There are only two edges between v_i and v_{i+2}, $0 \leqslant i \leqslant n+1$, which are the edges v_tv_{t+2} and $v_{t+1}v_{t+3}$. In this case, n is odd and t is even, and $v_{t+1}v_{t-2}, v_tv_{t-3}, \ldots, v_2v_{n+1}, v_1v_n, v_0v_{n-1}$, $v_{n+1}v_{n-2}, \ldots, v_{t+6}v_{t+3}, v_{t+5}v_{t+2}, v_tv_{t+2}, v_{t+1}v_{t+3} \in E(G)$. We set

$$\begin{aligned}
U_1 &= \{v_0, v_2, \ldots, v_{t-4}, v_{t-2}, v_t\} \cup \{v_{t+3}, v_{t+5}, v_{t+7}, \ldots, v_{n-2}, v_n\}, \\
U_2 &= \{v_1, v_3, \ldots, v_{t-3}, v_{t-1}, v_{t+1}\} \cup \{v_{t+2}, v_{t+4}, v_{t+6}, \ldots, v_{n-3}, v_{n-1}, v_{n+1}\}.
\end{aligned}$$

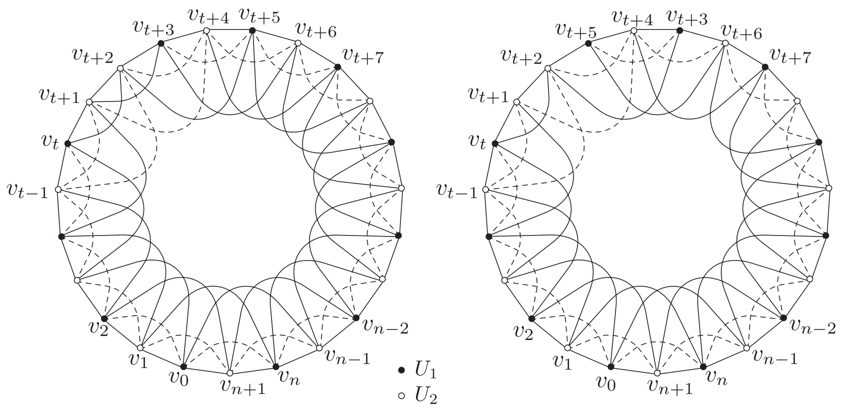

Fig. 13. The fundamental relation and the result of exchanging two vertices v_{t+3} and v_{t+5}.

We will show that each vertex in U_1 is adjacent to each vertex in U_2 and that there are no edges between U_1 and U_2.

If $v_{t-1}v_{t+2} \in E(G)$ or $v_{t+1}v_{t+4} \in E(G)$, then we can deduce that $P \dashrightarrow Q$ in the same way as above, so we assume that $v_{t-1}v_{t+2}, v_{t+1}v_{t+4} \notin E(G)$. Consequently, we have obtained the relations between two vertices v_i and v_j that satisfy $|i-j| \equiv 2, 3$, except $v_t v_{t+3}$. This is called *fundamental relation* (Fig. 13).

Here, let us view from another aspects by exchanging the two vertices v_{t+3} and v_{t+5}. To compare with the fundamental relation, we have three lacking relations: the pairs $v_{t+1}v_{t+5}$, $v_{t+3}v_{t+7}$ and $v_{t+3}v_{t+8}$. By considering the following n-path:

$$Y_1 = \langle v_{t+4}v_{t+3}v_{t+6}v_{t+7} \dots v_{t-1}v_t v_{t+2}v_{t+1}\rangle,$$

we deduce that $v_{t+1}v_{t+5} \in E(G)$ since $v_{t+1}v_{t+4} \notin E(G)$. If $v_{t+3}v_{t+7} \in E(G)$, we can define the following n-path:

$$Y_2 = \langle v_{t+6}v_{t+3}v_{t+7}v_{t+8} \dots v_t v_{t+1}v_{t+5}v_{t+4}\rangle,$$

however, Y_2 cannot move since v_{t+4} is adjacent neither to v_{t+2} nor to v_{t+6}, a contradiction. We hence conclude that $v_{t+3}v_{t+7} \notin E(G)$. By considering the following n-path:

$$Y_3 = \langle v_{t+7}v_{t+6}v_{t+9}v_{t+10} \dots v_t v_{t+1}v_{t+2}v_{t+5}v_{t+4}v_{t+3}\rangle,$$

we deduce that $v_{t+3}v_{t+8} \in E(G)$ since $v_{t+3}v_{t+7} \notin E(G)$.

As a consequence of the exchange, we have got the same form as before, however, which has a little advantage than the fundamental relation; we have found that $v_{t+1}v_{t+5}, v_{t+3}v_{t+8} \in E(G)$ and $v_{t+3}v_{t+7} \notin E(G)$. Symmetrically, we can deduce a similar fact by exchanging two vertices v_t, v_{t-2}.

Furthermore, by exchanging the two consecutive vertices v_{t+2i+1} and v_{t+2i+3} in U_1 for index i, $1 < i < (n-1)/2$, we can also find four lacking relations: the pairs $v_{t+2i+1}v_{t+2i+6}$, $v_{t+2i-2}v_{t+2i+3}$, $v_{t+2i+1}v_{t+2i+5}$ and $v_{t+2i-1}v_{t+2i+3}$. We set

$$Z_1 = \langle v_{t+2i+4}v_{t+2i+5}v_{t+2i+2}v_{t+2i+3}v_{t+2i}v_{t+2i-1} \dots v_{t+2i+7}v_{t+2i+6}\rangle.$$

Since $v_{t+2i+4}v_{t+2i+6} \notin E(G)$, v_{t+2i+6} is adjacent to v_{t+2i+1}. By considering the following n-path:

$$Z_2 = \langle v_{t+2i}v_{t+2i-1}v_{t+2i+2}v_{t+2i+1}v_{t+2i+4}v_{t+2i+5} \dots v_{t+2i-3}v_{t+2i-2}\rangle,$$

we similarly deduce that $v_{t+2i-2}v_{t+2i+3} \in E(G)$.

If $v_{t+2i+1}v_{t+2i+5} \in E(G)$, we can define the following n-path:

$$Z_3 = \langle v_{t+2i+4}v_{t+2i+1}v_{t+2i+5}v_{t+2i+6} \dots v_{t-2}v_{t-1}v_t v_{t+2}v_{t+1}v_{t+4}v_{t+3} \dots v_{t+2i-3}v_{t+2i-4}v_{t+2i-1}v_{t+2i-2}v_{t+2i+3}v_{t+2i+2}\rangle,$$

however, Z_3 cannot move since v_{t+2i+2} is adjacent neither to v_{t+2i} nor to v_{t+2i+4}, a contradiction. We hence deduce that $v_{t+2i+1}v_{t+2i+5} \notin E(G)$. We similarly deduce that $v_{t+2i-1}v_{t+2i+3} \notin E(G)$. As a consequence, the lacking pairs are supplied and the fundamental relation appears again.

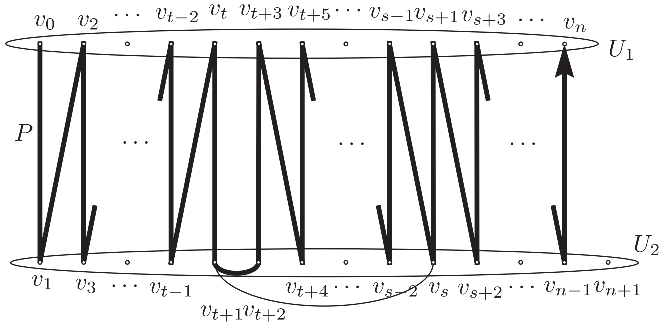

Fig. 14.

As we have seen above, the fundamental relation is obtained again by the results of exchanging the consecutive vertices of U_1. Step by step, exchanging the vertices of U_1 for all over its combination, we deduce that each vertex in U_1 is adjacent to each vertex in U_2 and that there are no edges in U_1.

If U_2 has no edges other than $v_{t+1}v_{t+2}$, then the graph is a complete bipartite graph $K_{(n+1)/2,(n+3)/2}$ with an edge lying in the not smaller partition set. However, this one is not n-reversible for $n \geqslant 5$, and is 3-transferable for $n = 3$. Therefore, U_2 has at least one edge other than $v_{t+1}v_{t+2}$. We consider the two cases whether such an edge is adjacent to $v_{t+1}v_{t+2}$ or not.

We first assume that the edge in U_2 is adjacent to $v_{t+1}v_{t+2}$. Without loss of generality, the edge has v_{t+1} as its end, and let $v_{t+1}v_s$, $s \geqslant t+4$, be such an edge (Fig. 14). In this case, we can deduce $P \dashrightarrow Q$ as follows:

$$
\begin{aligned}
P & \overset{v_{n+1}}{\rightarrow} \overset{v_0}{\rightarrow} \overset{v_1}{\rightarrow} \cdots \overset{v_t}{\rightarrow} \overset{v_{t+1}}{\rightarrow} \langle v_{t+3} v_{t+4} v_{t+5} \ldots v_{n-1} v_n v_{n+1} v_0 v_1 \ldots v_{t-1} v_t v_{t+1} \rangle \\
& \overset{v_{t+3}}{\rightarrow} \overset{v_{t+2}}{\rightarrow} \overset{v_{t+5}}{\rightarrow} \overset{v_{t+4}}{\rightarrow} \cdots \overset{v_{s-1}}{\rightarrow} \overset{v_{s-2}}{\rightarrow} \\
& \langle v_{s+1} v_{s+2} \ldots v_{n-1} v_n v_{n+1} v_0 v_1 \ldots v_{t-1} v_t v_{t+1} v_{t+3} v_{t+2} v_{t+5} v_{t+4} \ldots v_{s-4} v_{s-1} v_{s-2} \rangle \\
& \overset{v_{s+1}}{\rightarrow} \overset{v_{s+2}}{\rightarrow} \overset{v_{s+3}}{\rightarrow} \cdots \overset{v_n}{\rightarrow} \overset{v_{n+1}}{\rightarrow} \overset{v_0}{\rightarrow} \overset{v_1}{\rightarrow} \cdots \overset{v_t}{\rightarrow} \\
& \quad \langle v_{t+1} v_{t+3} v_{t+2} v_{t+5} v_{t+4} \ldots v_{s-4} v_{s-1} v_{s-2} v_{s+1} v_{s+2} \ldots v_{n-1} v_n v_{n+1} v_0 v_1 \ldots v_{t-1} v_t \rangle \\
& \overset{v_s}{\rightarrow} \overset{v_{t+1}}{\rightarrow} \overset{v_{t+2}}{\rightarrow} \overset{v_{t+5}}{\rightarrow} \overset{v_{t+4}}{\rightarrow} \cdots \overset{v_{s-1}}{\rightarrow} \overset{v_{s-2}}{\rightarrow} \\
& \quad \langle v_{s+1} v_{s+2} \ldots v_{n-1} v_n v_{n+1} v_0 v_1 \ldots v_{t-1} v_t v_s v_{t+1} v_{t+2} v_{t+5} v_{t+4} \ldots v_{s-4} v_{s-1} v_{s-2} \rangle \\
& \overset{v_{s+1}}{\rightarrow} \overset{v_{s+2}}{\rightarrow} \cdots \overset{v_{n-2}}{\rightarrow} \overset{v_{n-1}}{\rightarrow} \overset{v_{t+3}}{\rightarrow} \overset{v_{n+1}}{\rightarrow} \overset{v_n}{\rightarrow} \\
& \quad \langle v_1 v_2 \ldots v_{t-1} v_t v_s v_{t+1} v_{t+2} v_{t+5} v_{t+4} \ldots v_{s-4} v_{s-1} v_{s-2} v_{s+1} v_{s+2} \ldots v_{n-2} v_{n-1} v_{t+3} v_{n+1} v_n \rangle \\
& \overset{v_n}{\leftarrow} \overset{v_{n+1}}{\leftarrow} \overset{v_0}{\leftarrow} \overset{v_{n-1}}{\leftarrow} \overset{v_{n-2}}{\leftarrow} \cdots \overset{v_{s+1}}{\leftarrow} \\
& \quad \langle v_{s+1} v_{s+2} \ldots v_{n-2} v_{n-1} v_0 v_{n+1} v_n v_1 v_2 \ldots v_{t-1} v_t v_s v_{t+1} v_{t+2} v_{t+5} v_{t+4} \ldots v_{s-4} v_{s-1} v_{s-2} \rangle \\
& \overset{v_{s-2}}{\leftarrow} \overset{v_{s-1}}{\leftarrow} \cdots \overset{v_{t+5}}{\leftarrow} \overset{v_{t+2}}{\leftarrow} \\
& \quad \langle v_{t+2} v_{t+5} v_{t+4} \ldots v_{s-1} v_{s-2} v_{s+1} v_{s+2} \ldots v_{n-2} v_{n-1} v_0 v_{n+1} v_n v_1 v_2 \ldots v_{t-1} v_t v_s v_{t+1} \rangle \\
& \overset{v_{t+3}}{\leftarrow} \overset{v_{t+1}}{\leftarrow} \overset{v_t}{\leftarrow} \cdots \overset{v_2}{\leftarrow} \overset{v_1}{\leftarrow} \\
& \quad \langle v_1 v_2 \ldots v_{t-1} v_t v_{t+1} v_{t+3} v_{t+2} v_{t+5} v_{t+4} \ldots v_{s-1} v_{s-2} v_{s+1} v_{s+2} \ldots v_{n-2} v_{n-1} v_0 v_{n+1} v_n \rangle \\
& \overset{v_n}{\leftarrow} \overset{v_{n+1}}{\leftarrow} \overset{v_0}{\leftarrow} \overset{v_{n-1}}{\leftarrow} \cdots \overset{v_{s+2}}{\leftarrow} \overset{v_{s+1}}{\leftarrow} \\
& \quad \langle v_{s+1} v_{s+2} \ldots v_{n-2} v_{n-1} v_0 v_{n+1} v_n v_1 v_2 \ldots v_{t-1} v_t v_{t+1} v_{t+3} v_{t+2} v_{t+5} v_{t+4} \ldots v_{s-1} v_{s-2} \rangle \\
& \overset{v_s}{\leftarrow} \overset{v_{s-1}}{\leftarrow} \cdots \overset{v_{t+4}}{\leftarrow} \overset{v_{t+3}}{\leftarrow} \\
& \quad \langle v_{t+3} v_{t+4} \ldots v_{s-1} v_s v_{s+1} v_{s+2} \ldots v_{n-2} v_{n-1} v_0 v_{n+1} v_n v_1 v_2 \ldots v_{t-1} v_t v_{t+1} \rangle \\
& \overset{v_{t+2}}{\leftarrow} \overset{v_{t+1}}{\leftarrow} \cdots \overset{v_1}{\leftarrow} \overset{v_n}{\leftarrow} Q,
\end{aligned}
$$

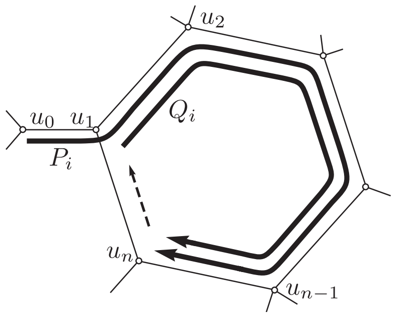

Fig. 15.

hence, the assertion holds. In the other case when the edge in U_2 is not adjacent to $v_{t+1}v_{t+2}$, we can also deduce that $P \dashrightarrow Q$. $\square$

Proposition 21. *Let G be an n-reversible graph and $P = \langle v_0 v_1 v_2 \dots v_{n-2} v_{n-1} v_n \rangle$, $Q = \langle v_n v_1 v_2 \dots v_{n-2} v_{n-1} v_0 \rangle$ two n-paths in G. Then $P \dashrightarrow Q$, that is, $P \propto Q$.*

Proof. The path P cannot be reversible if $V(P) = V(G)$, we therefore assume that there is a vertex not in $V(P)$. We have already seen in Theorems 9, 13, 19 and 20 that P can transfer to Q by a cross flip, so that we can conclude that $P \dashrightarrow Q$. $\square$

Proof of main theorem. The "only if" part is immediate from Definitions 1 and 2. We prove the "if" part by induction on n. The cases $n = 1, 2$ are already shown in Remark 2, so we assume that $n \geqslant 3$ and suppose that the assertion holds for $n - 1$.

We assume that G is n-reversible. We notice that G is $(n-1)$-reversible by Theorem 2, and is also $(n-1)$-transferable by induction.

Let P, P' be any two n-paths in G, and Q, Q' the subpaths of P, P' that have length $n-1$ with $h(P) = h(Q)$, $h(P') = h(Q')$. Since G is $(n-1)$-transferable, there is a sequence of $(n-1)$-paths $Q = Q_0 \to Q_1 \to \cdots \to Q_m = Q'$. For this sequence, if P also has the same sequence, then P can transfer to P' as synchronized with Q_i. However, this is not always possible (Fig. 15).

It happens when Q_i moves to $t(Q_i)$ for some i, then P_i can no longer keep step with Q_i directly. Therefore, we will search another route by taking a roundabout way instead of directly moving to $t(Q_i)$.

Let $P_i = \langle u_0 u_1 u_2 \dots u_n \rangle$ and $Q_i = \langle u_1 u_2 \dots u_n \rangle$. Since P_i is reversible, there is a vertex $w \in V(G) - V(Q_i)$ to which P_i can move by a step. On the other hand, since $P_i' = \langle u_0 u_1 u_n u_{n-1} \dots u_2 \rangle$ is reversible, there is a vertex $w' \in V(G) - V(Q_i)$ to which P_i' can move by a step. If $w \neq w'$, we have the following sequence:

$$\begin{aligned} P_i &\overset{w}{\to} \langle u_1 u_2 \dots u_n w \rangle \\ &\overset{w'}{>} \langle w' u_2 \dots u_n w \rangle \\ &\overset{u_1}{<} \langle w' u_2 \dots u_n u_1 \rangle =: P_{i+1}. \end{aligned}$$

Let the last n-path be P_{i+1}. The path P_{i+1} contains $Q_{i+1} = \langle u_2 \dots u_n u_1 \rangle$ as a subpath, so can keep step with Q_i. If $w = w'$,

$$\begin{aligned} P_i &\overset{w}{\to} \langle u_1 u_2 \dots u_n w \rangle \\ &\propto \langle w u_2 \dots u_n u_1 \rangle =: P_{i+1}. \end{aligned}$$

Let the last n-path be P_{i+1}. This path also contains Q_{i+1}. Anyway, we have a sequence $P = P_0 \dashrightarrow P_1 \dashrightarrow \cdots \dashrightarrow P_m$ such that $Q_i \subset P_i$, $h(P_i) = h(Q_i)$ for each i. We may last consider the case that P_m does not have the same tail as P', however, we can deduce $P_m \dashrightarrow P'$ by its tail flip.

As a consequence, any two n-paths in G can transfer from one to another. We establish this theorem. $\square$

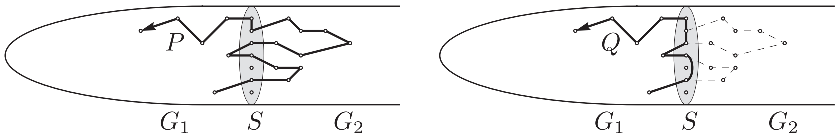

Fig. 16.

4. Union of graphs

If G is a graph with induced subgraphs G_1, G_2 and S such that $G = G_1 \cup G_2$ and $S = G_1 \cap G_2$, we say that G arises from G_1 and G_2 by *pasting* these graphs together along S.

Theorem 22. *If G is obtained from two n-transferable graphs G_1 and G_2 by pasting them together along their complete subgraphs, then G is n-transferable.*

Proof. Let P be an arbitrary n-path in G. It is sufficient to show that P is reversible. If P is fully contained in G_1 or G_2, then P is reversible, we thus assume that P crosses the complete subgraph S where they intersect. Without loss of generality, we assume that $h(P)$ is lying in G_1.

Replacing the subpaths of P buried under G_2 by edges of S, we obtain a new path Q (see Fig. 16). We notice that the length of Q, say l, is less than n. By Lemma 5, the path Q is contained in some $(l + 1)$-path in G_1 and let Q^+ be one of such paths.

If $t(Q) = t(Q^+)$, then P can take a step to $h(Q^+)$. If $h(Q) = h(Q^+)$, then there is a vertex in $V(G_1) - V(Q)$ to which Q^+ can move by a step, and then P can also take a step to the vertex. Anyway, continuing in this way, we will have an n-path in G_1 to which P can transfer. The path is reversible since G_1 is reversible, and by Proposition 1, P is reversible. $\square$

Acknowledgements

The author would like to express his thanks to professors S. Suzuki and K. Taniyama for their useful advice, and also to the referees for their helpful comments.

References

[1] H.J. Broersma, C. Hoede, Path graphs, J. Graph Theory 13 (4) (1989) 427–444.
[2] H.J. Broersma, X. Li, Directed path graphs, Preprint University of Twente, 1996.
[3] R. Diestel, Graph Theory, Springer, Berlin, 1993.
[4] N. Robertson, P. Seymour, R. Thomas, Kuratowski chains, J. Combin. Theory Ser. B 64 (2) (1995) 127–154.
[5] N. Robertson, P. Seymour, R. Thomas, Linkless embeddings of graphs in 3-space, Bull. Amer. Math. Soc. 28 (1) (1993) 84–89.