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NOTE

BIPARTITE REGULAR GRAPHS AND SHORTNESS PARAMETERS

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By constructing sequences of non-Hamiltonian graphs it is proved that (1) for $k \geqslant 4$, the class of k-connected k-valent bipartite graphs has shortness exponent less than one and (2) the class of cyclically 4-edge-connected trivalent bipartite graphs has shortness coefficient less than one.

1. Introduction

In this paper, a *graph* has neither loops nor multiple edges. A *multigraph* has no loops but may have multiple edges. For any graph G, $v(G)$ denotes the order (number of vertices) and $h(G)$ the circumference (length of a maximum cycle). For any infinite class of graphs $\mathcal{G}$, the *shortness exponent* $\sigma(\mathcal{G})$ and *shortness coefficient* $\rho(\mathcal{G})$ are defined as follows [3]:

$$\sigma(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\log h(G)}{\log v(G)}, \qquad \rho(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{h(G)}{v(G)}.$$

Both parameters lie between 0 and 1 inclusive and, since $\rho = 0$ when $\sigma < 1$, at most one of them is of interest for any particular class $\mathcal{G}$. We denote a path with end vertices u and v by $P(u, v)$. For other definitions and notation see, for instance, [1].

Let $\mathcal{B}_k$ denote the class of all k-connected regular k-valent bipartite graphs and $\mathcal{C}_r$ the class of all cyclically r-edge-connected graphs. Several small non-Hamiltonian graphs in $\mathcal{B}_3$ are known, of which the first was the one due to Horton (see [1, p. 240]), which has 96 vertices. The recent example due to Ellingham and Horton [2] has only 54 vertices and is in the class $\mathcal{B}_3 \cap \mathcal{C}_4$.

In this paper we prove two theorems;

Theorem 1. $\sigma(\mathcal{B}_k) < 1$ *for* $k \geqslant 4$.

Theorem 2. $\rho(\mathcal{B}_3 \cap \mathcal{C}_4) < 1$.

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The starting point of each proof is a non-Hamiltonian graph G_0 in $\mathscr{B}_3 \cap \mathscr{C}_4$. We denote the number of vertices in G_0 by g and call vertices in the two sets of the vertex bipartition the *x-vertices* and *y-vertices*. Every cycle in G_0 is even, of length at most $g-2$, and it misses at least one x-vertex and one y-vertex.

2. Proof of Theorem 1

Take G_0 (as above) and convert it into a k-valent bipartite multigraph H_0 by assigning multiplicity $k-2t$ to the edges of a 1-factor F and multiplicity t to all other edges, where $t=\lfloor (k+1)/3 \rfloor$. We require H_0 to be k-edge-connected and, when $k=4$ or $k \geqslant 6$, this follows from the fact that G_0 is cyclically 4-edge-connected, because the sum of multiplicities of any four edges is at least k. When $k=5$, the edges of F remain as single edges in H_0 and F must be chosen so that it does not contain four edges whose deletion would separate G_0 into two components, both with cycles. For instance, if G_0 is the graph shown in [2, Fig. 4] then it is a ring of four subgraphs joined by four pairs of edges and a suitable F would be the union of four 1-factors, chosen separately in the four subgraphs.

We now convert the multigraph H_0 into a graph J_0 by the method of Meredith [4], which we used previously in [5]. Denote the complete bipartite graph $K_{k,k-1}$ simply by K. Let the k vertices with valency $k-1$ and the $k-1$ vertices with valency k be called *l-vertices* and *m-vertices*, respectively. When a copy of K is an induced subgraph of a k-valent graph G, each l-vertex is incident with one of the edges that join K to $G-K$, so the l-vertices are the *linking vertices* of K. In the multigraph H_0, replace each vertex v by a copy of K, as shown (for $k=4$) in Fig. 1. The k edges originally incident at v become the edges incident at the l-vertices of the copy of K which replaces v. We denote the graph obtained after these substitutions by J_0 and note that it is k-valent and bipartite. By [4, Theorem 3], J_0 is also k-connected, so $J_0 \in \mathscr{B}_k$.

Lemma 1. *Every cycle in J_0 misses at least two m-vertices.*

Proof. Let C be a cycle in J_0 and suppose that it intersects all the copies of K, for

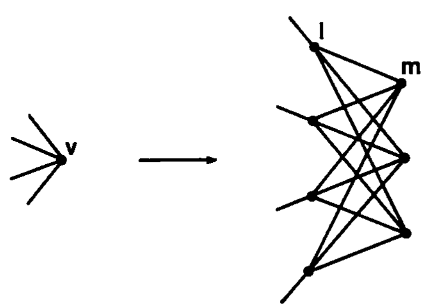

Fig. 1.

otherwise the result is immediate. If we contract all copies of K to single vertices, then J_0 becomes H_0 and C becomes a tour T that contains every vertex of H_0. Since H_0 (like G_0) is non-Hamiltonian, T is not a cycle. As H_0 is bipartite, T contains an even number of edges and hence its length is at least $g+2$. At least two vertices, an x-vertex and a y-vertex, must occur twice in T, so there are at least two copies of K in J_0 such that $C \cap K$ consists of at least two paths. Each component path in $C \cap K$ has l- and m-vertices alternately, with an l-vertex at each end, so it contains one more of the l-vertices than the m-vertices. Since the number of l-vertices in K exceeds the number of m-vertices by only one, C must miss at least one m-vertex in K whenever $C \cap K$ consists of two or more paths. The lemma follows. □

Choose an m-vertex m_0 in J_0 and define $X = J_0 - m_0$.

Lemma 2. *Let G be a graph that contains a copy of X and let C be a cycle in G that intersects both X and $G-X$. Then C misses at least two m-vertices in X.*

Proof. Let T_0 be a tour of J_0 such that $T_0 \cap X = C \cap X$. If $C \cap X$ consists of t paths, then m_0 occurs t times in T_0 and no other vertex occurs more than once. If $t=1$, then T_0 is a cycle and, by Lemma 1, misses at least two m-vertices. Both of these vertices are in X, so C misses two m-vertices in X. If $t>1$, then the $t-1$ extra occurrences of m_0 in T_0 force T_0 to miss $t-1$ more m-vertices in the same copy of K, so C misses more than two m-vertices in X. □

We use X to construct an infinite sequence of graphs $\langle J_n \rangle$, starting with J_0, as follows. For $n \geqslant 0$, let J_{n+1} be a graph obtained from J_n by replacing every m-vertex by a copy of X. For all n, J_n is bipartite, k-valent and k-connected, so $J_n \in \mathcal{B}_k$.

There are $(k-1)g$ m-vertices in J_0 and one fewer in X so J_n contains $(k-1)g[(k-1)g-1]^n$ m-vertices.

Lemma 3. *No cycle in J_n contains more than $[(k-1)g-2][(k-1)g-3]^n$ m-vertices.*

Proof. Let $f(n) = [(k-1)g-2][(k-1)g-3]^n$. By Lemma 1, no cycle in J_0 contains more than $f(0)$ m-vertices, so the lemma holds for $n=0$. Suppose (as induction hypothesis) that the lemma holds for some n and consider any cycle C_{n+1} in J_{n+1}. If we shrink all copies of X to single m-vertices, so that J_{n+1} becomes J_n, then C_{n+1} becomes a tour T_n in which no vertices except m-vertices can occur more than once. If T_n contains $g(n)$ different m-vertices, then C_{n+1} intersects $g(n)$ copies of X and, by Lemma 2, C_{n+1} contains at most $g(n)[(k-1)g-3]$ m-vertices.

In case T_n is not a cycle, we can convert it into a cycle C_n, with more than $g(n)$

m-vertices in it, by replacing repetitions of an m-vertex by different m-vertices belonging to the same copy of K and not already in T_n. This is possible because all $k-1$ of the m-vertices in K are adjacent to all the l-vertices and, as there are only k l-vertices, $T_n \cap K$ contains at most $k-1$ occurrences of m-vertices. By the induction hypothesis, C_n contains no more than $f(n)$ m-vertices, so $g(n) \leqslant f(n)$. Hence C_{n+1} contains no more than $f(n)[(k-1)g-3]=f(n+1)$ m-vertices; that is, the lemma holds for $n+1$. By induction, it holds generally. □

We now have

$$v(J_{n+1})-v(J_n)=(k-1)g[(k-1)g-1]^n(v(X)-1),$$

and, by Lemma 3,

$$h(J_{n+1})-h(J_n) \leqslant [(k-1)g-2][(k-1)g-3]^n(v(X)-1).$$

These recurrence relations have solutions of the form

$$v(J_n)=a[(k-1)g-1]^n+b, \qquad h(J_n) \leqslant c[(k-1)g-3]^n+d,$$

where a, b, c and d are constants such that $a>0$ and $c>0$. Therefore

$$\sigma(\mathscr{B}_k) \leqslant \log[(k-1)g-3]/\log[(k-1)g-1]<1,$$

and this completes the proof of Theorem 1. □

For example, if we take the graph shown in [2, Fig. 4] as G_0 and consider the 4-valent case, then we obtain the inequality of $\sigma(\mathscr{B}_4) \leqslant \log 159/\log 161$.

Although Theorem 1 remains true if $k=3$, there is a simpler construction which gives a better bound for σ. See the Note at the end.

3. Proof of Theorem 2

Let Z denote a subgraph obtained from G_0 by deleting any two adjacent vertices x_0 and y_0, that is, $Z=G_0-K_2$. Let G be any graph in which Z is an induced subgraph with x_1, x_2, y_1 and y_2 (see Fig. 2) as its linking vertices. Let C be any cycle in G that intersects both Z and $G-Z$.

Lemma 4. (1) *If* $C \cap Z$ *is of type* $P(x_1, y_1)$ *or* $P(x_1, y_1) \cup P(x_2, y_2)$, *then* C *misses at least two vertices of* Z.

(2) *If* $C \cap Z$ *is of type* $P(x_1, x_2)$, *then* C *misses a vertex of* Z.

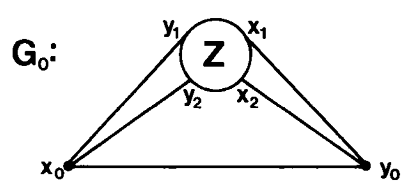

Fig. 2. The subgraph Z in G_0.

Proof. In each case, we define a cycle C_0 in G_0 such that $C_0 \cap Z = C \cap Z$.

(1) If $C \cap Z = P(x_1, y_1)$, let $C_0 = P(x_1, y_1) \cup y_1x_0y_0x_1$. If $C \cap Z = P(x_1, y_1) \cup P(x_2, y_2)$, let $C_0 = P(x_1, y_1) \cup y_1x_0y_2 \cup P(y_2, x_2) \cup x_2y_0x_1$, where we write $P(x_2, y_2)$ as $P(y_2, x_2)$ to exhibit C_0 as a cycle.

Since $v(C_0) \leq g-2$ and C_0 contains x_0 and y_0, C_0 must miss two vertices of Z. As $C \cap Z = C_0 \cap Z$, it follows that C also misses two vertices of Z.

(2) If $C \cap Z = P(x_1, x_2)$, let $C_0 = P(x_1, x_2) \cup x_2y_0x_1$.

Since $v(C_0) \leq g-2$ and C_0 only misses one vertex x_0 outside Z, it must miss a vertex of Z. As $C \cap Z = C_0 \cap Z$, it follows that C misses a vertex of Z. □

Note that Lemma 4 makes no assertion if $C \cap Z$ is of the type $P(x_1, x_2) \cup P(y_1, y_2)$.

For $n > 1$, let R_n consist of n copies of Z (called Z_i, $0 \leq i \leq n-1$) joined to form a ring by $2n$ edges which link the vertices x_1 and x_2 of each Z_i to the vertices y_1 and y_2 of Z_{i+1} (or Z_0, in the case $i = n-1$). The graph R_n is in the class $\mathscr{B}_3 \cap \mathscr{C}_4$. To find an upper bound for $h(R_n)$ we need only consider cycles C which intersect all the copies of Z. There are three cases:

(i) $C \cap Z_i$ is of type $P(x_1, y_1)$ for all i. By Lemma 4(1), $v(R_n) - v(C) \geq 2n$.

(ii) $C \cap Z_0$ is of type $P(x_1, x_2)$, $C \cap Z_i$ if of type $P(x_1, y_1) \cup P(x_2, y_2)$ for $1 \leq i \leq n-2$ and $C \cap Z_{n-1}$ is of type $P(y_1, y_2)$. By Lemma 4, $v(R_n) - v(C) \geq 1 + 2(n-2) + 1$.

(iii) $C \cap Z_0$ is of type $P(x_1, x_2) \cup P(y_1, y_2)$ and $C \cap Z_i$ is of type $P(x_1, y_1) \cup P(x_2, y_2)$ for $1 \leq i \leq n-1$. By Lemma 4, $v(R_n) - v(C) \geq 0 + 2(n-1)$.

Hence

$$h(R_n) \leq v(R_n) - 2(n-1).$$

But $v(R_n) = n(g-2)$, so it follows that

$$\rho(\mathscr{B}_3) \cap \mathscr{C}_4) \leq (g-4)/(g-2) < 1.$$

If we take G_0 to be the graph shown in [2, Fig. 4], then we obtain

$$\rho(\mathscr{B}_3 \cap \mathscr{C}_4) \leq 25/26.$$

4. Note

The author is grateful to a referee for pointing out an error in [6], which invalidates the first result of that paper. In fact (see [6, Fig. 2]) I can be spanned by a pair of cycles, one in H_1 and the other in H_2. Hence (see [6, Fig. 3]) each copy of L in J_1 can be spanned by a pair of paths, one entering and leaving through x-vertices and the other through y-vertices. It is easy to find a spanning cycle of J_1 which contains these paths, so J_1 is Hamiltonian.

The construction used for the second result in [6] remains valid, provided that we use a different graph as starting point. With the graph of Ellingham and

Horton [2, Fig. 4] in place of J_1 we obtain (in our new notation) the improved inequality

$$\sigma(\mathscr{B}_3) \leq \log 26/\log 27 < 1.$$

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