GRADED GENERALIZATIONS OF WEYL- AND CLIFFORD ALGEBRAS

Hans TILGNER
Institut für Mathematik III der Freien Universität Berlin, Germany

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Graded skew bilinear forms (,) on graded vector spaces \( V \) are defined such that their restrictions to the even resp. odd subspaces are skew resp. odd. Over such graded symplectic vector spaces a (universal) factor algebra of the tensor algebra of \( V \) is described which reduces to a Weyl resp. Clifford algebra if only one even resp. odd subspace is nontrivial. Introducing the total graduation on this polynomial algebra and graded symmetrization it is shown that the elements up to second power are closed under graded commutation. If the graduation is of type \( \mathbb{Z}_2 \) the elements of second power are a Lie-graded algebra and this is the only graduation for which this is true. The graded commutation relations of this algebra are calculated. It is isomorphic to the graded symplectic algebra of \((V,(,))\) which is contained in the graded derivation algebra of the graded Heisenberg algebra of elements up to first power.

1. Introduction to graded symplectic vector spaces and Lie-graded algebras

Let \( K \) be a field of characteristic zero, \( V \) be a graded \( K \)-vector space of type \( \Delta \) where \( \Delta \) is one of the commutative rings \( \mathbb{Z} \) or \( \mathbb{Z}_2 \). A bilinear form \((,\)\) on the direct sum

\[ V = \bigoplus_{i \in \Delta} V_i \]

is said to be graded skew if \( \{y, x_i\} = -(-1)^{i1}(x_i, y) \) for \( x_i \) in \( V_i \), \( y \) in \( V \). Examples on finite-dimensional vector spaces are given by taking \text{diag}(\ldots, I, \ldots) \) as matrix of \((,\)\) in some basis, where the square matrix \( I \) has the desired symmetry properties. In this case the above direct sum is \((,\)\)-orthogonal. \((V,(,))\) is said to be a graded symplectic vector space if \((,\)\) is graded skew and nondegenerate. A graded derivation of degree \( i \) of \((V,(,))\) is a graded endomorphism \( D^{(i)} \) on \( V \) of degree \( i \), i.e. \( D^{(i)} V_k \subset V_{k+i} \) for all \( k \) in \( \Delta \), such that for \( x_k \) in \( V_k \), \( a \) in \( V \)

\[ (D^{(i)} x_k, a) + (-1)^{ik} (x_k, D^{(i)} a) = 0. \]

The \( D^{(i)} \) span a subspace \( \text{der}_i(V,(,)) \) of the space \( \text{End}_i V \) of all graded endomorphisms of degree \( i \) on \( V \), hence

\[ \text{der}_i(V,(,)) = \bigoplus_{i \in \Delta} \text{der}_i(V,(,)) \]
is a subspace of
\[ \text{end}^* V = \bigoplus_{i \in \Delta} \text{end}_i V. \]

A Lie-graded algebra of type \( \Delta \) is a graded algebra \((V, [ , ]_z)\) of type \( \Delta \) such that for \( x_k \) in \( V_k \), \( y_l \) in \( V_l \) and \( a \) in \( V \)

\[ [x_k, y_l]_z = -(-1)^{|y_l|}[y_l, x_k]_z \quad \text{(graded antisymmetry)} \]

\[ [[x_k, y_l], a]_z = [x_k, [y_l, a]_z]_z - (-1)^{|y_l|}[y_l, [x_k, a]_z]_z \quad \text{(graded Jacobi identity)} \]

Lie-graded algebras were studied in [7], [12], [5], [10], [8], [2], [4], [13], [14]; more references are given in [4]. Examples are the following [3, proposition 1 in chap. III §10 no. 4] and [13]: (i) \( \text{end}^* V \) with the graded commutator \( 2[D^{(k)}, D^{(l)}]_z = D^{(k)}D^{(l)} - (-1)^{k+l}D^{(l)}D^{(k)} \); (ii) \( \text{der}^*(V, ( , )) \) for any bilinear form \(( , )\) on \( V \) is a subalgebra of \( \text{end} V \). If \((V, ( , ))\) is graded symplectic we write \( \text{sp}^*(V, ( , )) \) for this graded symplectic algebra of \((V, ( , ))\); (iii) if \((V, \Box)\) is a graded algebra, the algebra of graded derivations

\[ \text{der}^*(V, \Box) = \bigoplus_{i \in \Delta} \text{der}_i (V, \Box) \]

where \( \text{der}_i (V, \Box) \) is the subspace of \( \text{end}_i V \) spanned by the \( D^{(i)} \) with

\[ D^{(i)}(x_k \Box a) = (D^{(i)}x_k) \Box a + (-1)^{|x_k|} x_k \Box (D^{(i)}a), \]

is another type of subalgebra of \( \text{end}^* V \); (iv) \(( , )\) being graded skew and \( \gamma_k \) in the \( k \)th copy of \( K \) in \( \bigoplus_{i \in \Delta} K_{(i)} \), a \( \gamma \)-dependent Lie-graded structure on \( \bigoplus_{i \in \Delta} (K_{(i)} \oplus V_i) \) is defined by

\[ [\alpha_k \oplus x_k, \beta_l \oplus y_l]_z = \gamma_{k+l}(x_k, y_l). \]

Its zero-component obviously is the Heisenberg Lie algebra of the symplectic vector space \((V_0, ( , ))\); however it does not contain the defining Jordan brackets of a Clifford algebra [6, p. 232], for instance on \( V_1 \) with respect to the symmetric bilinear form \(( , )\). It is easy to prove that the trivially prolonged algebra \( \text{der}^*(V, ( , )) \) is a subalgebra of the graded derivation algebra of this example.

In the following a universal algebra over \((V, ( , ))\), \(( , )\) being graded skew, is constructed. It restricts to the Weyl- resp. Clifford algebra if only \( V_0 \) resp. \( V_1 \) are nontrivial. If \(( , )\) vanishes identically the graded symmetric algebra over \( V \) results, which in the two special cases reduces to the symmetric algebra over \( V_0 \) resp. the exterior algebra over \( V_1 \). Although this generalized Weyl algebra is not a graded algebra the elements up to second power are closed with respect to a suitable graded commutator. In the \( \mathbb{Z}_2 \)-case and if \( V_0 \oplus V_1 \) is \(( , )\)-orthogonal the subspace of graded symmetrized elements of second power is a Lie-graded algebra isomorphic to \( \text{der}^*(V, ( , )) \). The isomorphism is the restriction to \( V \) of the graded left multiplication \( \text{ad}^\Box \) with respect to the graded commutator. This is a graded
generalization of the well known linear canonical transformation theory for bosons and fermions in quantum field theory [1, chap. II and III].

2. The Weyl algebra over a graded symplectic vector space

Following [9], see also [11], we define a functor weyl\(^\oplus\) from the category of graded symplectic vector spaces into the category of associative algebras (over the same ground field \(\mathbb{K}\)): Let \(\mathcal{J}^{1-}\) be the two-sided ideal in the tensor algebra \(\text{ten} V\) over \(V\) generated by the elements \(x_k \otimes y_l - (-1)^{kl} y_l \otimes x_k - 2(x_k, y_l)\) where \(x_k\) is in \(V_k\), \(y_l\) in \(V_l\) (\(V_l\) being identified to its image in \(\text{ten} V\)). The Weyl algebra over \((V, \{,\})\) is the quotient \(\text{weyl}(V, \{,\} )\) of \(\text{ten} V\) by \(\mathcal{J}^{1-}\). This algebra can also be defined as solution of the following universal mapping problem: Given \((V, \{,\})\) one considers the pairs \((A, \omega)\) where \(A\) is an associative \(\mathbb{K}\)-algebra with identity element \(e\) and \(\omega\) a \(\mathbb{K}\)-linear mapping of \(V\) into \(A\) with

\[\omega(x_k)\omega(y_l) - (-1)^{kl}\omega(y_l)\omega(x_k) = 2(x_k, y_l)e.\]

The \(\text{weyl}(V, \{,\} )\) is a solution of the universal mapping problem defined by the \((A, \omega)\), i.e. every mapping \(\omega\) of type (4) factorizes uniquely through \(\text{weyl}(A, \omega)\), i.e. every mapping \(\omega\) of type (4) factorizes uniquely through \(\text{weyl}(V, \{,\} )\) in the form \(\omega = \theta \circ \iota\) where \(\iota\) is the injection \(V \rightarrow \text{ten} V \rightarrow \text{ten} V/\mathcal{J}^{1-}\) and \(\theta\) is a morphism of associative \(\mathbb{K}\)-algebras with identity elements. Especially \(\text{weyl}(V, 0)\) is the graded symmetric algebra \(S^\oplus(V)\) over \(V\). Since \(\text{weyl}(V, \{,\} )\) is the solution of a universal mapping problem, \(\text{weyl}(V, \{,\} )\) is a covariant functor from the category of graded symplectic vector spaces, whose morphisms are the \((,\)-preserving graded isomorphisms of degree 0, into the category of associative graded \(\mathbb{K}\)-algebras. In the following \(V\) and its image in \(\text{weyl}(V, \{,\} )\) will be identified.

[3, proposition 7 in chap. III §5 no. 5] shows that there is a unique graduation of type \(\Delta \times \mathbb{N}\) on \(\text{ten} V\), compatible with the algebra structure, which induces on \(\text{Ke}\) the trivial graduation, on \(V\) the given graduation \(\Delta\) and on the tensors of power \(n\), \(\text{ten}^n V\), the total graduation, loc. sit. chap. II §11 no. 1. However \(\mathcal{J}^{1-}\) is not a graded ideal unless \((,\) vanishes: \(x_k \otimes y_l - (-1)^{kl} y_l \otimes x_k\) is of degree \((k + l, 2)\) whereas the element \(2(x_k, y_l)e\) is of degree \((0, 0)\). Hence the canonical epimorphism onto \(\text{weyl}(V, \{,\} )\) is not graded unless \((,\) vanishes and unless it is \(S^\oplus(V)\), \(\text{weyl}(V, \{,\} )\) is not a graded algebra of type \(\Delta \times \mathbb{N}\) and not the solution of a universal graded mapping problem.

In the following multiplication in \(\text{weyl}(V, \{,\} )\) will be written without a symbol and homogeneous elements in \(V\) will have their degree in \(\Delta\) as an index. The definition of \(\mathcal{J}^{1-}\) implies

\[x_k y_l - (-1)^{kl} y_l x_k = 2(x_k, y_l)e.\]

Writing \(\sigma_0\) resp. \(\tau_1\) for the restrictions of \((,\) to \(V_0\) resp. \(V_1\), we get
\(x_0 y_0 - y_0 x_0 = 2\sigma_0(x_0, y_0)\)

\[6\]
\[x_1 y_1 + y_1 x_1 = 2\tau_1(x_1, y_1),\]

which shows that the subalgebras of \(\text{weyl}^\ast(\mathcal{V}, (., .))\) generated by \(\mathcal{V}_0\) resp. \(\mathcal{V}_1\) are Weyl resp. Clifford algebras over \((\mathcal{V}_0, \sigma_0)\) resp. \((\mathcal{V}_1, \tau_1)\).

3. Graded symmetrized elements of second power

For \(x_k\) in \(\mathcal{V}_k\) and \(y_\ell\) in \(\mathcal{V}_\ell\) we write

\[2x_k y_\ell = x_k y_\ell + (-1)^{kl} y_\ell x_k + x_k y_\ell - (-1)^{kl} y_\ell x_k = 2\lambda_{k\ell} y_\ell y_\ell + 2\{x_k, y_\ell\}\]

with \(\lambda_{k\ell} = (-1)^{kl} \lambda\). Let \(\Lambda \mathcal{V}_k \mathcal{V}_\ell\) denote the subspace of the Weyl algebra spanned by the \(\lambda_{k\ell}\), and \(\Lambda^2 \mathcal{V}\) the direct sum of all those subspaces. Then \(\Lambda^2 \mathcal{V}\) has the total graduation of type \(\Delta\)

\[7\]
\[\Lambda^2 \mathcal{V} = \bigoplus_{k \in \Delta} \Lambda \mathcal{V}_k \mathcal{V}_\ell,\]

[3, remark in chap. III §11 no. 5]. If \(\mathcal{V}\) is finite-dimensional we have \(\dim \Lambda \mathcal{V}_k \mathcal{V}_\ell = \dim \mathcal{V}_k \dim \mathcal{V}_\ell\) if \(k \neq \ell\) and \(\dim \Lambda \mathcal{V}_k \mathcal{V}_k = \frac{1}{2} \dim \mathcal{V}_k \dim \mathcal{V}_k \pm 1\) if \(k\) is even resp. odd. If \(\Delta = \mathbb{Z}\) and \(\mathcal{V}\) has \(p\) nontrivial subspaces \(\mathcal{V}_l\) then \(\Lambda^2 \mathcal{V}\) has \((p^2 - 1) = \frac{1}{2} (p + 1)\) nontrivial subspaces, if \(\Delta = \mathbb{Z}_2\) the number of nontrivial subspaces in \(\mathcal{V}\) and \(\Lambda^2 \mathcal{V}\) is the same.

Taking graded commutators with respect to the (total) graduation \(\Delta\) on the subspaces \(\mathcal{V}\) and \(\Lambda^2 \mathcal{V}\) in \(\text{weyl}^\ast(\mathcal{V}, (., .))\) we get

\[8\]
\[(A_{x_k} y_\ell) z_m = (-1)^{k+l+m} z_m (A_{x_k} y_\ell) = x_k [y_\ell, z_m]_z + (-1)^{kl} y_\ell [x_k, z_m]_z + (-1)^{lm} [x_k, z_m]_z y_\ell,\]

\[9\]
\[2[A_{x_k} y_\ell, z_m]_z = \{1 + (-1)^{kl+m}\} \{y_\ell, z_m\} x_k + (-1)^{kl}\{1 + (-1)^{kl+m}\} \{x_k, z_m\} y_\ell,\]

hence \([\Lambda^2 \mathcal{V}, \mathcal{V}]_z \subset \mathcal{V}\). From this we get

\[(A_{x_k} y_\ell) z_m w_r = (-1)^{k+l+m} z_m w_r (A_{x_k} y_\ell) =\]

\[= (\lambda)^{k+l+m} z_m [A_{x_k} y_\ell, w_r]_z + [A_{x_k} y_\ell, z_m]_z w_r.\]

From this and (9) a simple but tedious calculation gives

\[4[A_{x_k} y_\ell, A z_m w_r]_z = 2[A_{x_k} y_\ell, A z_m w_r, (-1)^{k+l+m} A z_m w_r, A_{x_k} y_\ell]_z =\]

\[= \{1 + (-1)^{kl+m}\} \{1 + (-1)^{l+m}\} \{y_\ell, z_m\} A x_k w_r + (-1)^{kl} \{1 + (-1)^{k+m}\} \{x_k, z_m\} A y_\ell w_r +\]

\[+ (-1)^{lm} \{1 + (-1)^{l+m}\} \{y_\ell, w_r\} A x_k z_m,\]

\[10\]
If the direct sums in $V$ are $(,)$-orthogonal the two last multiples of the identity element vanish, i.e. $[A^2V, A^2V]_z \subseteq A^2V$. In this case the factors $I \cdots$ all are equal to 2 and the graded commutation relations (10) reduce to those given for the linear transformations $S(x_k, y_l)$

$$S(x_k, y_l) = \{y_b, a\}x_k + (-1)^{k+l}(x_k, a)y_l$$

in [14]. If $\Delta = Z_2$ the total graduation $Z_2$ on $\Lambda^2V$ is given by $(\Lambda^2V)_0 = \Lambda V_0 \oplus \Lambda V_1$ and $(\Lambda^2V)_1 = \Lambda V_0 V_1$. In this case it is easy to check by writing down the special cases that $[(\Lambda^2V)_k, (\Lambda^2V)_l]_z \subseteq (\Lambda^2V)_{k+l}$. Hence under these two assumptions for $\Delta$ and $V$ (10) are the graded commutation relations of a Lie-graded algebra on $\Lambda^2V$. Conversely, let $\Lambda^2V$ be a Lie-graded algebra with respect to the total graduation $Z$ or $Z_n$ for some $n$. Then $\Lambda V_i V_i \subseteq (\Lambda^2V)_{2i}$ for all $i$. On the other hand, if the direct sums in $V$ are $(,)$-orthogonal (10) implies $[\Lambda V_i V_i, \Lambda V_i V_i]_z \subseteq \Lambda V_i V_i$, unless $(,)$ vanishes identically, i.e. $\Lambda V_i V_i \subseteq (\Lambda^2V)_{2i} \cap (\Lambda^2V)_i$ for all $i$, which means that a nontrivial $\Delta$ must be $Z_2$. If $(,)$ vanishes identically $\Lambda^2V$ has the trivial graded commutative Lie-graded structure for any of the above $\Delta$.

If the direct sums in $(,)$-orthogonal we see from (9) that

$$\text{ad}^*(\Lambda x_k y_l)|_V z_m = [\Lambda x_k y_l, z_m]_z = S(x_k, y_l)z_m,$$

and from (9) the $S(x_k, y_l)$ fulfill condition (1) for $D^{(k+l)}$. Moreover, an inspection of the special cases shows that if $\Delta = Z_2$ $S(x_k, y_l)$ is in $\text{end}_{x_k} V$ (again the $Z_2$ graduation is the only one for which this holds: this assumption implies $[\Lambda V_i V_i, \Lambda V_i V_i]_z \subseteq \Lambda V_i V_i$; on the other hand, unless $(,)$ vanishes identically, $[\Lambda V_i V_i, \Lambda V_i V_i]_z \subseteq V_i$ from (9), hence $\Delta = Z_2$), which implies $\text{ad}^*(\Lambda^2V)|_V \subseteq \text{der}^*(V, (,))$. If $V$ has the finite dimension $n$ the dimensions of $\Lambda^2V$ and $\text{der}^*(V, (,))$ are both $\frac{1}{2} n (n + 1) - \text{dim} V$. Summarizing it is proved that

(13) Theorem. If $\Delta = Z_2$, $V$ has finite dimension, $(,)$ is nondegenerate and the decomposition $V = V_0 \oplus V_1 (,)$-orthogonal, then

$$\text{ad}^*(\cdot)|_V : \Lambda^2V \rightarrow \text{der}^*(V, (,))$$

is an isomorphism of Lie-graded algebras.

Taking the graded commutator with respect to the total graduation on $K \oplus V$ we get a Lie-graded subalgebra of type (iv) in $\text{weyl}^*(V, (,))$ with only one copy of $K$ and $\gamma_0 = 1$. However since the total is the trivial graduation on $K$, brackets of type $[\alpha_0, x_k]$ are all skew. Under the above isomorphism $\Lambda^2V$ acts on this algebra as graded derivations and one can form the $\text{ad}^*(\cdot)|_V$-semidirect sum described in [13], which now contains all elements of the Weyl algebra up to second power.
Taking a complex structure \( J \) on \( V \) one can consider elements in \( \Lambda^2 V \) of the form
\[
\Lambda x_i y_i + Jx_i y_i J^{-1},
\]
and it is clear that the isomorphism (13) maps the subspace spanned by these elements in \( \Lambda^2 V \) onto the pseudo-unitary subalgebra of \( \text{der}^+(V,\{\},\)) described in [14]. Under the assumptions of Theorem (13) a complex structure in \( \text{der}_0(V,\{\},\)) can be found by taking two symplectic bases of the form \( \{q_1, \ldots, q_n, p^1, \ldots, p^n\} \) in \( V_0 \) and \( V_1 \) and choosing \( J = \text{ad}^+ (H) \big|_V \), where \( H \) in \( \Lambda^2 V \) is the sum of two elements of the form \( \sum_{k=0}^{n} (q_k q_k + p^k p^k), i = 0, 1. \)

References