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On non-trivial barrier solutions of the dividend problem for a diffusion under constant and proportional transaction costs

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Abstract

In Bai and Paulsen [L. Bai, J. Paulsen, Optimal dividend policies with transaction costs for a class of diffusion processes, SIAM J. Control Optim. 48 (2010) 4987–5008] the optimal dividend problem under transaction costs was analyzed for a rather general class of diffusion processes. It was divided into several subclasses, and for the majority of subclasses the optimal policy is a simple barrier policy; whenever the process hits an upper barrier $\bar{u}^*$, reduce it to $\bar{u}^* - \xi$ through a dividend payment. After transaction costs, the shareholder receives $k\xi - K$.

It was proved that a simple barrier strategy is not always optimal, and here these more difficult cases are solved. The optimal solutions are rather complicated, but interesting.

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1. Introduction and problem formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a probability space satisfying the usual conditions, i.e. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and P-complete. Assume that the uncontrolled surplus process

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follows the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

where W is a Brownian motion and $\mu(x)$ and $\sigma(x)$ are Lipschitz-continuous. Let the company pay dividends to its shareholders, but at a fixed transaction cost $K > 0$ and a tax rate $1 - k < 1$, so that $k > 0$. We will allow $k > 1$, opening up for other interpretations than that $1 - k$ is a tax rate. This means that if $\xi > 0$ is the amount the capital is reduced by, then the net amount of money the shareholders receive is $k\xi - K$. It can be argued that taxes are paid on dividends after costs, so an alternative would be to use $k(\xi - K) = k\xi - kK$, but clearly this is just a reparametrization. Furthermore, different investors may have different tax rates, so $1 - k$ should be interpreted as an average tax rate.

Since every dividend payment results in a fixed transaction cost, the company should not pay out dividends continuously, but only at discrete time epochs. Therefore, a strategy can be described by

$$\pi = (\tau_1^\pi, \tau_2^\pi, \ldots, \tau_n^\pi, \ldots; \xi_1^\pi, \xi_2^\pi, \ldots, \xi_n^\pi, \ldots),$$

where τ_n^π and ξ_n^π denote the times and amounts of dividends. Thus, when applying the strategy π, the resulting surplus process X_t^π is given by

$$X_t^\pi = x + \int_0^t \mu(X_s^\pi)ds + \int_0^t \sigma(X_s^\pi)dW_s - \sum_{n=1}^{\infty} 1_{\{\tau_n^\pi < t\}} \xi_n^\pi.$$

Note that this makes X^π left continuous, so that $\xi_n^\pi = X_{\tau_n^\pi}^\pi - X_{\tau_n^\pi +}^\pi$.

Definition 1.1. A strategy π is said to be admissible if

(i) $0 \le \tau_1^\pi$ and for $n \ge 1$, $\tau_{n+1}^\pi > \tau_n^\pi$ on $\{\tau_n^\pi < \infty\}$.
(ii) τ_n^π is a stopping time with respect to $\{\mathcal{F}_t\}_{t\ge 0}$, $n = 1, 2, \ldots$
(iii) ξ_n^π is measurable with respect to $\mathcal{F}_{\tau_n^\pi +}$, $n = 1, 2, \ldots$
(iv) $\tau_n^\pi \to \infty$ a.s. as $n \to \infty$.
(v) $0 < \xi_n^\pi \le X_{\tau_n}^\pi$.

We denote the set of all admissible strategies by Π.

Another natural admissibility condition is that net money received should be positive, that is $k\xi - K > 0$. However, we are looking for optimal policies, and a policy that allows $k\xi - K \le 0$ can never be optimal, so it can be dropped as a condition.

With each admissible strategy π we define the corresponding ruin time as

$$\tau^\pi = \inf\{t \ge 0 : X_t^\pi < 0\},$$

and the performance function $V_\pi(x)$ as

$$V_\pi(x) = E_x \left[\sum_{n=1}^{\infty} e^{-\lambda \tau_n^\pi} (k\xi_n^\pi - K) 1_{\{\tau_n^\pi \le \tau^\pi\}} \right],$$

where by P_x we mean the probability measure conditioned on $X_0 = x$. $V_\pi(x)$ represents the expected total discounted dividends received by the shareholders until ruin when the initial reserve is x.

Define the optimal return function by

$$V^*(x) = \sup_{\pi \in \Pi} V_\pi(x),$$

and the optimal strategy, if it exists, by π^* so that $V_{\pi^*}(x) = V^*(x)$.

Definition 1.2. A (simple) lump sum dividend barrier strategy $\pi = \pi_{\bar{u},\underline{u}}$ with parameters $0 \le \underline{u} < \bar{u}$, is given by:

- When $X_t^\pi < \bar{u}$, do nothing.
- When $X_t^\pi \ge \bar{u}$, reduce X_t^π to $\underline{u}$ through a dividend payment.

With a lump sum dividend barrier strategy $\pi_{\bar{u},\underline{u}}$, the corresponding value function will be denoted by $V_{\bar{u},\underline{u}}(x)$.

A two-level lump sum dividend barrier strategy $\pi = \pi_{(\bar{u}_1,\underline{u}_1)(u_2^c,\bar{u}_2,\underline{u}_2)}$ with parameters $0 \le \underline{u}_1 < \bar{u}_1 < u_2^c < \bar{u}_2$ and $0 \le \underline{u}_2 < \bar{u}_2$ is given by:

- When $X_t^\pi < \bar{u}_1$, do nothing.
- When $\bar{u}_1 \le X_t^\pi \le u_2^c$, reduce X_t^π to $\underline{u}_1$ through a dividend payment.
- When $u_2^c < X_t^\pi < \bar{u}_2$, do nothing.
- When $X_t^\pi \ge \bar{u}_2$, reduce X_t^π to $\underline{u}_2$ through a dividend payment.

With a two-level lump sum dividend barrier strategy $\pi_{(\bar{u}_1,\underline{u}_1)(u_2^c,\bar{u}_2,\underline{u}_2)}$, the corresponding value function will be denoted by $V_{(\bar{u}_1,\underline{u}_1)(u_2^c,\bar{u}_2,\underline{u}_2)}(x)$.

We will work under the following set of assumptions:

A1. $|\mu(x)| + |\sigma(x)| \le C(1+x)$ for all $x \ge 0$ and some $C > 0$.
A2. $\mu(x)$ and $\sigma(x)$ are continuously differentiable and Lipschitz continuous and the derivatives $\mu'(x)$ and $\sigma'(x)$ are Lipschitz continuous for all $x \ge 0$.
A3. $\sigma^2(x) > 0$ for all $x \ge 0$.
A4. There exists a number $x_\lambda \in [0,\infty)$ so that $\mu'(x) > \lambda$ for all $x < x_\lambda$ and $\mu'(x) \le \lambda$ for all $x \ge x_\lambda$. The number λ is a discounting rate.

Note that under A3, any dividend payment that reduces the capital to zero will result in ruin.

For $g \in C^2(0,\infty)$, define the operator L by

$$Lg(x) = \frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) - \lambda g(x).$$

It is well known, see e.g. [6], that under the assumptions A1–A3 any solution of $Lg(x) = 0$ is in $C^2(0,\infty)$. Let g_1 and g_2 be two independent solutions of $Lg(x) = 0$, chosen so that $g(x) = g_1(0)g_2(x) - g_2(0)g_1(x)$ has $g'(0) > 0$. Such a solution will be called a canonical solution. Then any solution of $LV(x) = 0$ with $V(0) = 0$ and $V'(0) > 0$ is of the form

$$V(x) = cg(x), \quad c > 0.$$

Under Assumptions A1–A4, it was proved in [3] that there are two basic possibilities for the canonical solution g.

P. There is an x^* with $0 \le x^* \le \infty$ so that g is concave on $[0, x^*]$ and convex on (x^*, ∞). In particular $x^* = 0$ if and only if $\mu(0) \le 0$, and by the definition of x^*, $g''(x^*) = 0$ when $0 < x^* < \infty$.

R. There are numbers x_1^* and x_2^* with $0 < x_1^* \le x_\lambda \le x_2^* \le \infty$ so that g is convex on $[0, x_1^*)$, concave on $[x_1^*, x_2^*)$ and, if $x_2^* < \infty$, convex on $[x_2^*, \infty)$.

A complete solution of case P is given in [3], where it was proved that the optimal policies are simple lump sum dividend strategies. This paper also covers the more complex case R, but without a complete solution. To be more concrete, a few definitions are needed. With g a canonical solution, define for $a_1 < a_2$,

$$I(a_1, a_2, c) = \int_{a_1}^{a_2} (k - cg'(x))dx = k(a_2 - a_1) - c(g(a_2) - g(a_1)).$$

Define the set C, possibly empty, by $C = C_1 \cup C_2$ where

$$\begin{aligned} C_1 &= \{c > 0 : \text{ there exists } 0 < \underline{u} < \bar{u} \text{ so that} \\ &\qquad cg'(\underline{u}) = cg'(\bar{u}) = k \text{ and } I(\underline{u}, \bar{u}, c) = K\}, \\ C_2 &= \{c > 0 : \text{ there exists } \bar{u} > 0 \text{ so that} \\ &\qquad cg'(\bar{u}) = k,\ cg'(0) < k \text{ and } I(0, \bar{u}, c) = K\}. \end{aligned}$$

Since I and g' are continuous, C is closed. Hence if $C \ne \emptyset$, $c^* = \max\{c : c \in C\}$ is well defined. However, for known c^*, the corresponding $\underline{u}$ and $\bar{u}$ may not be unique. Therefore we define

$$U = \{u : c^* g'(u) = k\}.$$

Let

$$\bar{u}^* = \max\{u \in U : I(\underline{u}, u, c^*) = K \text{ for some } \underline{u} \in \{0\} \cup U\}$$

and then

$$\underline{u}^* = \max\{u \in \{0\} \cup U : I(u, \bar{u}^*, c^*) = K\}.$$

The case R can be divided into several subclasses as follows:

R1. $0 < x_1^* < \underline{u}^* < x_2^* < \bar{u}^*,\ c^* g'(0) \ge k.$
R2. $0 < x_1^* < \underline{u}^* < x_2^* < \bar{u}^*,\ c^* g'(0) < k.$
R3. $0 = \underline{u}^* < x_1^* < x_2^* < \bar{u}^*,\ c^* g'(x_1^*) \ge k.$
R4. $0 = \underline{u}^* < x_1^* < x_2^* < \bar{u}^*,\ c^* g'(x_1^*) < k.$
R5. $0 = \underline{u}^* < \bar{u}^* < x_1^*,\ x_2^* < \infty,\ c^* g'(x_2^*) \ge k.$
R6. $0 = \underline{u}^* < \bar{u}^* < x_1^*,\ x_2^* < \infty,\ c^* g'(x_2^*) < k.$
R7. $x_2^* = \infty$, i.e. g is concave on $[x_1^*, \infty)$.
R8. None of the above.

As pointed out in Remark 2.2 in [3], the case R8 is pretty odd, so we drop it in this paper.

If it exists, let $(c^*, \bar{u}^*)$ be a pair that satisfies

$$c^* g'(\bar{u}^*) = k \quad \text{and} \quad c^* g(\bar{u}^*) = k\bar{u}^* - K, \tag{1.1}$$

so that in particular

$$c^* = \frac{k}{g'(\bar{u}^*)}.$$

Under R5 and R6, $(c^*, \bar{u}^*)$ always exists, and is as given in the definition of R5–R6 above. Under R7, if it does not exist, we can set $c^* = \bar{u}^* = 0$.

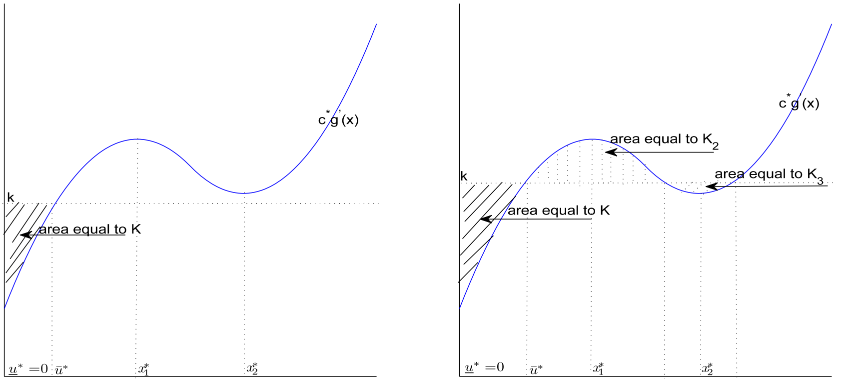

Fig. 1. Plots of cases R5 (left) and R6 (right). In case R6, $K_3 < \min\{K_2, K\}$.

By the properties of g, g' is ultimately increasing or decreasing, so we can define

$$g'_\infty = \lim_{x\to\infty} g'(x).$$

Under R1–R6, typically $g'_\infty = \infty$, while under R7, g'_∞ is always finite. Also define

$$c^\infty = \frac{k}{g'_\infty} \quad \text{with } c^\infty = 0 \text{ if } g'_\infty = \infty.$$

It was proved in [3] that for cases R1–R4 the optimal policies are always the simple barrier policies $\pi_{\bar{u}^*,\underline{u}^*}$ with corresponding value functions $V_{\bar{u}^*,\underline{u}^*}(x)$. It was proved that this is also the case if

$$\mu(x_\lambda)k - \lambda(kx_\lambda - K) \le 0 \tag{1.2}$$

and R5 or R6 apply, and also if R7 applies, (1.2) holds and $c^* \ge c^\infty$. When R7 applies and $c^* < c^\infty$ it was proved that there is no optimal policy, but the value function equals $V^*(x) = c^\infty g(x)$.

Therefore, what remains is R5–R7 when

$$\mu(x_\lambda)k - \lambda(kx_\lambda - K) > 0. \tag{1.3}$$

Fig. 1 gives a graphical illustration of cases R5 and R6. Note that in case R6, $K_3 \le \min\{K_2, K\}$, since otherwise it is possible to increase c^*, bringing us to one of the cases R1–R4. Also if $K_3 = K_2 = K$, by definition of c^*, we have case R3. Therefore $K_3 < \min\{K_2, K\}$. But then $c^* \ge c^\infty$, or equivalently $g'_\infty \ge g'(\bar{u}^*)$, for otherwise $K_3 = \infty$.

We can now formulate the two problems of this paper. A solution of these problems together with the results in [3] will give a complete solution to the dividend problem under A1–A4, with the exception of the odd case R8.

Problem 1. Assume A1–A4 and that R5 or R6 apply so that in particular $c^* \ge c^\infty$. Also assume (1.3). Find the optimal value function, and if it exists, the optimal dividend strategy.

Problem 2. Assume A1–A4 and that R7 applies. Also assume (1.3) and that $c^* \ge c^\infty$. Find the optimal value function, and if it exists, the optimal dividend strategy.

It was proved in [3] that a simple lump sum dividend strategy cannot be optimal for any of these two problems. This kind of result is not new for diffusion processes, see for example, Example 4.3 in [5], and for the case with no fixed transaction costs, see [1]. For more background information and details, the reader should consult [3] and references therein.

As pointed out in Remark 2.3 in [3], in order for R5 or R6 to apply it is necessary that $\mu(0) < -\frac{\lambda K}{k}$. It can thus be argued that these cases are less interesting from a practical point of view. However, from a theoretical point of view these cases are the most interesting, and certainly the most challenging, and as we shall see, the optimal solutions are highly nontrivial. To arrive at these solutions, or more concrete to arrive at the necessary variational inequalities, several new concepts and definitions are introduced. These are different from and more complicated than those in [3]. So although superficially this paper may resemble [3], in fact it is quite different.

A complete solution of Problem 1 is given in Section 3, and of Problem 2 in Section 4. However, before we can present the solutions, several definitions and preliminary results are needed. This is the topic of the next section.

2. Some preliminary results

The notation and definitions are the same as in Section 1. Throughout this section, the assumptions stated in Problem 1 or Problem 2 are assumed to hold, so in particular (1.1) is assumed to have a solution with $c^* > 0$. All results are proved in the Appendix.

The following lemma yields two solutions of $Lg(x) = 0$ that are useful for further analysis.

Lemma 2.1. *There exists two independent, positive solutions g_1 and g_2 of $Lg(x) = 0$ such that:*

(i) $g_1(0) = 1$, $g_1'(x) < 0$ *on* $[0, x_\lambda)$, $g_1'(x_\lambda) = 0$, $g_1'(x) > 0$ *on* (x_λ, ∞), *and* $g_1''(x) > 0$ *on* $[x_\lambda, \infty)$. *Furthermore, there exists at most one point* $z_0 > 0$ *such that* $g_1''(z_0) = 0$, *and at this point* $g_1'''(z_0) > 0$. *If* z_0 *exists then* $z_0 \in (0, x_1^*)$. *Also* $g_1(x) > 0$ *for all* $x \geq 0$.

(ii) g_2 *is strongly increasing on* $[0, \infty)$ *with* $g_2(0) = 1$ *and* $g_2'(0) > 0$.

It follows from Lemma 2.1 that such a z_0 exists if and only if $g_1''(0) < 0$. If the function g_1 in Lemma 2.1(i) satisfies $g_1''(0) \geq 0$ we set $z_0 = 0$. In this case, the interval $(0, z_0)$ is just the empty set.

With g_1 and g_2 as in Lemma 2.1, a canonical solution becomes

$$g(x) = g_2(x) - g_1(x).$$

In the sequel, the canonical solution $g(x)$ will always be this particular function. Define

$$g(x; \beta) = g_2(x) - \beta g_1(x),$$

so that $g(x; 1) = g(x)$, the canonical solution. When writing $g'(x; \beta)$ we shall mean $\frac{d}{dx} g(x; \beta) = g_2'(x) - \beta g_1'(x)$. Similarly with $g''(x; \beta)$ and $g'''(x; \beta)$.

From the fact that $g'(x) > 0$ and $g_2'(x) > 0$ it follows easily that $g'(x; \beta) > 0$ for $\beta \in [0, 1]$; hence we can define

$$\gamma(\beta, x) = k \left(\frac{g(x; \beta)}{g'(x; \beta)} - x \right) + K.$$

Note that

$$\gamma(1, \bar{u}^*) = 0. \tag{2.1}$$

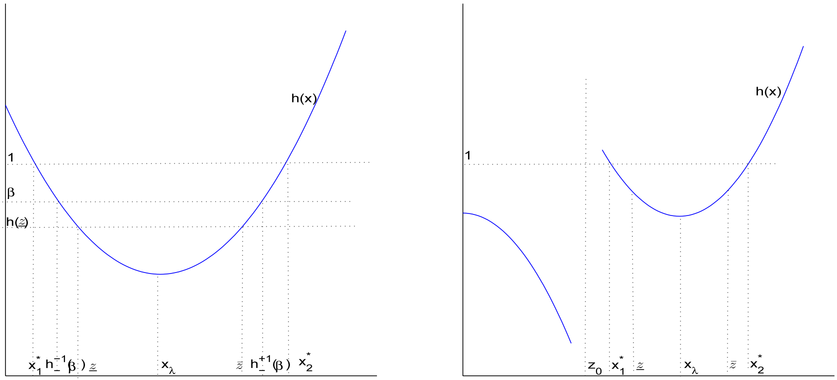

Fig. 2. Plots of $h(x)$. In the left plot, $g_1''(0) \geq 0$, while in the right plot $g_1''(0) < 0$.

The following function will play an important role in this paper.

$$h(x) = \frac{g_2''(x)}{g_1''(x)}, \quad x \neq z_0. \tag{2.2}$$

Here is a simple observation that will often be used

$$g''(x;\beta) = g_1''(x)(h(x) - \beta), \quad x \neq z_0. \tag{2.3}$$

The next result contains some important properties of the function h.

Lemma 2.2. *The function h is strongly decreasing on $(0, z_0) \cup (z_0, x_\lambda)$ and strongly increasing on (x_λ, ∞). Thus the limit $h_\infty = \lim_{x\to\infty} h(x)$ exists, but may be infinite. In cases* R5 *and* R6, *$h_\infty \geq 1$, while in case* R7, *$h_\infty \leq 1$.*

Using that $z_0 < x_1^*$ and the fact that $h(x_1^*) = 1$ (see Lemma A.1(c)), it follows from Lemma 2.2 that we can define a continuously differentiable, strongly decreasing function $h_-^{-1} : [h(x_\lambda), 1]$ onto $[x_1^*, x_\lambda]$ so that $h(h_-^{-1}(x)) = x$. Furthermore, we can define a continuously differentiable, strongly increasing function $h_+^{-1} : [h(x_\lambda), h_\infty) \to [x_\lambda, \infty)$ so that $h(h_+^{-1}(x)) = x$. See Fig. 2.

By Lemma 2.1 in [3] there is a unique $\underline{z} \in (x_1^*, x_\lambda)$ so that

$$\mu(\underline{z})k - \lambda(k\underline{z} - K) = 0 \tag{2.4}$$

and

$$\mu(x)k - \lambda(kx - K) < 0, \quad x < \underline{z}. \tag{2.5}$$

In Lemma 2.1 in [3] it is assumed that R5 or R6 apply, but looking at the proof shows that the only requirement is that $\bar{u}^* < x_1^*$. Therefore, (2.4) and (2.5) are valid under R7 as well whenever (1.1) holds.

Lemma 2.3. *Let $\underline{z}$ be as in (2.4). Then $h(\underline{z}) < 1$. Furthermore, if $g_1''(0) < 0$, or equivalently if $z_0 > 0$, then $h(0) \leq h(\underline{z})$. Finally, for $\beta \in (h(\underline{z}), 1]$,*

$$g''(x;\beta) > 0, \quad x \in (0, h_-^{-1}(\beta)).$$

By Lemmas 2.2 and 2.3, $h(\underline{z}) < h_\infty$ in cases R5 and R6, while in case R7 this inequality may not hold.

Lemma 2.4. *There is a unique, continuously differentiable, strongly decreasing function* $u_1 : [h(\underline{z}), 1] \to [\bar{u}^*, \underline{z}]$ *so that*

$$\gamma(\beta, u_1(\beta)) = 0.$$

Furthermore, $\bar{u}^* < u_1(\beta) < h_-^{-1}(\beta)$ *for* $\beta \in (h(\underline{z}), 1)$*, while* $u_1(h(\underline{z})) = \underline{z}$ *and* $u_1(1) = \bar{u}^*$.

It follows from Lemma 2.3 that for $\beta \in [h(\underline{z}), 1]$, as a function of x, $g'(x;\beta)$ increases on $(0, h_-^{-1}(\beta))$. Furthermore, if $h(\underline{z}) < h_\infty$ and $\beta \in [h(\underline{z}), h_\infty)$, it follows from (2.3) and the fact that $z_0 < x_1^* < h_-^{-1}(\beta)$, that $g'(x;\beta)$ decreases on $(h_-^{-1}(\beta), h_+^{-1}(\beta))$ and increases on $(h_+^{-1}(\beta), \infty)$. Therefore, when $h(\underline{z}) < h_\infty$ we can define

$$v(\beta) = \begin{cases} \dfrac{g'(h_+^{-1}(\beta);\beta)}{g'(u_1(\beta);\beta)}, & \beta \in [h(\underline{z}), 1] \cap [h(\underline{z}), h_\infty), \\ \infty, & \text{otherwise.} \end{cases} \tag{2.6}$$

Now $u_1(h(\underline{z})) = \underline{z} = h_-^{-1}(h(\underline{z}))$, so we get since $g'(x;\beta)$ has a local maximum at $h_-^{-1}(\beta)$,

$$v(h(\underline{z})) = \frac{g'(h_+^{-1}(h(\underline{z})); h(\underline{z}))}{g'(h_-^{-1}(h(\underline{z})); h(\underline{z}))} < 1.$$

Therefore, if $h(\underline{z}) < h_\infty$, define

$$\beta_0 = \sup\{\beta_1 \in (h(\underline{z}), 1] : v(\beta) \le 1 \text{ for all } \beta \in (h(\underline{z}), \beta_1]\}. \tag{2.7}$$

In case R7, $h_\infty \le 1$ by Lemma 2.2, so then $h_+^{-1}(1)$ is not defined, yielding that $\beta_0 < 1$. In cases R5 and R6,

$$v(1) = \frac{g'(h_+^{-1}(1))}{g'(u_1(1))} = \frac{g'(x_2^*)}{g'(\bar{u}^*)} = \frac{c^*}{k} g'(x_2^*).$$

Therefore, when there is a strict inequality in R5, $\beta_0 < 1$.

To prepare for Problem 2, assume that $g(x)$ is concave for $x > x_\lambda$. Also, for the moment we do not assume that $h(\underline{z}) < h_\infty$. By Lemma 2.2, $h_\infty \le 1$. Since g is ultimately concave, $g'_\infty < \infty$ and therefore

$$g'_{1,\infty} := \lim_{x\to\infty} g'_1(x) < \infty \quad \text{if and only if} \quad g'_{2,\infty} := \lim_{x\to\infty} g'_2(x) < \infty.$$

Assume that $g'_{1,\infty} < \infty$. By Lemma 2.4 we can define

$$G(\beta) = \frac{g'_{2,\infty} - \beta g'_{1,\infty}}{g'(u_1(\beta);\beta)} = \frac{\lim_{x\to\infty} g'(x;\beta)}{g'(u_1(\beta);\beta)}, \quad \beta \in [h(\underline{z}), 1].$$

Note that by Lemmas 2.1, 2.2 and 2.4, $g'(u_1(\beta);\beta) > g'(0,\beta) = 1 - \beta \ge 0$ for $\beta \in [h(\underline{z}), 1]$.

Lemma 2.5. *Assume that* g *is concave for* $x > x_\lambda$ *and that* $g'_{1,\infty} < \infty$*. Then the function* G *is continuously differentiable and strongly increasing on* $(h(\underline{z}), 1)$.

Let us return to the general case and again assume that $h(\underline{z}) < h_\infty$. Since $g'(x;\beta)$ increases for $x > h_+^{-1}(\beta)$, $\lim_{x\to\infty} g'(x;\beta)$ always exists, but may be infinite. In that case we set $G(\beta) = \infty$. Again by the ultimate increase of $g'(x;\beta)$, $\lim_{x\to\infty} g'(x;\beta) > g'(h_+^{-1}(\beta);\beta)$, hence

$$G(\beta) > v(\beta), \quad \beta \in [h(\underline{z}), \beta_0). \tag{2.8}$$

Finally, if it exists, define α_0 by

$$\alpha_0 = \begin{cases} h(\underline{z}) & \text{if } G(h(\underline{z})) \geq 1, \\ \alpha_1 & \text{if } G(h(\underline{z})) < 1, \end{cases} \tag{2.9}$$

where $\alpha_1 \in (h(\underline{z}), \beta_0)$ is the unique value that satisfies $G(\alpha_1) = 1$. So by definition, if it exists, $\alpha_0 < \beta_0$. By (2.8), such an α_0 exists whenever $\beta_0 < 1$. When $\beta_0 = 1, \alpha_0$ exists if and only if $G(1) > 1$, i.e. if and only if $g'_\infty > g'(\bar{u}^*)$, or equivalently if and only if $c^* > c^\infty$. In this case, by definition of c^*, (1.1) will necessarily hold.

Lemma 2.6. *Assume that $h(\underline{z}) < h_\infty$ and that α_0 in (2.9) exists. Then there are continuously differentiable functions u_i, $i = 1, 2, 3$, defined on (α_0, β_0), so that $\gamma(\beta, u_1(\beta)) = 0$ and*

$$g'(u_1(\beta);\beta) = g'(u_2(\beta);\beta) = g'(u_3(\beta);\beta). \tag{2.10}$$

Furthermore, u_1 is strongly decreasing and

$$\bar{u}^* < u_1(\beta) < h_-^{-1}(\beta) < \underline{z} < u_2(\beta) < h_+^{-1}(\beta) < u_3(\beta).$$

Also if $\alpha_0 = h(\underline{z})$,

$$\lim_{\beta\downarrow\alpha_0} u_1(\beta) = \lim_{\beta\downarrow\alpha_0} u_2(\beta) = \underline{z}, \tag{2.11}$$

and if $\alpha_0 > h(\underline{z})$,

$$\lim_{\beta\downarrow\alpha_0} u_3(\beta) = \infty. \tag{2.12}$$

Finally, if $\beta_0 < \min\{1, h_\infty\}$.

$$\lim_{\beta\uparrow\beta_0} u_2(\beta) = \lim_{\beta\uparrow\beta_0} u_3(\beta). \tag{2.13}$$

Define the function J by

$$J(\beta, u_1, u_2) = k\int_{u_1}^{u_2}\left(1 - \frac{g'(x;\beta)}{g'(u_2;\beta)}\right)dx = k\left(u_2 - u_1 - \frac{g(u_2;\beta) - g(u_1;\beta)}{g'(u_2;\beta)}\right).$$

When (2.10) is satisfied, we will use the simplified notation

$$J_1(\beta) = J(\beta, u_1(\beta), u_2(\beta)),$$
$$J_2(\beta) = J(\beta, u_2(\beta), u_3(\beta)),$$
$$J_{13}(\beta) = J(\beta, u_1(\beta), u_3(\beta)),$$

so that in particular $J_{13}(\beta) = J_1(\beta) + J_2(\beta)$.

Lemma 2.7. *Under the assumptions of Lemma 2.6, for $\beta \in (\alpha_0, \beta_0)$, both J_1 and J_{13} are strongly decreasing.*

When $G(h(\underline{z})) \geq 1$ then $\alpha_0 = h(\underline{z})$ and so $J_1(\alpha_0) = \lim_{\beta \downarrow \alpha_0} J_1(\beta) = 0$ by (2.11). When $G(h(\underline{z})) < 1, \alpha_0 > h(\underline{z})$ and then $J_1(\alpha_0) < 0$. By Lemma 2.6,

$$J_2(\alpha_0) = \lim_{\beta \downarrow \alpha_0} J_{13}(\beta) - \lim_{\beta \downarrow \alpha_0} J_1(\beta)$$

exists, but may be infinite because of (2.12). To get the precise solutions of our problems it is convenient to split into five different cases. Again, the assumption $h(\underline{z}) < h_\infty$ is in force.

G1. $G(h(\underline{z})) \geq 1$.
G2. $G(h(\underline{z})) < 1, \alpha_0$ exists and $J_{13}(\alpha_0) \geq 0$.
G3. $G(h(\underline{z})) < 1, \alpha_0$ exists, $J_{13}(\alpha_0) < 0$ and $J_2(\alpha_0) > K$.
G4. $G(h(\underline{z})) < 1, \alpha_0$ exists, $J_{13}(\alpha_0) < 0$ and $J_2(\alpha_0) \leq K$.
G5. α_0 does not exist, that is $c^* \leq c^\infty$.

In G1, α_0 exists by definition. Also, in Problem 1, G5 becomes $c^* = c^\infty$.

Remark. Under R5 or R6 the assumption that $G(h(\underline{z})) > 1$, or equivalently that $\lim_{x\to\infty} g'(x; h(\underline{z})) > g'(\underline{z}; h(\underline{z}))$ is extremely weak. For these cases, typically $g'_\infty = \infty$, and then

$$\lim_{x\to\infty} g'(x; h(\underline{z})) = g'_\infty + (1 - h(\underline{z})) \lim_{x\to\infty} g'_1(x) = \infty.$$

It is proved in [2, Proposition 2.5], that a sufficient condition for $g'_\infty = \infty$ is that there exist an $x_0 > 0$ and an $\varepsilon > 0$ so that $\mu'(x) < \lambda - \varepsilon$ for all $x > x_0$.

The next result is the basis for the optimality result Theorem 3.1 for Problem 1.

Proposition 2.1. *Given the assumptions of Problem 1 and either of* G1*,* G2 *or* G3*. Then there exists a* $\tilde{\beta} \in (\alpha_0, \beta_0)$ *and numbers* $\tilde{u}_i = u_i(\tilde{\beta}),\ i = 1, 2, 3,$ *with*

$$\bar{u}^* < \tilde{u}_1 < h_-^{-1}(\tilde{\beta}) < \underline{z} < \tilde{u}_2 < h_+^{-1}(\tilde{\beta}) < \tilde{u}_3,$$

so that

(i) $\gamma(\tilde{\beta}, \tilde{u}_1) = 0$.
(ii) $g'(\tilde{u}_1; \tilde{\beta}) = g'(\tilde{u}_2; \tilde{\beta}) = g'(\tilde{u}_3; \tilde{\beta})$.
(iii) *exactly one of the following two possibilities holds:*
(iii-a) $-J(\tilde{\beta}, \tilde{u}_1, \tilde{u}_2) = J(\tilde{\beta}, \tilde{u}_2, \tilde{u}_3) \leq K$.
(iii-b) $-J(\tilde{\beta}, \tilde{u}_1, \tilde{u}_2) > J(\tilde{\beta}, \tilde{u}_2, \tilde{u}_3) = K$.

In case of G3*,* (iii-b) *applies.*

Similarly to Proposition 2.1, the next result is the basis for Theorem 4.1 that partially covers Problem 2.

Proposition 2.2. *Given the assumptions of Problem 2 and either of* G1*,* G2 *or* G3*. Also assume one of the following two conditions:*

1. $h_\infty = 1$.
2. $h(\underline{z}) < h_\infty < 1$ *and* $G(h_\infty) > 1$.

Then the results in Proposition 2.1 still hold.

3. Solution of Problem 1

The notation and definitions are the same as in Sections 1 and 2. Throughout this section, the assumptions stated in Problem 1 are assumed to hold. All results are proved in the Appendix.

In the presentation of the results, we need one more definition. Let $\gamma \in [h(\underline{z}), 1]$ and define

$$c_\gamma = \frac{k}{g'(u_1(\gamma); \gamma)},$$

so in particular $c_1 = c^*$ if the latter is positive.

Theorem 3.1. *Given the same assumptions and notation as in Proposition 2.1:*

(a) *If* (i), (ii) *and* (iii-a) *of Proposition 2.1 hold, then the two-level lump sum dividend strategy $\pi_{(\bar{u}^*,0)(\tilde{u}_1,\tilde{u}_3,0)}$ is optimal. The corresponding value function is given as*

$$V_{(\bar{u}^*,0)(\tilde{u}_1,\tilde{u}_3,0)}(x) = \begin{cases} c^* g(x), & x \in [0, \bar{u}^*), \\ kx - K, & x \in [\bar{u}^*, \tilde{u}_1), \\ c_{\tilde{\beta}} g(x; \tilde{\beta}), & x \in [\tilde{u}_1, \tilde{u}_3), \\ kx - K, & x \in [\tilde{u}_3, \infty). \end{cases}$$

(b) *If* (i), (ii) *and* (iii-b) *of Proposition 2.1 hold, then the two-level lump sum dividend strategy $\pi_{(\bar{u}^*,0)(\tilde{u}_1,\tilde{u}_3,\tilde{u}_2)}$ is optimal. The corresponding value function is given as*

$$V_{(\bar{u}^*,0)(\tilde{u}_1,\tilde{u}_3,\tilde{u}_2)}(x) = \begin{cases} c^* g(x), & x \in [0, \bar{u}^*), \\ kx - K, & x \in [\bar{u}^*, \tilde{u}_1), \\ c_{\tilde{\beta}} g(x; \tilde{\beta}), & x \in [\tilde{u}_1, \tilde{u}_3), \\ c_{\tilde{\beta}} g(\tilde{u}_2; \tilde{\beta}) + k(x - \tilde{u}_2) - K, & x \in [\tilde{u}_3, \infty). \end{cases}$$

In both cases, the value function is continuously differentiable.

When Theorem 3.1(a) holds, the first payment will lead to ruin, while when Theorem 3.1(b) holds, the first payment results in ruin only if $X^{\pi^*}_{\tau_1^{\pi^*}} \in [\bar{u}_1^*, \tilde{u}_1]$.

Now turn to the case G4, and here the solution is different from that in Theorem 3.1.

Theorem 3.2. *Assume that case* G4 *applies. Then the optimal value function is given as*

$$V^*(x) = \begin{cases} c^* g(x), & x \in [0, \bar{u}^*), \\ kx - K, & x \in [\bar{u}^*, u_1(\alpha_0)), \\ c_{\alpha_0} g(x; \alpha_0), & x \in [u_1(\alpha_0), \infty). \end{cases}$$

When $x \in (0, u_1(\alpha_0)]$ the lump sum dividend barrier strategy $\pi_{\bar{u}^*,0}$ is optimal, while for $x > u_1(\alpha_0)$ there is no optimal strategy and $V^*(x) = \lim_{\bar{u}\to\infty} V_{(\bar{u},0)}(x)$.

Finally, we give the solution for G5.

Theorem 3.3. *Assume that case* G5 *applies so that $c^* = c^\infty$. Then there is no optimal policy, but the value function is given as*

$$V^*(x) = c^* g(x), \quad x \geq 0.$$

The result in Theorem 3.3 is just a limit of those in Theorems 3.1 and 3.2. For example in Theorem 3.2, since as $c^\infty \uparrow c^*$, $G(1) \downarrow 1$ and so $\alpha_0 \uparrow 1$. But then $u_1(\alpha_0) \downarrow u_1(1) = \bar{u}^*$ and by (A.4), $c_{\alpha_0} \uparrow c^*$. Hence the interval $[\bar{u}^*, u_1(\alpha_0)) \to \emptyset$ and $c_{\alpha_0} g(x; \alpha_0) \to c^* g(x)$. Therefore

$$V^*(x) \to c^* g(x).$$

Similarly, in Theorem 3.1 as $c^\infty \uparrow c^*$, $\tilde{\beta} \uparrow 1$, $\tilde{u}_1 \downarrow \bar{u}^*$, $c_{\tilde{\beta}} \uparrow c_1 = c^*$ and $\tilde{u}_3 \to \infty$.

4. Solution of Problem 2

The notation and definitions are again the same as in Sections 1 and 2. Throughout the section, the assumptions stated in Problem 2 are assumed to hold. All results are proved in the Appendix.

As in the last section, we start with the case $c^* > c^\infty$, or equivalently $g'_\infty > g'(\bar{u}^*)$. The case $c^* = c^\infty$ is then solved in Theorem 4.3.

Theorem 4.1. *Under the assumptions and notation of Proposition 2.2, the optimal policy and value function are as in Theorem 3.1.*

It remains to solve the cases not covered by Theorem 4.1, i.e. when either

H1. $h(\underline{z}) \geq h_\infty$.
H2. $h(\underline{z}) < h_\infty < 1$ and $G(h_\infty) \leq 1$.
H3. Case G4.
H4. $c^* = c^\infty$.

Note that there can be overlap between case H3 and either of H1 or H2. For H1, since $h(\underline{z}) < h(x_1^*) = 1$, the condition $h_\infty < 1$ is automatically satisfied.

Lemma 4.1. *Assume that one of* H1–H3 *apply. Then there is a unique $\hat{\beta} \in (h(\underline{z}), 1)$ that satisfies $G(\hat{\beta}) = 1$. Furthermore, $u_1(\hat{\beta}) < \underline{z}$.*

Theorem 4.2. *Assume that one of* H1–H3 *apply, and let $\hat{\beta}$ be as in Lemma 4.1 . Then the optimal value function is given as*

$$V^*(x) = \begin{cases} c^* g(x), & x \in [0, \bar{u}^*), \\ kx - K, & x \in [\bar{u}^*, u_1(\hat{\beta})), \\ c_{\hat{\beta}} g(x; \hat{\beta}), & x \in [u_1(\hat{\beta}), \infty). \end{cases}$$

When $x \in (0, u_1(\hat{\beta})]$ the lump sum dividend barrier strategy $\pi_{\bar{u}^*,0}$ is optimal, while for $x > u_1(\hat{\beta})$ there is no optimal strategy and $V^*(x) = \lim_{\bar{u}\to\infty} V_{(\bar{u},0)}(x)$.

Finally, here is the result when $c^* = c^\infty$.

Theorem 4.3. *Assume* H4, *i.e. that $c^* = c^\infty$. Then there is no optimal policy, but the value function is given as*

$$V^*(x) = c^* g(x), \quad x \geq 0.$$

Just as pointed out after Theorem 3.3, Theorem 4.4 is a limiting case of Theorem 4.3 as $c^\infty \uparrow c^*$.

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Appendix. Proofs

Proof of Lemma 2.1. The proof that $z_0 < x_1^*$ is deferred until after the proof of Lemma A.2. For the construction of the g_i, it is well known, see e.g. [4], that the equation $Lg(x) = 0$ has a strongly increasing as well as a strongly decreasing solution, both positive. Let g_2 be the strongly increasing solution, divided by $g_2(0)$ so that the new g_2 satisfies $g_2(0) = 1$.

Using the same arguments as in the proof of Lemma A.5 in [7], but starting at x_λ, we find that there is a solution g_1 satisfying

$$g_1(x_\lambda) = 1, \qquad g_1'(x_\lambda) = 0, \qquad g_1'(x) > 0 \quad \text{on } (x_\lambda, \infty) \quad \text{and}$$
$$g_1''(x) > 0 \quad \text{on } [x_\lambda, \infty).$$

We will show that $g_1'(x) < 0$ on $(0, x_\lambda)$, which clearly implies that g_1 is positive. Since $g_1'(x_\lambda) = 0$ and $g_1''(x_\lambda) > 0$, there is an $\varepsilon > 0$ so that $g_1'(x) < 0$ and $g_1''(x) > 0$ on $(x_\lambda - \varepsilon, x_\lambda)$. Define, if it exists

$$x_0 = \max\{x \le x_\lambda - \varepsilon : g_1'(x) = 0\}.$$

Since $g_1'(x) < 0$ for $x \in (x_0, x_\lambda)$, this implies that $g_1''(x_0) \le 0$ and that $g_1(x_0) > 1$. The relation $Lg_1(x) = 0$ gives

$$g_1''(x) = \frac{2\lambda}{\sigma^2(x)} g_1(x) - \frac{2\mu(x)}{\sigma^2(x)} g_1'(x). \tag{A.1}$$

Therefore, by the definition of x_0, $g_1''(x_0) = \frac{2\lambda}{\sigma^2(x)} g_1(x_0) > 0$, leading to a contradiction. Consequently, $g_1'(x) < 0$ on $(0, x_\lambda)$.

We now turn to the proof that there is at most one $z_0 < x_\lambda$ so that $g_1''(z_0) = 0$. Differentiating $Lg_1(x) = 0$ gives

$$g_1'''(x) = \frac{2(\lambda - \mu'(x))}{\sigma^2(x)} g_1'(x) - \frac{2}{\sigma^2(x)} (\mu(x) + \sigma(x)\sigma'(x)) g_1''(x). \tag{A.2}$$

If there is a $z_0 < x_\lambda$ so that $g_1''(z_0) = 0$, it follows from (A.2) that $g_1'''(z_0) > 0$. But then clearly there can only be one such z_0. Also $\min_{x \ge 0} g_1(x) = g_1(x_\lambda) = 1$ so $g_1(x) > 0$ for all $x \ge 0$. Finally, we rescale g_1 by dividing it by $g_1(0)$ so that the new g_1 satisfies $g_1(0) = 1$. □

Here are some simple results that will be useful in the sequel.

Lemma A.1. *Let g_1 and g_2 be as in Lemma 2.1, and let h be as in (2.2).*

(a) *The Wronskian $W(x) = g_1(x)g_2'(x) - g_2(x)g_1'(x) > 0$.*

(b) *$\frac{g_2(x)}{g_1(x)}$ is increasing in x.*

(c) *$h(x_1^*) = 1$ and if $x_2^* < \infty$, $h(x_2^*) = 1$.*

(d)

$$h'(x) = \frac{4\lambda(\lambda - \mu'(x))W(x)}{\sigma^4(x)(g_1''(x))^2}, \quad x \ne z_0.$$

(e)

$$\frac{g(x;h(x))}{g'(x;h(x))} = \frac{\mu(x)}{\lambda}, \quad x \neq z_0.$$

(f)

$$\frac{\partial}{\partial\beta}\gamma(\beta,x) = -k\frac{W(x)}{(g'(x;\beta))^2} < 0.$$

(g)

$$\frac{\partial}{\partial x}\gamma(\beta,x) = -k\frac{g(x;\beta)}{(g'(x;\beta))^2}g_1''(x)(h(x)-\beta) = -k\frac{g(x;\beta)}{(g'(x;\beta))^2}(g_2''(x)-\beta g_1''(x)).$$

Proof. For (a), we have that $W(0) = g'(0) > 0$, and since the Wronskian of two independent solutions never vanishes, the result follows. As for (b), direct differentiation gives

$$\frac{d}{dx}\frac{g_2(x)}{g_1(x)} = \frac{W(x)}{(g_1(x))^2} > 0$$

by (a). For (c), we have

$$0 = g''(x_1^*) = g_2''(x_1^*) - g_1''(x_1^*) \Rightarrow h(x_1^*) = 1.$$

Similarly, if $x_2^* < \infty$, $h(x_2^*) = 1$. For (d), differentiation gives

$$h'(x) = \frac{g_1''(x)g_2'''(x) - g_1'''(x)g_2''(x)}{(g_1''(x))^2}.$$

Using (A.2) for $g_i'''(x)$, $i = 1, 2$, yields

$$g_1''(x)g_2'''(x) - g_1'''(x)g_2''(x) = \frac{2(\lambda-\mu'(x))}{\sigma^2(x)}(g_1''(x)g_2'(x) - g_2''(x)g_1'(x)).$$

Using (A.1) for $g_i''(x)$, $i = 1, 2$, yields

$$g_1''(x)g_2'(x) - g_2''(x)g_1'(x) = \frac{2\lambda}{\sigma^2(x)}W(x). \tag{A.3}$$

Combining these results proves (d). For (e), multiplying by $\frac{1}{2}\sigma^2(x)g_1''(x)$ on both the numerator and the denominator gives

$$\begin{aligned}\frac{g(x;h(x))}{g'(x;h(x))} &= \frac{\frac{1}{2}\sigma^2(x)g_1''(x)g_2(x) - \frac{1}{2}\sigma^2(x)g_2''(x)g_1(x)}{\frac{1}{2}\sigma^2(x)g_1''(x)g_2'(x) - \frac{1}{2}\sigma^2(x)g_2''(x)g_1'(x)} \\ &= \frac{(\lambda g_1(x) - \mu(x)g_1'(x))g_2(x) - (\lambda g_2(x) - \mu(x)g_2'(x))g_1(x)}{(\lambda g_1(x) - \mu(x)g_1'(x))g_2'(x) - (\lambda g_2(x) - \mu(x)g_2'(x))g_1'(x)} \\ &= \frac{\mu(x)W(x)}{\lambda W(x)} \\ &= \frac{\mu(x)}{\lambda}.\end{aligned}$$

Finally, (f) and (g) follow by straightforward differentiation and use of the definition of $h(x)$. □

Proof of Lemma 2.2. The first part is a direct consequence of Lemma 2.1(i) and Lemma A.1(a) and (d). As for the second part, assume R5 or R6. Then for $x > x_2^*$, $0 < g''(x) = g_2''(x) - g_1''(x)$. But since $g_1''(x) > 0$ for $x > x_\lambda$, it follows that $h(x) > 1$ for $x > x_2^*$. This covers cases R5 and R6, and the result for case R7 is proved in the same way. □

Lemma A.2. *Assume* (1.3). *Then*

(a) $\gamma(h(x), x) < 0,\ x < \underline{z}$.
(b) $\gamma(h(\underline{z}), \underline{z}) = 0$.

Proof. To prove (a), we use Lemma A.1(e) together with (1.1), so that for $x < \underline{z}$,

$$\gamma(h(x), x) = k\frac{g(x; h(x))}{g'(x; h(x))} - kx + K = \frac{1}{\lambda}(k\mu(x) - \lambda(kx - K)) < 0.$$

Part (b) follows from the proof of (a) and the definition of $\underline{z}$. □

Proof that $z_0 < x_1^*$ in Lemma 2.1. By the comment right after Lemma 2.1, $g_1''(0) < 0$ and since g is convex on $[0, x_1^*]$, $0 \le g''(0) = g_2''(0) - g_1''(0)$, i.e. $h(0) \le 1$. By Lemma 2.1(i), $z_0 < x_\lambda$, so by Lemma A.1(d), h is decreasing on $(0, z_0)$, in particular $h(x) < 1$ on $(0, z_0)$. But by definition of z_0, $g_1''(x) < 0$ on $(0, z_0)$, and so $g''(x) = g_1''(x)(h(x) - 1) > 0$ on $(0, z_0)$. Consequently by continuity, $z_0 \le x_1^*$. If $z_0 = x_1^*$, then $h(x_1^*) = 1$ implies that $g_1''(z_0) = g_2''(z_0) = 0$, which in turn implies that $g_1''(z_0)g_2'(z_0) = g_2''(z_0)g_1'(z_0)$. But by (A.3), this is equivalent to the Wronskian $W(z_0) = 0$, a contradiction. Hence $z_0 < x_1^*$. Note that the proof is independent of (1.3) and (2.5). □

Proof of Lemma 2.3. Since $\underline{z} \in (x_1^*, x_\lambda)$ and h decreases in this interval, the fact that $h(x_1^*) = 1$ yields that $h(\underline{z}) < 1$.

Next assume that $z_0 > 0$, and also assume that $h(0) > h(\underline{z})$. We will show that this gives a contradiction. Let $\beta \in (h(\underline{z}), h(0))$. Then $\beta < 1$ since as we saw in the last proof, $h(0) \le 1$. We also saw in that proof that $g_2''(z_0) \ne 0$, and since $g_1''(x) < 0$ when $x < z_0$, $h(x) \to -\infty$ as $x \uparrow z_0$. Therefore, there is a unique $z_\beta < z_0$ so that $h(z_\beta) = \beta$. By Lemma A.1(g), as a function of x, $\gamma(\beta, x)$ increases on $(0, z_\beta)$ and decreases on (z_β, x_1^*). By Lemma A.2(a), the maximum $\gamma(h(z_\beta), z_\beta) < 0$. But by Lemma A.1(f) and (2.1), $\gamma(\beta, \bar{u}^*) > \gamma(1, \bar{u}^*) = 0$, a contradiction since $\bar{u}^* < x_1^* < \underline{z}$.

For the last part, assume that $\beta \in (h(\underline{z}), 1]$. Assume first that $x \in (z_0, h_-^{-1}(\beta))$. Then $g_1''(x) > 0$ and $h(x) > h(h_-^{-1}(\beta)) = \beta$. Therefore,

$$g''(x; \beta) = g_1''(x)(h(x) - \beta) > 0.$$

Assume next that $z_0 > 0$ and that $x \in (0, z_0)$. Then $g_1''(x) < 0$ and $h(x) < h(0) \le h(\underline{z}) < \beta$, where we used the result just proved. Therefore, $g''(x; \beta) > 0$ again. Finally, assume that $x = z_0$. Then $g''(z_0; \beta) = g_2''(z_0) = g''(z_0) > 0$ since $z_0 < x_1^*$ and $g_1''(z_0) = 0$. This ends the proof. □

Proof of Lemma 2.4. For any $\beta \in (h(\underline{z}), 1)$, by Lemma A.1(f) and (2.1), $\gamma(\beta, \bar{u}^*) > \gamma(1, \bar{u}^*) = 0$, and by Lemma A.2(a), $\gamma(\beta, h_-^{-1}(\beta)) < 0$. Therefore, since $\gamma(\beta, x)$ is continuous in x, there is at least one $u_1(\beta) \in (\bar{u}^*, h_-^{-1}(\beta))$ so that $\gamma(\beta, u_1(\beta)) = 0$. Since $u_1(\beta) < h_-^{-1}(\beta)$, it follows from Lemma A.3 that $g''(u_1(\beta); \beta) > 0$, and so by Lemma A.2(g),

$$\frac{\partial}{\partial x}\gamma(\beta, u_1(\beta)) < 0$$

for all $\beta \in (h(\underline{z}), 1)$. This implies in particular that $u_1(\beta)$ is unique, and furthermore by the implicit function theorem, $u_1(\beta)$ is continuously differentiable on $(h(\underline{z}), 1)$. Finally, using that $\frac{\partial}{\partial\beta}\gamma(\beta, u_1(\beta)) = 0$ and Lemma A.2(g) yields

$$u_1'(\beta) = -\frac{\frac{\partial}{\partial\beta}\gamma(\beta, u_1(\beta))}{\frac{\partial}{\partial x}\gamma(\beta, u_1(\beta))} < 0,$$

and so u_1 is strongly decreasing. □

Proof of Lemma 2.5. Taking the derivative w.r.t. β in $\gamma(\beta, u_1(\beta)) = 0$ gives after some simplification that

$$\frac{d}{d\beta} g'(u_1(\beta); \beta) = -\frac{g_1(u_1(\beta))g'(u_1(\beta); \beta)}{g(u_1(\beta); \beta)}. \tag{A.4}$$

Taking the derivative of $G(\beta)$ and using (A.4) gives

$$G'(\beta) = \frac{g'_{2,\infty} g_1(u_1(\beta)) - g'_{1,\infty} g_2(u_1(\beta))}{g'(u_1(\beta); \beta) g(u_1(\beta); \beta)}. \tag{A.5}$$

We have

$$\frac{g_2'(x)}{g_1'(x)} - \frac{g_2(x)}{g_1(x)} = \frac{W(x)}{g_1'(x) g_1(x)},$$

which is positive for $x > x_\lambda$. Consequently, by Lemma A.1(b),

$$\frac{g_2'(x)}{g_1'(x)} > \frac{g_2(x)}{g_1(x)} > \frac{g_2(u_1(\beta))}{g_1(u_1(\beta))}, \quad x > x_\lambda.$$

Letting $x \to \infty$ gives

$$\frac{g'_{2,\infty}}{g'_{1,\infty}} > \frac{g_2(u_1(\beta))}{g_1(u_1(\beta))},$$

and the result follows from (A.5). □

Proof of Lemma 2.6. The part about u_1 is proved in Lemma 2.4. Since $g'(x; \beta)$ has a local maximum at $x = h_-^{-1}(\beta)$ and a local minimum at $x = h_+^{-1}(\beta)$, for existence of u_2 and u_3 all we need to show is that for all $\beta \in (\alpha_0, \beta_0)$,

1. $g'(u_1(\beta); \beta) > g'(h_+^{-1}(\beta); \beta)$.
2. $\lim_{x\to\infty} g'(x; \beta) > g'(u_1(\beta); \beta)$.

But (1) is satisfied by the definition of β_0 in (2.7) since $\beta < \beta_0$, while (2) follows since $G(\alpha_0) \geq 1$ and G, whenever finite, is strictly increasing by Lemma 2.5. Furthermore, letting $w(\beta, u) = g'(u_1(\beta); \beta) - g'(u; \beta)$ gives $w(\beta, u_2(\beta)) = w(\beta, u_3(\beta)) = 0$, and then it easily follows from the implicit function theorem that u_2 and u_3 are continuously differentiable.

When $\alpha_0 = h(\underline{z})$ it follows from Lemma 2.4 that $u_1(\alpha_0) = \underline{z} = h_-^{-1}(\alpha_0)$ and so $g'(x; \alpha_0)$ has a maximum at $x = \underline{z}$, which implies that $u_1(\alpha_0) = u_2(\alpha_0) = \underline{z}$.

When $\alpha_0 > h(\underline{z})$, then since $G(\alpha_0) = 1$,

$$\frac{\lim_{x\to\infty} g'(x;\beta)}{g'(u_1(\beta);\beta)} \downarrow 1 \quad \text{as } \beta \downarrow \alpha_0.$$

Therefore, to find the $u_3(\beta)$ that satisfies $g'(u_3(\beta);\beta) = g'(u_1(\beta);\beta)$ we have to go further and further out as $\beta \downarrow \alpha_0$.

If $\beta_0 < \min\{1, h_\infty\}$, by continuity $v(\beta_0) = 1$, and so we can set $u_3(\beta_0) = u_2(\beta_0) = h_+^{-1}(\beta_0)$. □

Proof of Lemma 2.7. Using that $\gamma(\beta, u_1(\beta)) = 0$ and that $g'(u_2(\beta);\beta) = g'(u_1(\beta);\beta)$ we get

$$J_1(\beta) = J_1(\beta) - \gamma(\beta, u_1(\beta)) = k\left(u_2(\beta) - \frac{g(u_2(\beta);\beta)}{g'(u_1(\beta);\beta)}\right) - K.$$

Since

$$\frac{d}{d\beta} g(u_2(\beta);\beta) = g'(u_1(\beta);\beta)u_2'(\beta) - g_1(u_2(\beta))$$

we get

$$\frac{d}{d\beta} J_1(\beta) = \frac{k}{(g'(u_1(\beta);\beta))^2} v_1(\beta),$$

where

$$\begin{aligned} v_1(\beta) &= g'(u_1(\beta);\beta)g_1(u_2(\beta)) + g(u_2(\beta);\beta)\frac{d}{d\beta}g'(u_1(\beta);\beta) \\ &= g'(u_1(\beta);\beta)g_1(u_2(\beta)) - \frac{g(u_2(\beta);\beta)g_1(u_1(\beta))g'(u_1(\beta);\beta)}{g(u_1(\beta);\beta)}, \end{aligned}$$

where we used (A.4) in the second equality. Clearly $J_1(\beta)$ and $v_1(\beta)$ have the same sign, and since $g(u_1(\beta);\beta)$ and $g'(u_1(\beta);\beta)$ are both positive, the sign of $v_1(\beta)$ is the same as the sign of

$$\begin{aligned} v_2(\beta) &= g_1(u_2(\beta))g(u_1(\beta);\beta) - g_1(u_1(\beta))g(u_2(\beta);\beta) \\ &= g_1(u_2(\beta))g_2(u_1(\beta)) - g_2(u_2(\beta))g_1(u_1(\beta)) \\ &= g_1(u_1(\beta))g_1(u_2(\beta))\left(\frac{g_2(u_1(\beta))}{g_1(u_1(\beta))} - \frac{g_2(u_2(\beta))}{g_1(u_2(\beta))}\right) < 0, \end{aligned}$$

by Lemma A.1(b). Replacing $u_2(\beta)$ by $u_3(\beta)$ we get the same conclusion for $J_{13}(\beta)$. □

Proof of Proposition 2.1. Since either R5 or R6 apply, it follows from the comment after Lemma 2.3 that $h(\underline{z}) < h_\infty$. Therefore, by Lemma 2.6, for any $\beta \in [\alpha_0, \beta_0)$ there are numbers $u_i = u_i(\beta)$ $i = 1, 2, 3$, so that

$$g'(u_1;\beta) = g'(u_2;\beta) = g'(u_3;\beta),$$

where $u_1 = u_2 = \underline{z}$ if $\beta = \alpha_0 = h(\underline{z})$.

Assume G1 or G2. We will show that

$$J_{13}(\alpha_0) \geq 0 \quad \text{and} \quad J_{13}(\beta_0) < 0. \tag{A.6}$$

In case G1, $\alpha_0 = h(\underline{z})$ and by (2.11), $J_1(h(\underline{z})) = 0$. Also $J_2(h(\underline{z})) > 0$ and so $J_{13}(\alpha_0) = J_1(\alpha_0) + J_2(\alpha_0) > 0$. In case G2, $J_{13}(\alpha_0) \geq 0$ by assumption. To prove that $J_{13}(\beta_0) < 0$ assume first that $\beta_0 < 1$, which implies that $\beta_0 < \min\{1, h_\infty\}$ by Lemma 2.2. Therefore, $J_1(\beta_0) < 0$

while $J_2(\beta_0) = 0$ by (2.13), and so $J_{13}(\beta_0) < 0$. If $\beta_0 = 1$, then $J_{13}(1) = K_3 - K_2 < 0$, and so (A.6) is proved.

By Lemma 2.7, J_{13} is strongly decreasing on (α_0, β_0), and so by (A.6) there is a unique $\hat{\beta}$ so that $J_{13}(\hat{\beta}) = 0$, i.e. so that $-J_1(\hat{\beta}) = J_2(\hat{\beta})$. Assume first that $\beta_0 < 1$. If $J_2(\hat{\beta}) \leq K$, let $\tilde{\beta} = \hat{\beta}$. If $J_2(\hat{\beta}) > K$, since $J_2(\beta_0) = 0$, we can define $\tilde{\beta} < \beta_0$ by

$$\tilde{\beta} = \min\{\beta > \hat{\beta} : J_2(\beta) = K\}.$$

Then since $J_{13}(\beta)$ is decreasing,

$$0 > J_{13}(\tilde{\beta}) = K + J_1(\tilde{\beta});$$

hence $-J_1(\tilde{\beta}) > K$.

If $\beta_0 = 1$, then $J_2(1) = K_3 < K$ and we can use the same arguments.

Finally assume that G3 holds. Then $J_{13}(\alpha_0) < 0$, but since $J_2(\alpha_0) > K$ and

$$J_2(\beta_0) = \begin{cases} 0, & \beta_0 < 1, \\ K_3, & \beta_0 = 1, \end{cases}$$

and $K_3 < K$, we can define $\tilde{\beta} < \beta_0$ by

$$\tilde{\beta} = \min\{\beta > \alpha_0 : J_2(\beta) = K\}.$$

Since $0 > J_{13}(\alpha_0) > J_{13}(\tilde{\beta})$, $-J_1(\tilde{\beta}) > J_2(\tilde{\beta})$, and so case (iii-b) applies. □

Proof of Proposition 2.2. Assume first that $h_\infty = 1$. By letting $h_+^{-1}(1) = \lim_{\beta \to 1} h_+^{-1}(\beta) = \infty$ and using that $u_1(\beta) \to \bar{u}^*$ as $\beta \to 1$ we can use (2.6) to conclude that $v(\beta) \to v(1)$ as $\beta \to 1$, where

$$v(1) = \frac{g'_\infty}{g'(\bar{u}^*)} = \frac{c^*}{c^\infty} > 1,$$

by assumption. Therefore, $\beta_0 < 1$, and so trivially $\beta_0 < \min\{1, h_\infty\}$. Clearly α_0 can be well defined as well, and $\alpha_0 < \beta_0$ by (2.8). Hence the assumptions of Lemma 2.6 hold.

Assume next that $h(\underline{z}) < h_\infty < 1$ and that $G(h_\infty) > 1$. We have by (2.8),

$$1 < \frac{G(\beta)}{v(\beta)} = \frac{\lim_{x \to \infty} g'(x; \beta)}{g'(h_+^{-1}(\beta); \beta)} \to 1 \quad \text{as } \beta \uparrow h_\infty,$$

and since $G(h_\infty) > 1$, it follows that $v(\beta) > 1$ for β sufficiently close to h_∞. Therefore, $\beta_0 < h_\infty = \min\{1, h_\infty\}$ and again all assumptions of Lemma 2.6 hold.

The rest of the proof is now the same as the proof of Proposition 2.1 for the case with $\beta_0 < 1$. □

We will now turn to the proof of Theorem 3.1. The proof is standard once the necessary variational inequalities have been established, and that is the topic of the next two lemmas.

Lemma A.3. *Let V be the proposed value functions in Theorem 3.1. Then V is continuously differentiable on $(0, \infty)$ and twice continuously differentiable on the set $(0, \bar{u}^*) \cup (\bar{u}^*, \tilde{u}_1) \cup (\tilde{u}_1, \tilde{u}_3) \cup (\tilde{u}_3, \infty)$. Furthermore,*

$$LV(x) = 0 \quad \text{on } (0, \bar{u}^*) \cup (\tilde{u}_1, \tilde{u}_3) \quad \text{and} \quad LV(x) < 0 \quad \text{on } (\bar{u}^*, \tilde{u}_1) \cup (\tilde{u}_3, \infty).$$

Proof. Consider first case (iii-a). By definition, $c^*g(\bar{u}^*-) = k\bar{u}^* - K$ and $c^*g'(\bar{u}^*-) = k$; hence V is continuously differentiable at $\bar{u}^*$. Also, by definition

$$V(\tilde{u}_1) = \frac{k}{g'(\tilde{u}_1;\tilde{\beta})} g(\tilde{u}_1;\tilde{\beta}) = \gamma(\tilde{\beta},\tilde{u}_1) + k\tilde{u}_1 - K = k\tilde{u}_1 - K = V(\tilde{u}_1-)$$

and clearly $V'(\tilde{u}_1+) = k$; hence V is continuously differentiable at $\tilde{u}_1$. Similarly, V is continuously differentiable at $\tilde{u}_3$.

By definition, $LV(x) = 0$ for $x \in (0,\bar{u}^*) \cup (\tilde{u}_1,\tilde{u}_3)$. Let $x \in (\bar{u}^*,\tilde{u}_1)$. Then

$$LV(x) = \mu(x)k - \lambda(kx - K) < 0,$$

by (2.5) and the fact that $\tilde{u}_1 < \underline{z}$.

Finally, let $x \in (\tilde{u}_3,\infty)$. Then

$$\begin{aligned} LV(x) &= \mu(x)V'(x) - \lambda V(x) \\ &\le \mu(\tilde{u}_3)k - \lambda V(\tilde{u}_3) \\ &= \mu(\tilde{u}_3-)V'(\tilde{u}_3-) - \lambda V(\tilde{u}_3-) \\ &< \frac{1}{2}\sigma^2(\tilde{u}_3-)V''(\tilde{u}_3-) + \mu(\tilde{u}_3-)V'(\tilde{u}_3-) - \lambda V(\tilde{u}_3-) \\ &= LV(\tilde{u}_3-) = 0. \end{aligned}$$

Here the first inequality follows from the fact that for $x > \tilde{\mu}_3$, $\frac{d}{dx}(\mu(x)k - \lambda V(x)) = k(\mu'(x) - \lambda) \le 0$. The next equality is clear since V is continuously differentiable and $V'(\tilde{u}_3+) = k$. For the second inequality, note that as in (2.3),

$$g''(\tilde{u}_3;\tilde{\beta}) = g_1''(\tilde{u}_3)(h(\tilde{u}_3) - \tilde{\beta}).$$

But g_1 is convex on (x_λ,∞) and $\tilde{\beta} < h(\tilde{u}_3)$ by Proposition 2.1 since h_+ is increasing. Consequently $V''(\tilde{u}_3-) > 0$.

Now consider case (iii-b). That V is continuously differentiable at $\bar{u}^*$ and $\tilde{u}_1$ is proved as above. Furthermore,

$$\begin{aligned} V(\tilde{u}_3) &= c_{\tilde{\beta}} g(\tilde{u}_2;\tilde{\beta}) + k(\tilde{u}_3 - \tilde{u}_2) - K \\ &= c_{\tilde{\beta}} g(\tilde{u}_3;\tilde{\beta}) + J_2(\tilde{\beta}) - K \\ &= c_{\tilde{\beta}} g(\tilde{u}_3;\tilde{\beta}) \\ &= V(\tilde{u}_3-). \end{aligned}$$

Trivially, $V'(\tilde{u}_3+) = V'(\tilde{u}_3-) = k$. Finally, the signs of $LV(x)$ are shown just as above. □

For a function $\phi : [0,\infty) \mapsto [0,\infty)$, define the maximum utility operator M by

$$\begin{aligned} M\phi(x) &= \sup\{\phi(x-\eta) + k\eta - K : 0 \le \eta \le x\} \\ &= \sup\{\phi(z) + k(x-z) - K : 0 \le z \le x\}. \end{aligned}$$

Lemma A.4. *Let V be as in Lemma A.3. Then*

$$MV(x) = V(x) \quad \text{on } [\bar{u}^*,\tilde{u}_1] \cup [\tilde{u}_3,\infty) \quad \text{and} \quad MV(x) < V(x) \quad \text{on } (0,\bar{u}^*) \cup (\tilde{u}_1,\tilde{u}_3).$$

Proof. The following elementary argument will often be used in the proof. Assume we want to maximize $h(y) = \phi(y) + k(x-y) - K$ for $y \in [a_0, a_1]$ with $0 \le a_0 < a_1 \le x$. Then $h'(y) = \phi'(y) - k$, so if $\phi'(y) \le k$ on $[a_0, a_1]$, a maximum is at $y = a_0$. Conversely, if $\phi'(y) \ge k$ on $[a_0, a_1]$, a maximum is at $y = a_1$.

Consider first case (iii-a). For $x \in (0, \tilde{u}_1]$, $V'(y) \le k$ for $y \in (0, x)$; hence $MV(x) = V(0) + kx - K = kx - K$. Therefore, $MV(x) = V(x)$ on $[\bar{u}^*, \tilde{u}_1]$, while for $x \in [0, \bar{u}^*)$,

$$MV(x) = (V(\bar{u}^*) - k\bar{u}^* + K) + kx - K = V(x) + \int_x^{\bar{u}^*} (V'(y) - k)dy < V(x).$$

Let $x \in (\tilde{u}_1, \tilde{u}_2)$. Then

$$V'(x) \begin{cases} < 0, & y \in (0, \tilde{u}_1), \\ > 0, & y \in (\tilde{u}_1, x). \end{cases}$$

Therefore, $MV(x) = \max\{V(x) - K, kx - K\}$. Clearly $V(x) - K < V(x)$ and

$$\begin{aligned} V(x) - (kx - K) &= \int_{\tilde{u}_1}^x (V'(y) - k)dy + V(\tilde{u}_1) - (k\tilde{u}_1 - K) \\ &= \int_{\tilde{u}_1}^x (V'(y) - k)dy > 0, \end{aligned}$$

and so $MV(x) < V(x)$.

Let $x \in [\tilde{u}_2, \tilde{u}_3)$. Then, since $V'(y) > k$ on $(\tilde{u}_1, \tilde{u}_2)$ while $V'(y) < k$ on $(0, \tilde{u}_1) \cup (\tilde{u}_2, x)$, we get

$$MV(x) = \max\{V(\tilde{u}_2) + k(x - \tilde{u}_2) - K, kx - K\}. \tag{A.7}$$

Now,

$$V(\tilde{u}_2) - k\tilde{u}_2 = V(\tilde{u}_1) - k\tilde{u}_1 - J_1(\tilde{\beta}) \le V(\tilde{u}_1) - k\tilde{u}_1 + K = 0,$$

and therefore $V(\tilde{u}_2) + k(x - \tilde{u}_2) - K \le kx - K$, implying that $MV(x) = kx - K$. But then

$$MV(x) - V(x) = k(x - \tilde{u}_1) - (V(x) - V(\tilde{u}_1)) = J(\tilde{\beta}, \tilde{u}_1, x) < J(\tilde{\beta}, \tilde{u}_1, \tilde{u}_3) = 0,$$

and consequently $MV(x) < V(x)$.

Finally, let $x \in [\tilde{u}_3, \infty)$. Then $MV(x)$ is again given by (A.7), and as there $MV(x) = kx - K = V(x)$.

We continue with case (iii-b). When $x \in [0, \tilde{u}_2)$ the proof is just as above, so assume that $x \in [\tilde{u}_2, \tilde{u}_3)$. Then again $MV(x)$ is given by (A.7), but now

$$V(\tilde{u}_2) - k\tilde{u}_2 = V(\tilde{u}_1) - k\tilde{u}_1 - J_1(\tilde{\beta}) > V(\tilde{u}_1) - k\tilde{u}_1 + K = 0,$$

so $MV(x) = V(\tilde{u}_2) + k(x - \tilde{u}_2) - K$ and therefore

$$MV(x) - V(x) = J(\tilde{\beta}, \tilde{u}_2, x) - K < J(\tilde{\beta}, \tilde{u}_2, \tilde{u}_3) - K = 0,$$

and so $MV(x) < V(x)$.

When $x \in [\tilde{u}_3, \infty)$, as above $MV(x) = V(\tilde{u}_2) + k(x - \tilde{u}_2) - K = V(x)$. □

Proof of Theorem 3.1. Using the variational inequalities in Lemma A.3 and A.4, the proof is exactly as the proof of Theorem 2.2 in [3]. □

Proof of Theorem 3.2. The proof is exactly like the proof of the (more interesting) Theorem 4.2. □

Proof of Theorem 3.3. We can set $u_3(1) = \infty$ and then since it is case R6, $J_{13}(1) = K_3 - K_2 < 0$ and $J_2(1) = K_3 < \min\{K_2, K\}$. With $V(x) = c^* g(x)$, clearly $LV(x) = 0$. The proof of the variational inequality $MV(x) < V(x)$ is as in Lemma A.4. Then the rest is as in the proof of Theorem 4.3. □

Proof of Theorem 4.1. That is exactly like the proof of Theorem 3.1. □

Proof of Lemma 4.1. Since $G(1) = \frac{g'_\infty}{g'(\bar{u}^*)} = \frac{c^*}{c^\infty} > 1$ by assumption, it follows from Lemma 2.5 that if it exists, $\hat{\beta} < 1$. In case (1), note that by Lemma 2.4, $g'(u_1(h(\underline{z})); h(\underline{z})) = g'(\underline{z}; h(\underline{z}))$ and by (2.3),

$$g''(x; h(\underline{z})) = g_1''(x)(h(x) - h(\underline{z})) < 0, \quad x > \underline{z},$$

since $\underline{z} > z_0$ and $h(\underline{z}) \geq h_\infty$. Therefore,

$$\lim_{x\to\infty} g'(x; h(\underline{z})) < g'(u_1(h(\underline{z})); h(\underline{z})),$$

i.e. $G(h(\underline{z})) < 1$ so $\hat{\beta}$ exists and $\hat{\beta} > h(\underline{z})$. In case (2) $G(h_\infty) \leq 1$ by assumption, and the result follows. Also $\hat{\beta} \geq h_\infty > h(\underline{z})$. In case (3), $G(\alpha_0) = 1$; hence $\hat{\beta} = \alpha_0$. Since in all three cases $\hat{\beta} > h(\underline{z})$,

$$u_1(\hat{\beta}) < h_-^{-1}(\hat{\beta}) < h_-^{-1}(h(\underline{z})) = \underline{z},$$

and this ends the proof. □

Proof of Theorem 4.2. The proof that the proposed value function V (we write V instead of V^*) is continuously differentiable is straightforward, and is omitted.

We must prove the variational inequalities

$$LV(x) = 0 \quad \text{on } (0, \bar{u}^*) \cup (\hat{u}_1, \infty) \quad \text{and} \quad LV(x) < 0 \quad \text{on } (\bar{u}^*, \hat{u}_1). \tag{A.8}$$

The first part is obvious, and the second is identical to that in the proof of Lemma A.3.

The next step is to prove the variational inequalities

$$MV(x) < 0 \quad \text{on } (0, \bar{u}^*) \cup (\hat{u}_1, \infty) \quad \text{and} \quad MV(x) = 0 \quad \text{on } (\bar{u}^*, \hat{u}_1). \tag{A.9}$$

We begin by proving

$$c_{\hat{\beta}} g'(x; \hat{\beta}) > k, \quad x > \hat{u}_1, \tag{A.10}$$

where we throughout the proof write $\hat{u}_1$ for $u_1(\hat{\beta})$. This is equivalent to

$$m(x) \stackrel{\text{def}}{=} g'(x; \hat{\beta}) - g'(\hat{u}_1; \hat{\beta}) > 0, \quad x > \hat{u}_1.$$

Note that $m(\hat{u}_1) = 0$ and since $G(\hat{\beta}) = 1$, $\lim_{x\to\infty} m(x) = 0$. However, $m'(x) = g''(x; \hat{\beta}) = g_1''(x)(h(x) - \hat{\beta})$, so in particular

$$m'(\hat{u}_1) = g_1''(\hat{u}_1)(h(\hat{u}_1) - \hat{\beta}).$$

Clearly $g_1''(\hat{u}_1) > 0$. Also, $\hat{u}_1 < h_-^{-1}(\hat{\beta})$, and since h is decreasing on (x_1^*, x_λ), $h(\hat{u}_1) > h(h_-^{-1}(\hat{\beta})) = \hat{\beta}$. Consequently, $m'(\hat{u}_1) > 0$. Furthermore, $m'(x)$ has only one zero on $(\hat{u}_1, \infty)$

and so (A.10) is proved. To prove (A.9), we only prove that $MV(x) < 0$ on $(\hat{u}_1, \infty)$, since the rest is as in Lemma A.4. We get for $x > \hat{u}_1$,

$$MV(x) = \max\{V(x) - K, kx - K\}.$$

Now

$$\begin{aligned} kx - K - V(x) &= \int_0^x (k - V'(y))dy - K \\ &= \int_0^{\bar{u}^*} (k - c^* g'(y))dy + \int_{\hat{u}_1}^x (k - c_{\hat{\beta}} g'(y; \hat{\beta}))dy - K \\ &= \int_{\hat{u}_1}^x (k - c_{\hat{\beta}} g'(y; \hat{\beta}))dy < 0, \end{aligned}$$

where we used (A.10) in the last inequality. Therefore, $MV(x) < V(x)$ on $(\hat{u}_1, \infty)$. Using the variational inequalities (A.8) and (A.9), standard arguments, see e.g. [3], shows that $V_{\bar{u}^*,0}(x) = V(x) = V^*(x)$ for $x \le \hat{u}_1$.

For the lump sum dividend barrier strategy $\pi_{\bar{u},0}$ with $\bar{u} > \hat{u}_1$, it is easy to prove that its value function is

$$V_{\bar{u},0}(x) = \begin{cases} \delta(\bar{u})g(x), & x \in [0, \bar{u}), \\ kx - K, & x \in [\bar{u}, \infty), \end{cases}$$

where $\delta(\bar{u}) = \frac{k\bar{u} - K}{g(\bar{u})}$. Then, by the same arguments as in Theorem 2.1(c) in [3], $V_{\bar{u},0}(x)$ is increasing and $\lim_{\bar{u}\to\infty} V_{\bar{u},0}(x) = V(x)$ for $x > \hat{u}_1$.

Assume that there exists an optimal strategy π^* when the initial surplus $x > \hat{u}_1$. Since V is twice continuously differentiable except at a finite number of points, Itô's formula used on $e^{-\lambda t} V(X_{t+})$ stopped at the time of ruin τ^{π^*} gives

$$\begin{aligned} e^{-\lambda(t\wedge\tau^{\pi^*})} V(X^{\pi^*}_{(t\wedge\tau^{\pi^*})+}) &= V(x) + \int_0^{t\wedge\tau^{\pi^*}} e^{-\lambda s} LV(X_s^{\pi^*})ds \\ &\quad + \int_0^{t\wedge\tau^{\pi^*}} e^{-\lambda s} \sigma(X_s^{\pi^*}) V'(X_s^{\pi^*}) dW_s \\ &\quad + \sum_{s\le t\wedge\tau^{\pi^*}} e^{-\lambda s} \left(V(X_{s+}^{\pi^*}) - V(X_s^{\pi^*}) \right) \\ &< V(x) + \int_0^{t\wedge\tau^{\pi^*}} e^{-\lambda s} \sigma(X_s^{\pi^*}) V'(X_s^{\pi^*}) dW_s \\ &\quad - \sum_{n=1}^{\infty} e^{-\lambda \tau_n^{\pi^*}} (k\xi_n^{\pi^*} - K) 1_{\{\tau_n^{\pi^*} \le t\wedge\tau^{\pi^*}\}}. \end{aligned}$$

The inequality follows from the fact that if $X^{\pi^*}_{\tau_1^*} < \hat{u}_1$ then by (A.8),

$$P_x \left(\int_0^{t\wedge\tau^{\pi^*}} e^{-\lambda s} LV(X_s^{\pi^*}) ds < 0 \right) = 1,$$

while if $X^{\pi^*}_{\tau_1^*} \geq \hat{u}_1$ then by (A.9),

$$P_x\left(\sum_{s\leq t\wedge\tau^{\pi^*}} e^{-\lambda s}\left(V(X^{\pi^*}_{s+}) - V(X^{\pi^*}_s)\right) < -\sum_{n=1}^{\infty} e^{-\lambda\tau_n^{\pi^*}}(k\xi_n^{\pi^*} - K)1_{\{\tau_n^{\pi^*}\leq t\wedge\tau^{\pi^*}\}}\right) = 1.$$

Taking expectations and letting $t \to \infty$ gives $0 < V(x) - V_{\pi^*}(x)$, a contradiction since $\lim_{\bar{u}\to\infty} V_{\bar{u},0}(x) = V(x)$ for $x > \hat{u}_1$. □

Proof of Theorem 4.3. The proof is basically the same as that of Theorem 3.3, and is omitted. □

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