Extensions of inequalities for unitarily invariant norms via log majorization

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A capital letter means $n \times n$ matrix. T is said to be positive definite (denoted by $T > 0$) if $T$ is positive semidefinite and invertible. We shall show the following central results via log majorization obtained by an order preserving operator inequality.

Theorem. If $A > 0$ and $B \geq 0$, then for $0 \leq \alpha \leq 1$, $t \in [0, 1]$ and $r \geq t$

$$\left\{ A^{1-t} A^t \right\}^{\frac{1}{s-1-t}} \geq A^w \left( A^t B^w \right)^{\frac{1}{s}}$$

holds for

$$\frac{(1 - \alpha)(r - t)}{1 - \alpha t} + 1 \geq s \geq 1,$$

where $w = (1 - \alpha)(s - r) + \alpha(1 - t)s$.

Our result extends the following recent elegant inequality by Matharu and Aujila.

Let $A, B$ be positive definite and $\alpha \in [0, 1]$. Then

$$\prod_{i=1}^{k} \lambda_j \left( A^{1-\alpha} B^\alpha \right) \geq \prod_{i=1}^{k} \lambda_j \left( A^w B^w \right) 1 \leq k \leq n.$$ 

Also some results associated with log majorization are shown.

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1. Introduction

Throughout this paper, a capital letter means \( n \times n \) matrix. \( T \) is said to be positive definite (denoted by \( T > 0 \)) if \( T \) is positive semidefinite and invertible.

Following after Ando and Hiai [1], let us write \( A \succ (\log) B \) for positive semidefinite matrices \( A, B \geq 0 \) and call the log majorization if

\[
\prod_{i=1}^{k} \lambda_i(A) \geq \prod_{i=1}^{k} \lambda_i(B), \quad k = 1, 2, \ldots, n-1,
\]

and

\[
\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B), \quad \text{i.e.,} \quad \det A = \det B,
\]

where \( \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \) and \( \lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B) \) are the eigenvalues of \( A \) and \( B \) respectively arranged in decreasing order. By \( s_1(X) \geq s_2(X) \geq \cdots \geq s_n(X) \), we denote the eigenvalues of \( |X| = (X^*X)^{1/2} \), i.e., singular values of \( X \).

\( A \succ (\log) B \) ensures \( \|A\| \geq \|B\| \) for every unitarily invariant norm, in fact, the log majorization gives a powerful technique for matrix norm inequalities.

When \( 0 \leq \alpha \leq 1 \), the \( \alpha \)-power mean of \( A, B > 0 \) is defined by

\[
A \#_{\alpha} B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}}.
\]

Further \( A \#_{\alpha} B \) for \( A, B \geq 0 \) is defined by \( A \#_{\alpha} B = \lim_{\epsilon \downarrow 0} (A + \epsilon I) \#_{\alpha} (B + \epsilon I) \).

Theorem A [10, Theorem 2.10]. Let \( A, B \) be positive definite and \( \alpha \in [0, 1] \). Then

\[
\prod_{i=1}^{k} \lambda_j(A^{1-\alpha} B^\alpha) \geq \prod_{i=1}^{k} \lambda_j(A_i^{\alpha} B) \quad 1 \leq k \leq n.
\] (1.1)

Theorem B [10, Corollary 2.13]. Let \( A, B \) be positive definite and \( \alpha \in [0, 1] \). Then

\[
\prod_{i=1}^{k} s_j((B^{1/2} A B^{1/2})^{1/2}) \geq \prod_{i=1}^{k} s_j(A_i^{\alpha} B) \quad 1 \leq k \leq n.
\] (1.2)

This implies

\[
\|||B^{1/2} A B^{1/2}||| \geq |||A_i^{\alpha} B|||
\] (1.3)

for all unitarily invariant norm \(||.||\).

Also merely (1.3) of Theorem B for positive semidefinite matrices is shown in [8, Proposition 3.7] motivated by the comparison between two different types of Young’s inequality for matrix version.

We shall show further extensions of Theorem A, Theorem B and related results via log majorization obtained by Theorem C which is an order preserving operator inequality.

2. Log majorization via an order preserving operator inequality

In this section, we shall show some results on log majorization obtained by Theorem C which can be considered as an order preserving operator inequality.
Löwner–Heinz inequality [9,7] asserts that if \( A \succeq B \succeq 0 \), then \( A^\alpha \succeq B^\alpha \) for any \( \alpha \in [0,1) \). As an extension of Löwner–Heinz inequality, we have the following result [4, Theorem 1], if \( A \succeq B \succeq 0 \), then for \( p \geq 0 \)
\[
A^{(p+r)\alpha} \succeq (A^{\frac{r}{p}} B^p A^{\frac{1}{p}})^\alpha
\]
holds for \( r \geq 0 \) and \( 0 \leq \alpha \leq 1 \) with \( 1 + r \geq (p + r)\alpha \).

In order to interpolate both the inequality stated above and an inequality equivalent to the main result on log majorization by Ando–Hiai [1, Theorem 3.5], we state the following Theorem C.

**Theorem C.** If \( A \succeq B \succeq 0 \) with \( A \succ 0 \), then for \( t \in [0,1] \), \( p \geq 1 \) and \( s \geq 1 \),
\[
A^{(p-t)s+r)\alpha} \succeq \left\{ A^{\frac{t}{p}} \left( A^{\frac{1}{p}} B^p A^{\frac{1}{p}} \right)^{\frac{r}{p}} A^{\frac{1}{p}} \right\}^\alpha
\]
holds for \( r \geq t \) and \( 0 \leq \alpha \leq 1 \) with \( 1 - t + r \geq ((p - t)s + r)\alpha \).

The original proof of Theorem C is shown in [5, Corollary 1.2] and alternative proofs are given in [3,6, 133 p.] and best possibility of the exponential power of (\( \ast \)) is proved in [11]. We need the following variation of Theorem C to give a proof of Theorem 2.1 which is our main result.

**Lemma D.** If \( A \succeq B \succeq 0 \) with \( A \succ 0 \), then for \( t \in [0,1] \) and \( 0 < \alpha \leq 1 \),
\[
A^{(1-\alpha)t)s+r)\alpha} \succeq \left\{ A^{\frac{t}{p}} \left( A^{\frac{1}{p}} B^p A^{\frac{1}{p}} \right)^{\frac{r}{p}} A^{\frac{1}{p}} \right\}^\alpha
\]
holds for \( r \geq t \) with \( \frac{(1 - \alpha)(r - t)}{1 - \alpha t} + 1 \geq s \geq 1 \).

**Proof.** In Theorem C, put \( p = \frac{1}{\alpha} \geq 1 \) for \( 0 < \alpha \leq 1 \). Then \( p \geq 1 \) and \( s \geq 1 \) with \( 1 - t + r \geq ((p - t)s + r)\alpha \) in Theorem C can be rewritten as follows:

\[
1 - t + r \geq ((p - t)s + r)\alpha \iff 1 - t + r \geq (1 - \alpha) s + \alpha r
\]

and combining the last inequality with \( s \geq 1 \) in Theorem C,

\[
\frac{1 - t + r(1 - \alpha)}{1 - \alpha t} \geq s \geq 1 \iff 1 + \frac{(1 - \alpha)(r - t)}{1 - \alpha t} \geq s \geq 1
\]

and (\( \ast \)) ensures (\( \ast \ast \)) under the conditions stated in Lemma D by Theorem C. \( \square \)

**Theorem 2.1.** If \( A \succ 0 \) and \( B \succeq 0 \), then for \( 0 \leq \alpha \leq 1 \), \( t \in [0,1] \) and \( r \geq t \)

\[
\left\{ A^{\frac{1}{p}} (A^{p}_\alpha B) A^{\frac{1}{p}} \right\}^s \succeq A^{\frac{w}{p}} (A^{p}_\alpha B^p) A^{\frac{w}{p}}
\]
holds for \( \frac{(1 - \alpha)(r - t)}{1 - \alpha t} + 1 \geq s \geq 1 \), where \( w = (1 - \alpha)(s - r) + \alpha(1 - t)s \).

**Proof of Theorem 2.1.** We may assume \( 0 < \alpha \leq 1 \) since the result is trivial in case \( \alpha = 0 \). Considering the order of homogeneity of \( A \) and \( B \), we have only to prove that \( I \succeq A^{\frac{1}{p}} (A^{p}_\alpha B) A^{\frac{1}{p}} \) ensures

\[
I \succeq A^{\frac{w}{p}} (A^{p}_\alpha B^p) A^{\frac{w}{p}}
\]

and the condition \( I \succeq A^{\frac{1}{p}} (A^{p}_\alpha B) A^{\frac{1}{p}} \) is equivalently to \( A^{-1} \succeq (A^{\frac{1}{p}} BA^{\frac{1}{p}})^\alpha \).

Put \( A_1 = A^{-1} \) and \( B_1 = (A^{\frac{1}{p}} BA^{\frac{1}{p}})^\alpha \) and \( A_1 \succeq B_1 \geq 0 \) with \( A_1 > 0 \) holds.
Then (★ ★) in Lemma D yields the following inequality: for \( 0 < \alpha \leq 1, t \in [0, 1] \) and \( r \geq t \)
\[
A_t^{(1-\alpha)s+\alpha r} \geq A_t^{s} \left( \frac{A_t^{-1} B_t^{-1} A_t^{-1}}{A_t^{-1} B_t^{-1} A_t^{-1}} \right)^{s} A_t^{r} \]
that is,
\[
A^{-[(1-\alpha)s+\alpha r]} \geq A^{-\frac{s}{2}} \left( A^{-\alpha} B^{-\alpha} \right) A^{-\frac{r}{2}} \tag{2.3}
\]
holds for \( \frac{(1 - \alpha)(r - t)}{1 - \alpha t} + 1 \geq s \geq 1 \) and (2.3) is equivalent to the desired (2.2). □

**Corollary 2.2.** If \( A > 0 \) and \( B \geq 0 \), then the following inequalities (i) and (ii) hold for every \( \alpha \in (0, 1] \) and \( r \geq 0 \):

(i) \( (A^\alpha B A^\alpha)^\alpha > \frac{1}{(\log)} A^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} > A^{-\frac{1}{2} \alpha} (A^{-\frac{1}{2}} B^{-\frac{1}{2}})^\alpha A^{-\frac{1}{2}} \)

and

(ii) \( (B^\alpha A^{-\alpha} B A^{-\alpha})^\alpha > \frac{1}{(\log)} A^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} > A^{-\frac{1}{2} \alpha} (A^{-\frac{1}{2}} B^{-\frac{1}{2}})^\alpha A^{-\frac{1}{2}} \).

**Proof.**

(i) For the second log majorization, we have only to put \( s = 1 \) and \( t = 0 \) in Theorem 2.1. For the first log majorization, by homogeneity of order, we have only to prove that \( A^{-\frac{1}{\alpha}} \geq B^\alpha \) which is nothing but Löwner–Heinz inequality.

(ii) Since the second log majorization is shown in (i), for the first log majorization, we have only to prove that \( B^{-\frac{1}{\alpha}} \geq A^{-\frac{1}{\alpha}} \) (which is equivalent to \( A^{-\frac{1}{\alpha}} \geq B^\alpha \)) ensures \( A^{-1} \geq B^\alpha \), which is nothing but Löwner–Heinz inequality. □

**Remark 2.1.** The first inequality in (i) of Corollary 2.2 is the famous Araki inequality \([2]\).

**3. Further extensions of Theorem A, Theorem B and related results associated with log majorization**

We shall show further extensions of Theorem A, Theorem B and related results via log majorization obtained in Section 2.

**Corollary 3.1.** If \( A > 0 \) and \( B \geq 0 \), then the following inequalities (i)–(iv) hold for every \( \alpha \in (0, 1] \), and \( c \geq \alpha \):

(i) \( (A^{\alpha} B A^{\alpha})^{\alpha} > \frac{1}{(\log)} A^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} > A^{-\frac{1}{2} \alpha} (A^{-\frac{1}{2}} B^{-\frac{1}{2}})^\alpha A^{-\frac{1}{2}} \)

(ii) \( (B A^{-\alpha} B^{\alpha})^{\alpha} > \frac{1}{(\log)} A^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} > A^{-\frac{1}{2} \alpha} (A^{-\frac{1}{2}} B^{-\frac{1}{2}})^\alpha A^{-\frac{1}{2}} \)

(iii) \( ||| (A^{\alpha} B A^{\alpha})^{\alpha} ||| > ||| A^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} ||| > ||| A^{-\frac{1}{2} \alpha} (A^{-\frac{1}{2}} B^{-\frac{1}{2}})^\alpha A^{-\frac{1}{2}} ||| \)

and

(iv) \( ||| (B A^{-\alpha} B^{\alpha})^{\alpha} ||| > ||| A^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} ||| > ||| A^{-\frac{1}{2} \alpha} (A^{-\frac{1}{2}} B^{-\frac{1}{2}})^\alpha A^{-\frac{1}{2}} ||| \)

where \( ||| \cdot ||| \) means a unitarily invariant norm.

**Proof.** For the proofs of (i) and (ii), we have only to replace \( A \) by \( A^{\alpha} \) and \( r = \frac{1}{c - \alpha} > 0 \) for \( c > \alpha \) in Corollary 2.2 since the result is trivial in the case \( c = \alpha \). Moreover (iii) follows by (i) and also (iv) follows by (ii) respectively since \( X \succ Y \) ensures \( |||X||| \geq |||Y||| \). □
Remark 3.1. Put $c = 1$ in (i) of Corollary 3.1, then
\[
\left( A^{1-\alpha} B A^{1-\alpha} \right)^{\alpha} \succ \left( A^{1-\alpha} B A^{1-\alpha} \right) \quad \text{for } A, B \geq 0 \text{ and } \alpha \in (0, 1]
\] (3.1)

and the second log majorization of (3.1) easily implies (1.1) itself of Theorem A. We remark that the first inequality of (3.1) is Araki inequality [2].

Remark 3.2. Put $c = 1$ and $\alpha = \frac{1}{2}$ in (ii) and (iv) of Corollary 3.1, then
\[
\left( B^{1/2} A B^{1/2} \right)^{\frac{1}{2}} \succ \left( A^{1/2} B A^{1/2} \right)^{\frac{1}{2}} \quad \text{for } A, B \geq 0
\] (3.2)

and
\[
||| \left( B^{1/2} A B^{1/2} \right)^{\frac{1}{2}} ||| \geq ||| \left( A^{1/2} B A^{1/2} \right)^{\frac{1}{2}} ||| \geq ||| A^{1/2} B ||| \quad \text{for } A, B \geq 0
\] (3.3)

that is, (3.2) and (3.3) are more precise estimations than (1.2) and (1.3) of Theorem B respectively, that is, (ii) and (iv) of Corollary 3.1 are further extensions of Theorem B.

At the end of this short paper we shall show other applications of Theorem 2.1, Corollary 2.2 and Corollary 3.1 as follows.

Corollary 3.2. For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$
\[
\left( A^{1-\alpha} B \right)^{\alpha} \succ \left( A^{1-\alpha} B \right) \quad \text{for } r \geq s \geq 1.
\]

Proof. we have only to put $t = 1$ in Theorem 2.1. □

Corollary 3.2 implies the following well known result by putting $r = s$.

Theorem E [1, Theorem 2.1]. For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$
\[
\left( A^{1-\alpha} B \right)^{\alpha} \succ \left( A^{1-\alpha} B \right) \quad \text{for } r \geq 1.
\]

At the second log majorization of (i) of Corollary 2.2, replace $\alpha$ by $\frac{q}{p}$ for $0 < q \leq p$, $r$ by $p$ and finally replace $B$ by $A^{p-\alpha} B^{p}$, we have the following result.

Theorem F [1, Theorem 3.3]. If $A > 0$ and $B \geq 0$, then
\[
A^{1/2} (A^{p-\alpha} B^{p})^{\frac{q}{p}} A^{1/2} \succ A^{1/2} \left( A^{p-\alpha} B^{p} A^{p-\alpha} \right)^{\frac{q}{p}} A^{1/2} \quad \text{for every } 0 \leq \alpha \leq 1 \text{ and } 0 < q \leq p.
\]

Remark 3.3. Also replace $A$ by $A^\alpha$ and put $r = \frac{1}{\alpha}$ in (i) of Corollary 2.2 or put $c = 2\alpha$ in (i) of Corollary 3.1. Then we have the following result which is shown in [1, Corollary 3.4].
\[
\left( A^{1/2} B A^{1/2} \right)^{\alpha} \succ A^{\frac{q}{p}} B^\alpha A^{\frac{q}{p}} \succ A^{\alpha} (A^{1/2} B A^{1/2})^{\alpha} A^\alpha \quad \text{for } A > 0, B \geq 0 \text{ and } \alpha \in [0, 1].
\]

 Needless to say, the first inequality is Araki inequality [2].
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References

[4] T. Furuta, $A \geq B \geq 0$ assures $(B^{\prime} A^p B^{\prime})^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc. 101 (1987) 85–88.