VARIATIONS ON THE GALLAI–MILGRAM THEOREM

I. BEN-ARROYO HARTMAN*

Dept. of Computer Science, University of Toronto, Toronto, Ontario M5S 1A4, Canada

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We strengthen the Gallai–Milgram theorem for digraphs with independence number two, and propose various problems and conjectures strengthening the Gallai–Milgram theorem and the Gallai–Roy theorem for strong digraphs.

1. Introduction

Let \( D = (V, E) \) be a directed graph. A path \( P \) in \( D \) is a sequence of distinct vertices \( (v_1, \ldots, v_n) \) such that \( (v_i, v_{i+1}) \in E, \ i = 1, 2, \ldots, n - 1 \). Denote the initial vertex \( v_1 \) of \( P \) by \( \text{in}(P) \) and the terminal vertex \( v_n \) by \( \text{ter}(P) \). A path partition is a partition of \( V \) into disjoint paths. Let \( \pi(D) \) denote the minimum number of paths in a path partition of \( D \). The Gallai–Milgram Theorem [7] states that for any digraph \( D \) with independence number \( \alpha(D) \), \( \pi(D) \leq \alpha(D) \). For \( \alpha = 1 \), this result is known as Rédei’s theorem [10], stating that every tournament contains a Hamilton path. If \( D \) is also strongly connected (i.e. for any two vertices \( x, y \in V(D) \) there exists a path \( P \) with \( \text{in}(P) = x \) and \( \text{ter}(P) = y \)), then a theorem of Camion [3] implies a stronger result than Rédei’s theorem: a strong tournament contains a Hamilton cycle. In view of Camion’s theorem it is natural to attempt to strengthen the Gallai–Milgram theorem for all \( \alpha \) for strong digraphs. Chen and Manalastas [4] have done so for \( \alpha = 2 \).

Theorem 1 [4]. Every strongly connected digraph \( D \), with \( \alpha(D) = 2 \), is spanned by two consistent cycles.

Two cycles \( C_1 \) and \( C_2 \) are consistent if their intersection is empty, or a single vertex, or a subpath of \( C_1 \) and \( C_2 \) (see Fig. 1).

An immediate result of Theorem 1 is

Corollary 2. Every strongly connected digraph with \( \alpha = 2 \) contains a Hamilton path.

A natural extension of the above corollary is the following:

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Conjecture 3. For every strongly connected digraph $D$ with $\alpha \geq 2$

$$\pi(D) \leq \alpha(D) - 1.$$  

This conjecture implies the Gallai–Milgram theorem. This can easily be seen by constructing a new digraph $D^*$ by adding a new vertex $v$, and joining it (both ways) to the vertices of $D$. Clearly $D^*$ is strong, and a path partition with $\alpha - 1$ paths in $D^*$ implies the existence of a path partition with $\alpha$ paths in $D$.

In the next section we prove the following strengthening of the Gallai–Milgram Theorem for $\alpha = 2$.

Theorem 4. Let $D$ be a digraph with $\alpha = 2$ and let $x$ be a vertex in $D$ such that for any vertex $y \neq x$, there exists a path in $D$ from $x$ to $y$. Then $D$ is spanned by two disjoint paths, $P_1$ and $P_2$, such that $\text{in}(P_1) = x$.

As a corollary, we deduce Conjecture 3 in the case when $\alpha = 3$ and $D$ contains a cut vertex.

2. Strengthening Gallai–Milgram for $\alpha = 2$

We begin by introducing the following terminology. Let $P$ be a directed path (or cycle) and let $x, y \in V(P)$. Then $P(x, y)$ denotes the subpath of $P$ from $x$ to $y$. We write $P(y, x)$ for $P(\text{in}(P), y)$ and $P(x, y)$ for $P(x, \text{ter}(P))$. Assume $a \in P$. Then $a^-(P)(a^+(P))$ denotes the vertex on $P$ which dominates (is dominated by) $a$. If $P$ and $Q$ are two paths were $\text{ter}(P)$ dominates $\text{in}(Q)$ or $\text{ter}(P) = \text{in}(Q)$, then $P * Q$ is the concatenation of $P$ and $Q$.

Let $S = \{A_1, A_2, \ldots\}$ be the set of strong components of $D$. There is a natural partial order $P(S, \succeq)$ defined by:

$$A_i \succeq A_j$$ if and only if there exists a path in $D$ from $A_i$ to $A_j$.

A component $A_i$ is maximal (minimal) if it is maximal (minimal) in the order $P$. Two components $A_1$ and $A_2$ are adjacent if there exist vertices $x, y$ such that
x ∈ V(A₁), y ∈ V(A₂) and x, y are adjacent in the graph. Similarly, we say that a component A₁ dominates A₂ if there exists x ∈ V(A₁) and y ∈ V(A₂) such that (x, y) ∈ E(D). Denote by ℓ(D) the length of the longest chain in the partial order P. Equivalently, l(D) is the maximum number of strong components that any given path in D may meet. Other terminology and notation undefined here can be found in [2].

The following theorem is slightly stronger than Theorem 4.

**Theorem 5.** Let D be a digraph with α = 2 and let A be a maximal strong component in D. Then for every vertex x ∈ V(A), there exist two disjoint spanning paths P₁ and P₂ in D such that in(P₁) = x.

Before proving Theorem 5, we present the following two technical lemmas:

**Lemma 6.** Let D be a strong digraph with α = 2. Then D contains two Hamilton paths Q₁ and Q₂ such that the vertices a₁ = in(Q₁) and a₂ = in(Q₂) are independent, and b₁ = ter(Q₁) and b₂ = ter(Q₂) are independent.

**Proof.** By Theorem 1, D is spanned by two consistent cycles C₁ and C₂. Choose such cycles such that |C₁ ∩ C₂| is maximum. Assume, first, that C₁ ∩ C₂ ≠ ∅ (see Fig. 2a). Let (t₁, t₂, . . . , tₖ) be the subpath common to C₁ and C₂, and let a₁ = tᵢ⁺(C₁), a₂ = tᵢ⁻(C₂), b₁ = tᵢ⁻(C₂) and b₂ = tᵢ⁺(C₁). By the maximality of |C₁ ∩ C₂| the vertices a₁ and a₂ are independent, and so are b₁ and b₂. The following paths Q₁ and Q₂ satisfy the conditions of the Lemma:

\[
Q₁ = C₁(a₁, \ldots , tₖ) * C₂(a₂, \ldots , b₁),
\]
\[
Q₂ = C₂(a₂, \ldots , tₖ) * C₁(a₁, \ldots , b₂).
\]

If C₁ ∩ C₂ = ∅ (Fig. 2b), then since G is strong, there exist edges e₁ = (x₁, y₁) ∈ E(D) and e₂ = (y₂, x₂) ∈ E(D) where x₁, x₂ ∈ V(C₁) and y₁, y₂ ∈ V(C₂). By the maximality of |C₁ ∩ C₂|, the vertices a₁ = x₁⁺(C₁) and a₂ = y₂⁺(C₂) are independent.
as well as the vertices \( b_1 = y_1^{-}(C_2) \) and \( b_2 = x_2^{-}(C_1) \). The paths

\[
Q_1 = C_1(a_1, \ldots, x_1) \cdot y_1 \cdot C_2(y_1, \ldots, b_1)
\]
\[
Q_2 = C_2(a_2, \ldots, y_2) \cdot x_2 \cdot C_1(x_2, \ldots, b_2)
\]
satisfy the conditions of the lemma.

**Lemma 7.** Let \( D \) be a directed graph with \( \alpha = 2 \) and \( l(D) = 2 \). Assume that \( D \) contains a maximum strong component \( A \) with \( \alpha(A) = 2 \). Then for every vertex \( x \in V(A) \) there exist two disjoint paths \( P_1 \) and \( P_2 \) spanning \( D \) which satisfy:

(i) \( \text{in}(P_i) = x \); and

(ii) \( \text{ter}(P_1) \) and \( \text{ter}(P_2) \) are in different strong components of \( D \).

**Proof.** By Theorem 1, \( A \) is spanned by two consistent cycles. Pick such cycles \( C_1 \) and \( C_2 \) so that \( |C_1 \cap C_2| \) is maximum, and define the vertices \( a_1, a_2, b_1 \) and \( b_2 \) and the paths \( Q_1 \) and \( Q_2 \) as in the proof of the previous lemma. We may assume that \( x \in V(C_1) \) and let \( z = x^{-}(C_1) \).

Consider the graph \( D' \) spanned by \( V \setminus V(A) \). Since \( \alpha(D') \leq 2 \) it follows that \( D' \) contains at most two strong components.

Assume first that \( D' \) contains exactly two strong components \( B_1 \) and \( B_2 \). Since \( l(D) = 2 \), \( B_1 \) and \( B_2 \) are incomparable, and hence \( \alpha(B_1) = \alpha(B_2) = 1 \). Let \((u_1, u_2, \ldots, u_m, u_1)\) and \((v_1, v_2, \ldots, v_n, v_1)\) be Hamilton cycles in \( B_1 \) and \( B_2 \), respectively. Now, since \( \alpha = 2 \), vertex \( z \) must dominate some vertex in \( B_1 \) or \( B_2 \). Assume without loss of generality that \((2, u_1) \in E(D)\). Since the vertices \( b_1 \) and \( b_2 \) in \( A \) are independent, \( v_1 \) must be dominated by one of them. If \((b_1, v_1) \in E\), we define

\[
P_1 = Q_1(x, b_1) \cdot (v_1, \ldots, v_n)
\]
\[
P_2 = Q_1(a_1, z) \cdot (u_1, \ldots, u_m).
\]

Otherwise, \((b_2, v_1) \in E\) and we define

\[
P_1 = Q_2(x, b_2) \cdot (v_1, \ldots, v_n)
\]
\[
P_2 = Q_2(a_2, z) \cdot (u_1, \ldots, u_m)
\]

If \( D' \) contains exactly one strong component \( B \), we proceed to define \( Q_1 \) and \( Q_2 \) as in the previous case by letting \( B = B_2 \) and \( B_1 = \emptyset \).

**Proof of Theorem 5.** Assume, first, that \( \alpha(A) = 1 \), and let \( C(a_1, \ldots, a_m, a_1) \) be a Hamilton cycle in \( A \), where \( a_1 = x \). Define the graph \( D^* \) from \( D \), by replacing every edge \((a_m, a_j) \in E(D)\) by the edge \((a_j, a_m)\). Since \( \alpha(D^*) = \alpha \), \( D^* \) is spanned by at most two disjoint paths \( P_1^* \) and \( P_2^* \), where, without loss of generality, \( a_m \in V(P_1^*) \). Since \( a_m \) is a sink in \( D^*[V(A)] \), we have \( a_m^*(P_1^*) \notin V(A^*) \). Let \( b \) be
the first vertex of $P_2^*$ which is not in $A^*$. The following paths satisfy the theorem:

\[ P_1 = C(a_1, \ldots, a_m) \ast P_1^*(a_m) \]
\[ P_2 = P_2^*(b) \]

($P_2$ is empty if $b$ does not exist).

Assume now that every maximal strong component $A$ in $D$ has independence number two. Since $\alpha = 2$ it follows that $A$ is unique. If $D$ is strong, then by Corollary 2 the theorem follows. Otherwise we prove by induction on $l = l(G)$ that $D$ is spanned by two disjoint paths $P_1$ and $P_2$ satisfying:

(i) $\text{in}(P_1) = x$

and

(ii) $t_1 = \text{ter}(P_1)$ and $t_2 = \text{ter}(P_2)$ are in different strong components of $D$.

For $l = 2$, such paths exist by Lemma 7. Let $D$ be a graph with $l(D) = l > 2$ and let $\beta$ be the set of minimal strong components in $D$. (Note that $\beta$ contains at most two strong components.) Define $H = G[V - V(U\beta)]$. By the induction hypothesis, $H$ is spanned by two disjoint paths, $P_1$ and $P_2$ satisfying (i) and (ii) above.

Case (a). Assume that $D$ contains exactly two minimal strong components $B_1$ and $B_2$. Since $\alpha(D) = 2$, it follows that $\alpha(B_1) = \alpha(B_2) = 1$.

Let $(u_1, \ldots, u_m, u_1)$ and $(v_1, \ldots, v_n, v_1)$ be Hamilton cycles in $B_1$ and $B_2$ respectively.

Assume, first, that $H$ contains a minimal strong component $A$ which is adjacent both to $B_1$ and to $B_2$. Assume, without loss of generality, that $t_1 = \text{ter}(P_1) \in V(A)$, and $t_2 = \text{ter}(P_2) \in V(A_1)$, where $A_1$ is a strong component different from $A$. Since $A$ is a minimal component, it is spanned by a subpath $(x_1, x_2, \ldots, x_s, t_1)$ of $P_1$. If $t_1$ is adjacent to $B_i$ ($i = 1$ or 2) and $t_2$ is adjacent to $B_j$ ($j \neq i$, $j = 1$ or 2) then $P_1$ and $P_2$ can be easily extended to two paths which span $D$ and satisfy (i) and (ii) above. Otherwise, one of $B_i$, $i = 1, 2$, say $B_1$, is adjacent neither to $t_1$ not to $t_2$ (see Fig. 3). Let $x_j \in V(A)$ be the last vertex of $P_1$ which dominates some vertex $u_k$ in $B_1$ (since $A$ is adjacent to $B_1$, such a vertex exists). Hence $x_j + 1$ is not adjacent to any vertex in $V(B_1)$, implying that $(t_2, x_j + 1) \in E(D)$. We may now define the following two paths $\tilde{P}_1$ and $\tilde{P}_2$ which satisfy (i) and (ii) above.

\[ \tilde{P}_1 = P_1((x_j) \ast (u_k, u_{k + 1}, \ldots, u_{k - 1}) \]
\[ \tilde{P}_2 = P_2((t_2) \ast (x_j + 1, x_{j + 2}, \ldots, t_1) \ast (v_1, \ldots, v_n) \]

Assume now, that no minimum strong component in $H$ is adjacent both to $B_1$ and to $B_2$ (see Fig. 4).

Then $H$ contains two minimal components $A_1$ and $A_2$, and we may assume that $A_1$ dominates $B_1$ and is not adjacent to $B_2$, and $A_2$ dominates $B_2$ and is not adjacent to $B_1$. But since $B_1$ and $B_2$ are not adjacent, $t_1$ dominates all the vertices
in $B_1$ and $t_2$ dominates all the vertices in $B_2$. The paths $P_1$ and $P_2$ can be easily extended as depicted in Fig. 4.

Case (b). Assume that $D$ contains a unique minimal strong component $B$, and let $A$ be a minimum component in $H$ (see Fig. 5). Without loss of generality, $t_1 = \text{ter}(P_1) \in V(A)$, and $A$ is spanned by the subpath $(x_1, x_2, \ldots, x_i, t_1)$ of $P_1$. If $\alpha(B) = 2$ then $B$ contains by Lemma 7 two independent vertices $u_1$ and $u_2$ which are initial vertices of Hamilton paths. Vertex $t_1$ must dominate one of $a_i$ ($i = 1, 2$) and $P_1$ can be extended appropriately. Otherwise $\alpha(B) = 1$ and $B$ contains a Hamilton cycle $(u_1, u_2, \ldots, u_n, u_1)$. If either $t_1$ or $t_2 = \text{ter}(P_2)$ are adjacent to $B$, then $P_1$ and $P_2$ can be extended to include $B$. If not, let $x_j \in V(A)$ be the last
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vertex of $P_1$ adjacent to some vertex $u_k \in V(B)$ (since $A$ is a minimum component in $H$ such a vertex exists). As in Case (a), it follows that $(t_2, x_{j+1}) \in E(D)$ and $\tilde{P}_1$ and $\tilde{P}_2$ below satisfy (i) and (ii).

$$\tilde{P}_1 = P_1((u_k, u_{k+1}, \ldots, u_{k-1})$$

$$\tilde{P}_2 = P_2((x_{j+1}, \ldots, t_1).$$

This completes the proof of the theorem. $\square$

As a corollary we prove a special case of Conjecture 3:

**Corollary 8.** Let $D$ be a strong digraph with independence number $\alpha = 3$. Assume also that $D$ contains a cut vertex. Then $\pi(D) \leq 2$.

**Proof.** Let $u$ be a cut vertex of $D = (V, E)$. Let $H_1$ and $H_2$ be two induced subgraphs of $D$ such that $V(H_1) \cap V(H_2) = \{u\}$, and $V(H_1) \cup V(H_2) = V$. Clearly, $H_1$ and $H_2$ are strong and $3 \leq \alpha(H_1) + \alpha(H_2) \leq 4$.

There are two cases to consider:

**Case 1.** $\alpha(H_1) + \alpha(H_2) = 3$. Without loss of generality $\alpha(H_1) = 1$ and $\alpha(H_2) = 2$. Let $K = H_1 - u$. By Rédei's theorem and Corollary 2 the subgraphs $K$ and $H_2$ are spanned by Hamilton paths $P_1$ and $P_2$ respectively, which form a path partition of $D$.

**Case 2.** $\alpha(H_1) + \alpha(H_2) = 4$. In this case, we may assume that every maximum independent set in $H_1$ contains $u$. Hence $1 \leq \alpha(K) = \alpha(H_1) - 1 \leq 2$. If $\alpha(H_2) = 2$ then $\alpha(K) = 1$ and as in Case 1 the Hamilton paths in $K$ and $H_2$ form a path.
partition of $D$. Otherwise, $\alpha(H_2) = 1$ and $H_2$ contains a Hamilton cycle $C = (v_1, v_2, \ldots, v_i, v_1)$, where $v_i = u$. Now since $H_1$ is strong, there exists a vertex $x$ in a maximum strong component of $K$ which is dominated by $u$. By Theorem 5, $K$ is spanned by disjoint paths $P_1$ and $P_2$ where $\text{in}(P_1) = x$. Let $\tilde{P}_1 = (v_1, v_2, \ldots, v_i) \ast P_1$. The paths $\tilde{P}_1$ and $P_2$ form a path partition of $D$. \qed

3. Other conjectures and related problems

Consider an undirected graph $G$ with connectivity $k$ and independence number $\alpha(G) \leq k$. The Chvátal–Erdős Theorem [5] states that $G$ contains a Hamilton cycle. As a corollary, it follows that if $k < \alpha$, then $\pi(G) \leq \alpha(G) - k$. This can be seen by constructing a new graph $G^*$ by adding $\alpha - k$ new vertices $\{v_1, \ldots, v_{\alpha - k}\}$ to $G$ and joining them to each vertex in $V(G)$. The resultant graph $G^*$, has connectivity $k = \alpha$ and contains a Hamilton cycle if and only if $V(G)$ can be partitioned into $\alpha - k$ paths. The Chvátal–Erdős theorem and its corollary clearly strengthen the Gallai–Milgram theorem for undirected graphs.

Can we have such a strengthening for directed graphs? A directed graph $D = (V, E)$ is $k$-connected if $D[V - V']$ is strong for every $V' \subseteq V$, $|V'| < k$.

The following counter-examples show that the Chvátal–Erdős theorem cannot be necessarily extended to directed graphs. The graph in Fig. 6 (suggested by B. Jackson) has $k = \alpha = 2$ and contains no Hamilton cycle. The graph in Fig. 7 has $k = 3 \alpha = 2$ and no Hamilton cycle. Other counter-examples for $k = 2$ and $k = 3$ also exist.

The following is a more reasonable extension of the Chvátal–Erdős theorem to directed graphs.

**Conjecture 9.** Let $D$ be a $k$-connected directed graph with $\alpha(D) > k$. Then $\pi(D) \leq \alpha(D) - k$.

This conjecture is best possible as seen by the graph in Fig. 8. For each $i$, $1 \leq i \leq \alpha$, $T_i$ is a tournament, and for each $j$, $1 \leq j \leq k$, $v_j$ is connected (both ways) to all the vertices in each $T_i$, $1 \leq i \leq \alpha$.

For other possible extension of the Chvátal–Erdős theorem to directed graphs see [8] and [9].

![Fig. 6.](image)

(K*<sub>n</sub> denotes the complete symmetric digraph.)
Now let $D$ be a transitive digraph, i.e. $(x, y) \in E(D)$ and $(y, z) \in E(D)$ imply that $(x, z) \in E(D)$. It is not difficult to show that in this case $\pi(D) = \alpha(D)$. Note that a path in $D$ corresponds to an independent set in $D^c$ (the compliment of $D$) and vice versa, any independent set of vertices in $D$ induces a path in $D^c$. Hence, it follows that for all transitive digraphs $\lambda(D) = \chi(D)$, where $\chi(D)$ is the chromatic number of $D$ and $\lambda(D)$ is the number of vertices in a longest path in $D$. The Gallai–Roy theorem demonstrates that this "dualism" between paths and independent sets holds, to some extent, for all digraphs.

**Theorem 10** (Gallai [6] Roy [11]). For all digraphs $D$, $\lambda(D) \geq \chi(D)$.

If $D$ is strongly connected, then a stronger result holds.

**Theorem 11** (Bondy [1]). Let $D$ be a strongly connected digraph. Then $D$ contains a cycle of length at least $\chi(D)$.

**Corollary 12.** For any strong digraph $D = (V, E)$ with chromatic number $\chi$, $\lambda(D) \geq \min\{|V|, \chi + 1\}$.

We see that the "dual" version of Conjecture 3 holds for $k = 1$. For $k \geq 1$ we use the following lemma.

**Lemma 13.** A $k$-connected digraph $D = (V, E)$ contains a cycle of length at least $\min\{|V|, 2k\}$.

**Proof.** Assume $D$ is a non-Hamiltonian and let $C$ be the longest cycle in $D$. Let $x$ be a vertex not on $C$. Since $D$ is $k$-connected there exist $k$ internally disjoint paths $P_1, \ldots, P_k$ with $\text{in}(P_i) = x$ and $\text{ter}(P_i) = a_i \in V(C)$, $1 \leq i \leq k$, where all the vertices $\{a_i\}$ are distinct. Similarly, there exist $k$ internally disjoint paths $Q_1, \ldots, Q_k$ with $\text{in}(Q_i) = b_i \in V(C)$ all $\{b_i\}$ are distinct, and $\text{ter}(Q_i) = x$. Now, if for some $i, j$
(b_j, a_i) \in E(C) \text{ then } C \text{ would not be a longest cycle. Hence, no vertex } b_j, 1 \leq j \leq k, \text{ directly precedes } a_i, 1 \leq i \leq k \text{ on } C \text{ and thus } |C| \geq 2k. \quad \square

\textbf{Theorem 14.} Let } D \text{ be a } k\text{-connected digraph and assume } k \geq \chi - 1. \text{ Then } D \text{ contains a long cycle } C \text{ and a long path } P \text{ of lengths } |C| \geq \min\{|V|, 2k\} \text{ and } |P| \geq \min\{|V|, 2k + 1\}. \text{ Furthermore, this result is best possible.}

\textbf{Proof.} By the previous lemma, such a cycle and a path exist. The example in Fig. 9 shows it is best possible. } T_{\chi - 1} \text{ denotes a tournament with } \chi - 1 \text{ vertices. We let } |I| \geq k \text{ and each vertex in } A \text{ is adjacent (both ways) to each vertex in } I. \quad \square

\text{For } k \leq \chi - 1, \text{ the following Conjecture is a natural generalization of Corollary 12 and a “dual” version of Conjecture 9:}

\textbf{Conjecture 15.} A } k\text{-connected digraph } D = (V, E) \text{ with } k \leq \chi - 1 \text{ contains a path } P \text{ with } |P| \geq \min\{|V|, \chi + k\}.

\text{The graph in Fig. 10 demonstrates that this conjecture is best possible.}

Let } V(T_{\chi - 1}) = (x_1, x_2, \ldots, x_{\chi - 1}) \text{, and let } I \text{ be a set of at least } k + 1 \text{ independent vertices. For each } 1 \leq j \leq k, x_j \text{ is adjacent (both ways) to each vertex in } I \text{ and for each } k + 1 \leq j \leq \chi - 1, x_j \text{ dominates each vertex in } I.

\text{Note that for } k = \chi - 1, \text{ Conjecture 15 holds by the previous theorem.}
References