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Initial-boundary value problems for a class of pseudoparabolic equations with integral boundary conditions ☆

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Abstract

In this paper, we deal with a class of pseudoparabolic problems with integral boundary conditions. We will first establish an a priori estimate. Then, we prove the existence, uniqueness and continuous dependence of the solution upon the data. Finally, some extensions of the problem are given.

1. Introduction

In this article, we are concerned with a second order pseudoparabolic equation with integral conditions in a region with curved lateral boundaries, which arise from various physical phenomena such as the dynamics of ground moisture. The precise statement of the problem is as follows.

Let $\Gamma_p(\tau)$ $(p = 1, 2)$, $0 < \tau < T_1$, be nonintersecting curves in the (ξ, τ) plane. In $Q = \{(\xi, \tau) \in \mathbb{R}^2\colon\ \Gamma_1(\tau) < \xi < \Gamma_2(\tau),\ 0 < \tau < T_1\}$, we consider the following initial-boundary value problem for a pseudoparabolic equation:

$$\mathcal{L}\theta = \frac{\partial\theta}{\partial\tau} - \frac{\partial}{\partial\xi}\left(a(\xi,\tau)\frac{\partial\theta}{\partial\xi}\right) - \eta\frac{\partial^2}{\partial\tau\,\partial\xi}\left(a(\xi,\tau)\frac{\partial\theta}{\partial\xi}\right) + b(\xi,\tau)\theta = h(\xi,\tau), \tag{1.1}$$

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$$\ell\theta = \theta(\xi, 0) = \theta_0(\xi), \tag{1.2}$$

$$\int_{\Gamma_1(\tau)}^{\Gamma_2(\tau)} \xi^i \theta(\xi, \tau)\, d\xi = M_i(\tau) \quad (i = 0, 1) \tag{1.3}$$

with the compatibility conditions

$$\int_{\Gamma_1(0)}^{\Gamma_2(0)} \xi^i \theta_0(\xi)\, d\xi = M_i(0) \quad (i = 0, 1),$$

where $\Gamma_p(\tau)$ $(p = 1, 2)$, h, θ_0, M_i $(i = 0, 1)$, a, b are sufficiently smooth given functions. Conditions (1.3) represent the moisture moments.

In this paper, we establish the existence, uniqueness and continuous dependence upon the data of problem (1.1)–(1.3). The proof is based on an energy inequality and on the density of the range of the operator generated by the abstract formulation of the stated problem. This problem has not been studied previously. Concerning problems with integral condition(s) for other equations, we refer the reader to [1,3–12,19] and references cited therein. We mention that more general pseudoparabolic equations with classical boundary conditions can be found, for instance, in [2,13–18].

2. Preliminaries

Problem (1.1)–(1.3) can be converted to an equivalent problem, while the integral conditions are homogeneous, by introducing a new unknown function $v(\xi, \tau)$ which is representing the deviation of the function $\theta(\xi, \tau)$ from the function

$$U(\xi, \tau) = \frac{M_0(\tau)}{\Gamma_2(\tau) - \Gamma_1(\tau)} + \frac{6(2M_1(\tau) - (\Gamma_2(\tau) + \Gamma_1(\tau))M_0(\tau))}{(\Gamma_2(\tau) - \Gamma_1(\tau))^4} \times \left(3\left(\xi - \Gamma_1(\tau)\right)^2 - 2\left(\Gamma_2(\tau) - \Gamma_1(\tau)\right)\left(\xi - \Gamma_1(\tau)\right)\right).$$

Thus problem (1.1)–(1.3) becomes

$$\mathcal{L}v = \frac{\partial v}{\partial \tau} - \frac{\partial}{\partial \xi}\left(a(\xi, \tau)\frac{\partial v}{\partial \xi}\right) - \eta \frac{\partial^2}{\partial \tau \partial \xi}\left(a(\xi, \tau)\frac{\partial v}{\partial \xi}\right) + b(\xi, \tau)v = \tilde{h}(\xi, \tau), \tag{2.1}$$

$$\ell v = v(\xi, 0) = v_0(\xi), \tag{2.2}$$

$$\int_{\Gamma_1(\tau)}^{\Gamma_2(\tau)} \xi^i v(\xi, \tau)\, d\xi = 0 \quad (i = 0, 1) \tag{2.3}$$

with

$$\int_{\Gamma_1(0)}^{\Gamma_2(0)} \xi^i v_0(\xi)\, d\xi = 0 \quad (i = 0, 1).$$

We now introduce new variables x and t connected with ξ and τ, by the relations

$$x=\frac{(\beta-\alpha)\,\xi+(\alpha\Gamma_2(\tau)-\beta\Gamma_1(\tau))}{\Gamma_2(\tau)-\Gamma_1(\tau)},\qquad t=\tau.$$

Under this transformation, the region Q is reduced to the rectangle $\Omega=\{(x,t)\in\mathbb{R}^2\colon\ \alpha<x<\beta,\ 0<t<T\}$. In the new variables, problem (2.1)–(2.3) becomes

$$\mathcal{L}u=\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(p(x,t)\frac{\partial u}{\partial x}\right)-\eta\frac{\partial^2}{\partial t\,\partial x}\left(p(x,t)\frac{\partial u}{\partial x}\right)+q(x,t)u=f(x,t), \tag{2.4}$$

$$\ell u=u(x,0)=u_0(x), \tag{2.5}$$

$$\int\limits_{\Gamma_1(t)}^{\Gamma_2(t)} x^i u(x,t)\,dx=0\quad (i=0,1) \tag{2.6}$$

with

$$\int\limits_{\Gamma_1(0)}^{\Gamma_2(0)} x^i u_0(x)\,dx=0\quad (i=0,1).$$

Assumption A1. For all $(x,t)\in[\alpha,\beta]\times[0,T]$, we shall assume

$$0<c_0\leqslant p(x,t)\leqslant c_1,\qquad \left|q(x,t)\right|\leqslant c_2,$$
$$\left|\frac{\partial p}{\partial t}\right|\leqslant c_3,\qquad \left|\frac{\partial p}{\partial x}\right|\leqslant c_4,\qquad \left|\frac{\partial^2 p}{\partial x\,\partial t}\right|\leqslant c_5,\qquad \left|\frac{\partial^2 p}{\partial x^2}\right|\leqslant 0.$$

Assumption A2. For all $(x,t)\in[\alpha,\beta]\times[0,T]$, we shall assume

$$\frac{\partial p}{\partial t}\geqslant c_6,\qquad \left|\frac{\partial^2 p}{\partial t^2}\right|\leqslant c_7.$$

In Assumptions A1 and A2 and throughout, c_i $(i=0,1,\ldots)$ are positive constants.

Let us introduce function spaces needed in our investigation. We denote by $B_2^m(\alpha,\beta)$ the so-called Bouziani space, first defined by the author in [4–9]. It is considered as a completion of the space $C_0(\alpha,\beta)$, of continuous functions with compact support in (α,β), for the scalar product

$$(u,w)_{B_2^m(\alpha,\beta)}:=\int\limits_{\alpha}^{\beta}\Im_x^m u\cdot\Im_x^m w\,dx$$

and the associated norm

$$\|u\|_{B_2^m(\alpha,\beta)}:=\left(\int\limits_{\alpha}^{\beta}\left(\Im_x^m u\right)^2dx\right)^{1/2},$$

where

$$\Im_x^m u = \frac{1}{(m-1)!} \int_\alpha^x (x-\xi)^{m-1} u(\xi)\, d\xi, \quad m \geqslant 1.$$

For $m \geqslant 1$, we have

$$\|u\|_{B_2^m(\alpha,\beta)} \leqslant \frac{|\beta-\alpha|}{\sqrt{2}} \|u\|_{B_2^{m-1}(\alpha,\beta)}. \tag{2.7}$$

The space $B_2^0(\alpha,\beta)$ coincides with $L^2(\alpha,\beta)$. Let X be a Banach space with the norm $\|u\|_X$, and let $u:(0,T) \to X$ be an abstract function. By $\|u(\cdot,t)\|_X$ we denote the norm of the element $u(\cdot,t) \in X$ at a fixed t. We denote by $L^2(0,T;X)$ the set of all measurable abstract functions $u(\cdot,t)$ from $(0,T)$ into X such that

$$\|u\|_{L^2(0,T;X)} = \left(\int_0^T \big\|u(\cdot,t)\big\|_X \, dt \right)^{1/2} < \infty.$$

If X is a Hilbert space, so is the space $L^2(0,T;X)$. We denote by $C(0,T;X)$ the set of all continuous functions $u:(0,T) \to X$ with

$$\|u\|_{C(0,T;X)} = \max_{t\in[0,T]} \big\|u(\cdot,t)\big\|_X < \infty.$$

Let us, now, reformulate problem (2.4)–(2.6) as the problem to determine a solution $u = u(x,t)$ of the operator equation

$$Lu = (f, u_0), \quad \forall u \in D(L), \tag{2.8}$$

where $L = (\mathcal{L}, \ell)$, $D(L)$ is the set of all functions u belonging to $L^2(0,T;B_2^1(\alpha,\beta))$ for which $\partial u/\partial t, \partial u/\partial x, \partial^2 u/\partial x^2 \in L^2(0,T;B_2^1(\alpha,\beta))$ and satisfying conditions (2.6). The operator L acts from B to F, where B is the Banach space which is the closure of $D(L)$ in the norm

$$\|u\|_B = \left(\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(0,T;B_2^1(\alpha,\beta))} + \|u\|^2_{C(0,T;L^2(\alpha,\beta))} \right)^{1/2},$$

and F is the Hilbert space $L^2(0,T;B_2^1(\alpha,\beta)) \times L^2(\alpha,\beta)$ consisting of all elements (f,u_0) for which the norm

$$\big\|(f,u_0)\big\|_F = \left(\|f\|^2_{L^2(0,T;B_2^1(\alpha,\beta))} + \|u_0\|^2_{L^2(\alpha,\beta)} \right)^{1/2}$$

is finite.

Finally, we summarize some properties of the following smoothing operators [7,19]:

$$\varrho_\varepsilon^{-1} z = \frac{1}{\varepsilon} \Im_t \big(e^{(\tau-t)/\varepsilon} z\big), \quad \varepsilon > 0,$$

$$\big(\varrho_\varepsilon^{-1}\big)^* z = \frac{1}{\varepsilon} \Im_t^* \big(e^{(t-\tau)/\varepsilon} z\big), \quad \varepsilon > 0,$$

where $\Im_t g = \int_0^t g(x,\tau)\, d\tau$, $\Im_t^* g = \int_t^T g(x,\tau)\, d\tau$. They furnish the solutions of

$$\varepsilon\frac{\partial\varrho_\varepsilon^{-1}z}{\partial t}+\varrho_\varepsilon^{-1}z=z,\quad \varrho_\varepsilon^{-1}z(x,0)=0,$$
$$-\varepsilon\frac{\partial(\varrho_\varepsilon^{-1})^*z}{\partial t}+\left(\varrho_\varepsilon^{-1}\right)^*z=z,\quad \left(\varrho_\varepsilon^{-1}\right)^*z(x,T)=0,$$

respectively. These operators have, for all $z\in L^2(0,T;X)$, the following properties:

(P1) The functions $\varrho_\varepsilon^{-1}z$ and $(\varrho_\varepsilon^{-1})^*z\in H^1(0,T,X)$ with $\varrho_\varepsilon^{-1}z(x,0)=0$ and $(\varrho_\varepsilon^{-1})^*\times z(x,T)=0$.

(P2) The operator $(\varrho_\varepsilon^{-1})^*$ is conjugate to ϱ_ε^{-1}, i.e.,

$$\int_0^T\left(\varrho_\varepsilon^{-1}z,\omega(\cdot,t)\right)_X dt=\int_0^T\left(z(\cdot,t),\left(\varrho_\varepsilon^{-1}\right)^*\omega\right)_X dt\quad\text{for all }\omega\in L^2(0,T;X).$$

(P3)
$$\varrho_\varepsilon^{-1}\frac{\partial z}{\partial\tau}=\frac{\partial}{\partial t}\varrho_\varepsilon^{-1}z-\frac{1}{\varepsilon}e^{-t/\varepsilon}z(x,0).$$

(P4)
$$\int_0^T\left\|\varrho_\varepsilon^{-1}z\right\|_X^2dt\leqslant\int_0^T\|z\|_X^2dt\quad\text{and}\quad\int_0^T\left\|\varrho_\varepsilon^{-1}z-z\right\|_X^2dt\to 0$$

when $\varepsilon\to 0$.

(P5)
$$\int_0^T\left\|\left(\varrho_\varepsilon^{-1}\right)^*z\right\|_X^2dt\leqslant\int_0^T\|z\|_X^2dt\quad\text{and}\quad\int_0^T\left\|\left(\varrho_\varepsilon^{-1}\right)^*z-z\right\|_X^2dt\to 0,$$

when $\varepsilon\to 0$.

(P6) Set

$$P(t)z=\frac{\partial}{\partial x}\left(p(x,t)\frac{\partial u}{\partial x}\right)\quad\text{and}\quad P'(t)z=\frac{\partial}{\partial x}\left(\frac{\partial p(x,t)}{\partial t}\frac{\partial z}{\partial x}\right).$$

Then

$$P(t)\varrho_\varepsilon^{-1}z=\varrho_\varepsilon^{-1}P(\tau)z+\varepsilon\varrho_\varepsilon^{-1}P'(\tau)\varrho_\varepsilon^{-1}z.$$

3. An energy inequality

In this section, we establish an energy inequality for operator L and we give some of its consequences. For this purpose, we start by taking the inner product in $L^2(\alpha,\beta)$ of Eq. (2.4) and the integrodifferential operator $Mu=-\Im_x^2(\partial u/\partial t)$, where $\Im_x^2\cdot=\Im_x(\Im_\xi\cdot)$ with $\Im_x$ coincides with $\Im_x^1$,

$$-\left(\frac{\partial u(\cdot,t)}{\partial t},\Im_x^2\frac{\partial u(\cdot,t)}{\partial t}\right)_{L^2(\alpha,\beta)}+\left(\frac{\partial}{\partial x}\left(p\frac{\partial u(\cdot,t)}{\partial x}\right),\Im_x^2\frac{\partial u(\cdot,t)}{\partial t}\right)_{L^2(\alpha,\beta)}$$

$$+\eta\left(\frac{\partial^2}{\partial t\partial x}\left(p\frac{\partial u(\cdot,t)}{\partial x}\right),\Im_x^2\frac{\partial u(\cdot,t)}{\partial t}\right)_{L^2(\alpha,\beta)}-\left(qu(\cdot,t),\Im_x^2\frac{\partial u(\cdot,t)}{\partial t}\right)_{L^2(\alpha,\beta)}$$
$$=-\left(f(\cdot,t),\Im_x^2\frac{\partial u(\cdot,t)}{\partial t}\right)_{L^2(\alpha,\beta)}. \tag{3.1}$$

It is easy to see that the following hold:

$$\frac{\partial}{\partial x}\Im_x\frac{\partial u}{\partial t}=\frac{\partial u}{\partial t} \tag{3.2}$$

and

$$\Im_\alpha u=\Im_\alpha^2 u=\Im_\beta u=\Im_\beta^2 u=0. \tag{3.3}$$

Integrating by parts the first three terms on the left-hand side of (3.1) by making use relations (3.2) and (3.3), we obtain

$$-\left(\frac{\partial u(\cdot,t)}{\partial t},\Im_x^2\frac{\partial u(\cdot,t)}{\partial t}\right)_{L^2(\alpha,\beta)}=\int_\alpha^\beta\left(\Im_x\frac{\partial u}{\partial t}\right)^2dx, \tag{3.4}$$

$$\left(\frac{\partial}{\partial x}\left(p\frac{\partial u(\cdot,t)}{\partial x}\right),\Im_x^2\frac{\partial u(\cdot,t)}{\partial t}\right)_{L^2(\alpha,\beta)}=-\int_\alpha^\beta p\frac{\partial u}{\partial x}\Im_x\frac{\partial u}{\partial t}\,dx$$
$$=\int_\alpha^\beta pu\frac{\partial u}{\partial t}\,dx+\int_\alpha^\beta\frac{\partial p}{\partial x}u\Im_x\frac{\partial u}{\partial t}\,dx$$
$$=\frac{1}{2}\frac{\partial}{\partial t}\int_\alpha^\beta pu^2\,dx-\frac{1}{2}\int_\alpha^\beta\frac{\partial p}{\partial t}u^2\,dx+\int_\alpha^\beta\frac{\partial p}{\partial x}u\Im_x\frac{\partial u}{\partial t}\,dx, \tag{3.5}$$

$$\eta\left(\frac{\partial^2}{\partial t\partial x}\left(p\frac{\partial u(\cdot,t)}{\partial x}\right),\Im_x^2\frac{\partial u(\cdot,t)}{\partial t}\right)_{L^2(\alpha,\beta)}=-\eta\int_\alpha^\beta\frac{\partial}{\partial t}\left(p\frac{\partial u}{\partial x}\right)\Im_x\frac{\partial u}{\partial t}\,dx$$
$$=\eta\int_\alpha^\beta p\left(\frac{\partial u}{\partial t}\right)^2dx+\eta\int_\alpha^\beta\frac{\partial p}{\partial t}u\frac{\partial u}{\partial t}\,dx$$
$$+\eta\int_\alpha^\beta\frac{\partial p}{\partial x}\frac{\partial u}{\partial t}\Im_x\frac{\partial u}{\partial t}\,dx+\eta\int_\alpha^\beta\frac{\partial^2 p}{\partial x\partial t}u\Im_x\frac{\partial u}{\partial t}\,dx$$
$$=\eta\int_\alpha^\beta p\left(\frac{\partial u}{\partial t}\right)^2dx-\frac{\eta}{2}\int_\alpha^\beta\frac{\partial^2 p}{\partial x^2}\left(\Im_x\frac{\partial u}{\partial t}\right)^2dx$$
$$+\eta\int_\alpha^\beta\frac{\partial p}{\partial t}u\frac{\partial u}{\partial t}\,dx+\eta\int_\alpha^\beta\frac{\partial^2 p}{\partial x\partial t}u\Im_x\frac{\partial u}{\partial t}\,dx, \tag{3.6}$$

$$-\left(f(\cdot,t),\Im_x^2\frac{\partial u(\cdot,t)}{\partial t}\right)_{L^2(\alpha,\beta)}=\int_\alpha^\beta \Im_x f\,\Im_x\frac{\partial u}{\partial t}\,dx. \tag{3.7}$$

Substituting (3.4)–(3.7) into (3.1) and integrating the obtained equality over $(0,\tau)$, with $0\leqslant\tau\leqslant T$, it yields

$$\begin{aligned}
&2\eta\int_0^\tau\int_\alpha^\beta p\left(\frac{\partial u}{\partial t}\right)^2dx\,dt+2\int_0^\tau\int_\alpha^\beta\left(\Im_x\frac{\partial u}{\partial t}\right)^2dx\,dt\\
&\quad-\eta\int_0^\tau\int_\alpha^\beta\frac{\partial^2 p}{\partial x^2}\left(\Im_x\frac{\partial u}{\partial t}\right)^2dx\,dt+\int_\alpha^\beta p(x,\tau)u^2(x,\tau)\,dx\\
&\quad=2\int_0^\tau\int_\alpha^\beta \Im_x f\,\Im_x\frac{\partial u}{\partial t}\,dx\,dt+\int_\alpha^\beta p(x,0)u_0^2(x)\,dx\\
&\quad+\int_0^\tau\int_\alpha^\beta\frac{\partial p}{\partial t}u^2\,dx\,dt-2\int_0^\tau\int_\alpha^\beta\left(\frac{\partial p}{\partial x}+\eta\frac{\partial^2 p}{\partial x\partial t}\right)u\Im_x\frac{\partial u}{\partial t}\,dx\,dt\\
&\quad-2\eta\int_0^\tau\int_\alpha^\beta\frac{\partial p}{\partial t}u\frac{\partial u}{\partial t}\,dx\,dt+2\int_0^\tau\int_\alpha^\beta qu\Im_x^2\frac{\partial u}{\partial t}\,dx\,dt.
\end{aligned}\tag{3.8}$$

Let us apply inequality (2.7) for $m=2$ and the Cauchy inequality so that the integral over $\partial u/\partial t$ in the right-hand side will be simplified with the same in the left-hand side, and the integrals over $\Im_x(\partial u/\partial t)$ will be absorbed in the left-hand side. Then equality (3.8) becomes, by taking Assumption A1 into account,

$$\begin{aligned}
&\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{B_2^1(\alpha,\beta)}dt+\|u(\cdot,\tau)\|^2_{L^2(\alpha,\beta)}\\
&\quad\leqslant c_9\left(\int_0^\tau\|f(\cdot,t)\|^2_{B_2^1(\alpha,\beta)}dt+\|u_0\|^2_{L^2(\alpha,\beta)}\right)+c_{10}\int_0^\tau\|u(\cdot,t)\|^2_{L^2(\alpha,\beta)}dt,
\end{aligned}$$

where

$$c_9=\frac{\max(4,c_1)}{\min(1,c_0)}\quad\text{and}\quad c_{10}=\frac{4c_2^2+c_3+8c_4^2+8\eta^2c_5^2+\eta c_3^2/2c_0}{\min(1,c_0)}.$$

Thanks to a lemma of Gronwall's type [3, Lemma 1], the last inequality implies the following:

$$\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{B_2^1(\alpha,\beta)}dt+\|u(\cdot,\tau)\|^2_{L^2(\alpha,\beta)}$$

$$\leqslant c_9 \exp(c_{10}T)\left(\int_0^\tau \|f(\cdot,t)\|^2_{B_2^1(\alpha,\beta)}\,dt + \|u_0\|^2_{L^2(\alpha,\beta)}\right).$$

As the right-hand side of the above inequality is independent of τ, we take the upper bound for τ from 0 to T in the left-hand side, we obtain

$$\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;B_2^1(\alpha,\beta))} + \|u\|^2_{C(0,T;L^2(\alpha,\beta))} \leqslant C^2\big(\|f\|^2_{L^2(0,T;B_2^1(\alpha,\beta))} + \|u_0\|^2_{L^2(\alpha,\beta)}\big).$$

Thus we have established the following theorem.

Theorem 3.1. *Suppose that Assumption* A1 *holds, then there is a constant $C > 0$ independent of u, such that*

$$\|u\|_B \leqslant C\|Lu\|_F, \quad \forall u \in D(L). \tag{3.9}$$

It follows from (3.9) that there is a bounded inverse L^{-1} on the range LB of L. However, since we have no information concerning LB except that $LB \subset F$, we must extend L so that inequality (3.9) holds for the extension and its range is the whole space.

We first state the following proposition.

Proposition 3.1. *The operator L from B into F has a closure.*

Proof. Let us verify the closure criterion in Banach spaces. Suppose

$$u_n \to 0 \quad \text{in } B, \tag{3.10}$$

where $u_n \in D(L)$ and

$$Lu_n \to (f, u_0) \quad \text{in } L^2\big(0,T;B_2^1(\alpha,\beta)\big) \times L^2(\alpha,\beta) \tag{3.11}$$

as $n \to \infty$. We must show that $f = 0$ and $u_0 = 0$.

According to (3.10), we have

$$u_n \to 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{3.12}$$

where $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. The continuity of derivation of $\mathcal{D}'(\Omega)$ in $\mathcal{D}'(\Omega)$ implies that (3.12) becomes

$$\mathcal{L}u_n \to 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.13}$$

It follows from (3.11) that

$$\mathcal{L}u_n \to f \quad \text{in } L^2\big(0,T;B_2^1(\alpha,\beta)\big),$$

therefore

$$\mathcal{L}u_n \to f \quad \text{in } \mathcal{D}'(\Omega). \tag{3.14}$$

By virtue of the uniqueness of the limit in $\mathcal{D}'(\Omega)$, we deduce from (3.13) and (3.14) that $f = 0$.

Moreover, we see via (3.11), that

$$\ell u_n \to u_0 \quad \text{in } L^2(\alpha, \beta). \tag{3.15}$$

Passing to the limit in the inequality

$$\|\ell u_n\|_{L^2(\alpha,\beta)} \leqslant \|u_n\|_{C(0,T;L^2(\alpha,\beta))} \leqslant \|u_n\|_B, \quad \forall n,$$

by taking into account (3.10), we get

$$\ell u_n \to 0 \quad \text{in } L^2(\alpha, \beta). \tag{3.16}$$

Since the limit in $L^2(\alpha, \beta)$ is unique, it follows from (3.15) and (3.16) that $u_0 = 0$. Proposition 3.1 is proved. □

Definition 3.1. A solution of the operator equation

$$\bar{L}u = (f, u_0)$$

is called a *strong solution* of problem (2.8).

Since points of the graph of $\bar{L}$ are limits of sequences of points of the graph of L, then we pass to the limit in (3.9), from which we conclude

Corollary 3.1. *Under hypotheses of Theorem* 3.1*, there is a constant $C > 0$ independent of u such that*

$$\|u\|_B \leqslant C\|\bar{L}u\|_F, \quad \forall u \in D(\bar{L}).$$

Corollary 3.1 *asserts that $\bar{L}$ is injective, the linear operator $\bar{L}^{-1}$ exists and is continuous from $\bar{L}B$ onto B.*

Corollary 3.2. *The range $\bar{L}B$ of the operator $\bar{L}$ is closed in F, and equals to the closure $\overline{LB}$ of LB.*

4. Solvability of the problem

In this section, we state and prove the main result.

Theorem 4.1. *Under Assumptions* A1 *and* A2*, problem* (2.4)–(2.6) *has a unique strong solution $u = \bar{L}^{-1}(f, u_0) = \overline{L^{-1}}(f, u_0)$, that depends continuously upon the data, for all $f \in L^2(0, T; B_2^1(\alpha, \beta))$ and $u_0 \in L^2(\alpha, \beta)$.*

Proof. From Corollary 3.2, we deduce that to establish the solvability of problem (2.4)–(2.6) it suffices to show that LB is dense in F. We first prove the density of L_0B in F, for all $u \in D_0(L_0)$ and $(f, u_0) \in F$, where $L_0 = (\mathcal{L}_0, \ell)$, $\mathcal{L}_0$ be the principal part of $\mathcal{L}$, i.e.,

$$\mathcal{L}_0 u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(p(x,t)\frac{\partial u}{\partial x}\right) - \eta\frac{\partial^2}{\partial t \partial x}\left(p(x,t)\frac{\partial u}{\partial x}\right),$$

and $D_0(L_0) = D_0(L)$ be the set of all u belonging to $D(L)$ for which $\ell u = 0$.

Proposition 4.1. *Suppose that hypotheses of Theorem* 4.1 *are fulfilled. If*

$$(\mathcal{L}_0 u, \omega)_{L^2(0,T;B_2^1(\alpha,\beta))} = 0 \tag{4.1}$$

for some $\omega \in L^2(0,T;B_2^1(\alpha,\beta))$ *and for all* $u \in D_0(L_0)$*, then* ω *vanishes almost everywhere in* Ω.

Proof. Equality (4.1) can be written as follows:

$$\left(\frac{\partial u}{\partial t}, \omega\right)_{L^2(0,T;B_2^1(\alpha,\beta))} = \left(\left(I + \eta\frac{\partial}{\partial t}\right)Pu, \omega\right)_{L^2(0,T;B_2^1(\alpha,\beta))}. \tag{4.2}$$

From (4.2), with u replaced by $\varrho_\varepsilon^{-1}u$, we have

$$\left(\frac{\partial \varrho_\varepsilon^{-1}u}{\partial t}, \omega\right)_{L^2(0,T;B_2^1(\alpha,\beta))} = \left(\left(I + \eta\frac{\partial}{\partial t}\right)P\varrho_\varepsilon^{-1}u, \omega\right)_{L^2(0,T;B_2^1(\alpha,\beta))}.$$

According to (P3) and (P6), we get

$$\left(\varrho_\varepsilon^{-1}\frac{\partial u}{\partial \tau}, \omega\right)_{L^2(0,T;B_2^1(\alpha,\beta))}$$
$$= \left(\varrho_\varepsilon^{-1}\left(I + \eta\frac{\partial}{\partial \tau}\right)\left(Pu + \varepsilon P'\varrho_\varepsilon^{-1}u\right), \omega\right)_{L^2(0,T;B_2^1(\alpha,\beta))}.$$

By virtue of (P2), it yields

$$\left(\frac{\partial u}{\partial t}, \left(\varrho_\varepsilon^{-1}\right)^*\omega\right)_{L^2(0,T;B_2^1(\alpha,\beta))}$$
$$= \left(\left(Pu + \varepsilon P'\varrho_\varepsilon^{-1}u\right), \left(I + \eta\frac{\partial}{\partial \tau}\right)^*\left(\varrho_\varepsilon^{-1}\right)^*\omega\right)_{L^2(0,T;B_2^1(\alpha,\beta))}.$$

Therefore

$$\left(\frac{\partial u}{\partial t}, \left(\varrho_\varepsilon^{-1}\right)^*\Im_x^*\Im_\xi\omega\right)_{L^2(0,T;L^2(\alpha,\beta))}$$
$$= \left(\left(Pu + \varepsilon P'\varrho_\varepsilon^{-1}u\right), \left(I + \eta\frac{\partial}{\partial \tau}\right)^*\left(\varrho_\varepsilon^{-1}\right)^*\Im_x^*\Im_\xi\omega\right)_{L^2(0,T;L^2(\alpha,\beta))}. \tag{4.3}$$

The operator $P'(t)\varrho_\varepsilon^{-1}\cdot$ can be written as

$$P'(t)\varrho_\varepsilon^{-1}u = P'(t)\varrho_\varepsilon^{-1}P^{-1}(t)P(t)u = P_\varepsilon(t)P(t)u. \tag{4.4}$$

The calculations of $P^{-1}(t)$ and $P_\varepsilon(t)$ are simple but somewhat long. We only give the final result of computations,

$$P^{-1}(t)g = c_1(t) + c_2(t)\int_\alpha^x \frac{d\xi}{p(\xi,t)} + \int_\alpha^x \frac{d\xi}{p(\xi,t)}\int_\alpha^\xi g(\zeta,t)\,d\zeta, \tag{4.5}$$

where

$$c_1(t) = \frac{1}{\Delta(t)}\Bigg[\Bigg(\int_\alpha^\beta \frac{(\beta^2 - x^2)}{p(x,t)}\,dx\Bigg)\Bigg(\int_\alpha^\beta \frac{(\beta - x)}{p(x,t)}\,dx \int_\alpha^x g(\xi,t)\,d\xi\Bigg)$$
$$- \Bigg(\int_\alpha^\beta \frac{(\beta - x)}{p(x,t)}\,dx\Bigg)\Bigg(\int_\alpha^\beta \frac{(\beta^2 - x^2)}{p(x,t)}\,dx \int_\alpha^x g(\xi,t)\,d\xi\Bigg)\Bigg],$$
$$c_2(t) = -\frac{(\beta-\alpha)}{\Delta(t)}\Bigg(\int_\alpha^\beta \frac{(\alpha - x)(\beta - x)}{p(x,t)}\,dx \int_\alpha^x g(\xi,t)\,d\xi\Bigg),$$
$$\Delta(t) = (\beta-\alpha)\int_\alpha^\beta \frac{(\alpha - x)(\beta - x)}{p(x,t)}\,dx,$$

and

$$P_\varepsilon(t)g = \Bigg(\frac{\partial^2 p(x,t)}{\partial t\partial x}\varrho_\varepsilon^{-1} - \frac{\partial p(x,t)}{\partial t}\varrho_\varepsilon^{-1}\frac{1}{p(x,\tau)}\frac{\partial p(x,\tau)}{\partial x}\Bigg)\frac{1}{p(x,\tau)}$$
$$\times\Bigg(c_2(\tau) + \int_\alpha^x g(\xi,\tau)\,d\xi\Bigg) + \frac{\partial p(x,t)}{\partial t}\varrho_\varepsilon^{-1}\frac{1}{p(x,\tau)}g(x,\tau). \tag{4.6}$$

It then follows from (4.3)–(4.6) that

$$-\Bigg(u, \frac{\partial(\varrho_\varepsilon^{-1})^*\Im_x^*\Im_\xi\omega}{\partial t}\Bigg)_{L^2(0,T;L^2(\alpha,\beta))}$$
$$= \Bigg(Pu, \big(I + \varepsilon P_\varepsilon^*\big)\Bigg(I + \eta\frac{\partial}{\partial\tau}\Bigg)^*\big(\varrho_\varepsilon^{-1}\big)^*\Im_x^*\Im_\xi\omega\Bigg)_{L^2(0,T;L^2(\alpha,\beta))}, \tag{4.7}$$

where $P_\varepsilon^*(t)$, the adjoint of $P_\varepsilon(t)$, is defined by

$$P_\varepsilon^*\Bigg(I + \eta\frac{\partial}{\partial\tau}\Bigg)^*\big(\varrho_\varepsilon^{-1}\big)^*\Im_x^*\Im_\xi\omega$$
$$= -\frac{\int_x^\beta (\alpha-\xi)(\beta-\xi)/p(\xi,t)\,d\xi}{\int_\alpha^\beta (\alpha-x)(\beta-x)/p(x,t)\,dx}$$
$$\times\int_x^\beta \frac{1}{p}\Bigg(\big(\varrho_\varepsilon^{-1}\big)^*\frac{\partial^2 p}{\partial\xi\partial\tau} - \frac{\partial p}{\partial\xi}\frac{1}{p}\big(\varrho_\varepsilon^{-1}\big)^*\frac{\partial p}{\partial\tau}\Bigg)\Bigg(I + \eta\frac{\partial}{\partial\tau}\Bigg)^*\big(\varrho_\varepsilon^{-1}\big)^*\Im_x^*\Im_\xi\omega\,d\xi$$
$$+ \frac{1}{p}\big(\varrho_\varepsilon^{-1}\big)^*\frac{\partial p}{\partial\tau}\Bigg(I + \eta\frac{\partial}{\partial\tau}\Bigg)^*\big(\varrho_\varepsilon^{-1}\big)^*\Im_x^*\Im_\xi\omega. \tag{4.8}$$

We set in (4.7),

$$\Lambda_\varepsilon\Bigg(\Bigg(I + \eta\frac{\partial}{\partial\tau}\Bigg)^*\big(\varrho_\varepsilon^{-1}\big)^*\Im_x^*\Im_\xi\omega\Bigg) = \big(I + \varepsilon P_\varepsilon^*\big)\Bigg(I + \eta\frac{\partial}{\partial\tau}\Bigg)^*\big(\varrho_\varepsilon^{-1}\big)^*\Im_x^*\Im_\xi\omega. \tag{4.9}$$

Since the left-hand side of (4.7) is a continuous linear functional of u, then the operator Λ_ε has the derivatives $\partial\Lambda_\varepsilon/\partial x$ and $\partial^2\Lambda_\varepsilon/\partial x^2$ belonging to $L^2(\Omega)$, and verifying the following conditions:

$$\Lambda_\varepsilon|_{x=\alpha} = \Lambda_\varepsilon|_{x=\beta} = \left.\frac{\partial\Lambda_\varepsilon}{\partial x}\right|_{x=\alpha} = \left.\frac{\partial\Lambda_\varepsilon}{\partial x}\right|_{x=\beta} = 0. \tag{4.10}$$

The norm of the operator $\varepsilon P_\varepsilon^*(t)$ on $L^2(\Omega)$ is smaller than 1, for sufficiently small ε; hence the operator $(I+\varepsilon P_\varepsilon^*)(I+\eta(\partial/\partial\tau))^*(\varrho_\varepsilon^{-1})^*$ possesses a continuous inverse on $L^2(\Omega)$, i.e., $\Im_x^*\Im_\xi\omega \in L^2(\Omega)$.

The differentiation of (4.9) with respect to x leads to

$$\begin{aligned}
&\frac{\partial\Lambda_\varepsilon}{\partial x}\left(I+\eta\frac{\partial}{\partial\tau}\right)^*(\varrho_\varepsilon^{-1})^*\Im_x^*\Im_\xi\omega \\
&\quad= \varepsilon\frac{\partial P_\varepsilon^*}{\partial x}\left(I+\eta\frac{\partial}{\partial\tau}\right)^*(\varrho_\varepsilon^{-1})^*\Im_x^*\Im_\xi\omega \\
&\qquad- \left(I+\varepsilon P_\varepsilon^*\right)\left(I+\eta\frac{\partial}{\partial\tau}\right)^*(\varrho_\varepsilon^{-1})^*\Im_x\omega,
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
&\frac{\partial P_\varepsilon^*}{\partial x}\left(I+\eta\frac{\partial}{\partial\tau}\right)^*(\varrho_\varepsilon^{-1})^*\Im_x^*\Im_\xi\omega \\
&\quad= \frac{(\alpha-x)(\beta-x)/p(x,t)}{\int_\alpha^\beta(\alpha-x)(\beta-x)/p(x,t)\,dx} \\
&\qquad\times\int_\alpha^\beta\frac{1}{p}\left((\varrho_\varepsilon^{-1})^*\frac{\partial^2 p}{\partial x\partial t}-\frac{\partial p}{\partial x}\frac{1}{p}(\varrho_\varepsilon^{-1})^*\frac{\partial p}{\partial\tau}\right)\left(I+\eta\frac{\partial}{\partial\tau}\right)^*(\varrho_\varepsilon^{-1})^*\Im_x^*\Im_\xi\omega\,dx.
\end{aligned} \tag{4.12}$$

From (4.12) we conclude that $(\partial P_\varepsilon^*/\partial x)(I+\eta(\partial/\partial\tau))^*(\varrho_\varepsilon^{-1})^*\cdot$ is bounded on $L^2(\Omega)$. Thus (4.11) implies that $\Im_x\omega \in L^2(\Omega)$, for sufficiently small ε; in other words that $\omega \in L^2(0,T;B_2^1(\alpha,\beta))$. Analogously, using the boundedness of $(\partial P_\varepsilon^*/\partial x)(I+\eta(\partial/\partial\tau))^*\times(\varrho_\varepsilon^{-1})^*\cdot$ on $L^2(\Omega)$, deduce that $\omega\in L^2(\Omega)$.

From (4.8)–(4.12), we get

$$\left.\left(I+\varepsilon\frac{1}{p}(\varrho_\varepsilon^{-1})^*\frac{\partial p}{\partial\tau}\right)\left(I+\eta\frac{\partial}{\partial\tau}\right)^*(\varrho_\varepsilon^{-1})^*\Im_x^*\Im_\xi\omega\right|_{x=\alpha} = 0, \tag{4.13}$$

$$\left.\left(I+\varepsilon\frac{1}{p}(\varrho_\varepsilon^{-1})^*\frac{\partial p}{\partial\tau}\right)\left(I+\eta\frac{\partial}{\partial\tau}\right)^*(\varrho_\varepsilon^{-1})^*\Im_x^*\Im_\xi\omega\right|_{x=\beta} = 0, \tag{4.14}$$

$$\left.\left(I+\varepsilon\frac{1}{p}(\varrho_\varepsilon^{-1})^*\frac{\partial p}{\partial\tau}\right)\left(I+\eta\frac{\partial}{\partial\tau}\right)^*(\varrho_\varepsilon^{-1})^*\Im_x\omega\right|_{x=\alpha} = 0, \tag{4.15}$$

$$\left.\left(I+\varepsilon\frac{1}{p}(\varrho_\varepsilon^{-1})^*\frac{\partial p}{\partial\tau}\right)\left(I+\eta\frac{\partial}{\partial\tau}\right)^*(\varrho_\varepsilon^{-1})^*\Im_x\omega\right|_{x=\beta} = 0. \tag{4.16}$$

For each fixed $x \in [\alpha, \beta]$ and sufficiently small ε, the norm of operator $\varepsilon(1/p)(\varrho_\varepsilon^{-1})^* \times (\partial p/\partial \tau)$ on $L^2(\Omega)$ is smaller than 1. Therefore operator $I + \varepsilon(1/p)(\varrho_\varepsilon^{-1})^*(\partial p/\partial \tau)$ has continuous inverse on $L^2(\Omega)$, then from (4.13)–(4.16), we obtain

$$\Im_x^* \Im_\xi \omega|_{x=\alpha} = 0, \tag{4.17}$$

$$\Im_x^* \Im_\xi \omega|_{x=\beta} = 0, \tag{4.18}$$

$$\Im_x \omega|_{x=\alpha} = 0, \tag{4.19}$$

$$\Im_x \omega|_{x=\beta} = 0. \tag{4.20}$$

Let us set

$$\omega(x,t) = s(x,t). \tag{4.21}$$

It is easy to observe from (4.17)–(4.21) that

$$\Im_\alpha^2 s = \Im_\beta^2 s = \Im_\alpha s = \Im_\beta s = 0. \tag{4.22}$$

Replace ω in (4.2) by its representation (4.21), yields

$$\left(\frac{\partial u}{\partial t}, s\right)_{L^2(0,T;B_2^1(\alpha,\beta))} = \left(\left(I + \eta\frac{\partial}{\partial t}\right)Pu, s\right)_{L^2(0,T;B_2^1(\alpha,\beta))}, \tag{4.23}$$

and put

$$u = \Im_t(e^{c\tau} s), \tag{4.24}$$

where c is a constant such that

$$c = \left[\frac{c_3 + 2c_4^2 + 2c_5^2 + c_7}{c_0 + \eta c_6}, +\infty\right[, \tag{4.25}$$

with c_i $(i = 0, 3, 4, 5, 6, 7)$ are as in Assumptions A1 and A2.

Substituting (4.24) into (4.23), we get

$$\int_0^T\int_\alpha^\beta e^{ct}(\Im_x s)^2\,dx\,dt = \left(\left(I + \eta\frac{\partial}{\partial t}\right)P\big(\Im_t(e^{c\tau}s)\big), s\right)_{L^2(0,T;B_2^1(\alpha,\beta))}. \tag{4.26}$$

Integrating by parts the right-hand side of (4.26), we obtain

$$\begin{aligned}
&\left(\left(I + \eta\frac{\partial}{\partial t}\right)P\big(\Im_t(e^{c\tau}s)\big), s\right)_{L^2(0,T;B_2^1(\alpha,\beta))} \\
&\quad = -\frac{1}{2}\int_\alpha^\beta e^{-cT}\left(p(x,T) + \eta\frac{\partial p(x,T)}{\partial t}\right)\big(\Im_T(e^{ct}s)\big)^2\,dx \\
&\qquad - \frac{1}{2}\int_\Omega e^{-ct}\left(c\left(p + \eta\frac{\partial p}{\partial t}\right) - \frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial t^2}\right)\big(\Im_t(e^{c\tau}s)\big)^2\,dx\,dt \\
&\qquad - \eta\int_\Omega e^{ct}ps^2\,dx\,dt - \int_\Omega\left(\frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial x\,\partial t}\right)\Im_t(e^{c\tau}s)\Im_x s\,dx\,dt.
\end{aligned} \tag{4.27}$$

Applying the Cauchy inequality to the last term on the right-hand side of (4.27), we get

$$-\int_\Omega \left(\frac{\partial p}{\partial x} + \eta \frac{\partial^2 p}{\partial x \partial t}\right) \Im_t(e^{c\tau}s)\Im_x s\,dx\,dt$$
$$\leqslant \frac{1}{2}\int_\Omega e^{ct}(\Im_x s)^2\,dx\,dt + \int_\Omega \left(\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial^2 p}{\partial x \partial t}\right)^2\right)\big(\Im_t(e^{c\tau}s)\big)^2\,dx\,dt. \tag{4.28}$$

It then follows from (4.26)–(4.28) that

$$\frac{1}{2}\int_\Omega e^{ct}(\Im_x s)^2\,dx\,dt$$
$$\leqslant -\frac{1}{2}\int_\alpha^\beta e^{-cT}\left(p(x,T) + \eta\frac{\partial p(x,T)}{\partial t}\right)\big(\Im_T(e^{ct}s)\big)^2\,dx$$
$$-\frac{1}{2}\int_\Omega e^{-ct}\left(c\left(p + \eta\frac{\partial p}{\partial t}\right) - \frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial t^2} - 2\left(\frac{\partial p}{\partial x}\right)^2 - 2\left(\frac{\partial^2 p}{\partial x \partial t}\right)^2\right)$$
$$\times \big(\Im_t(e^{c\tau}s)\big)^2\,dx\,dt.$$

By virtue of Assumptions A1 and A2, we have

$$\frac{1}{2}\int_\Omega e^{ct}(\Im_x s)^2\,dx\,dt$$
$$\leqslant -\frac{1}{2}(c_0 + \eta c_6)\int_\alpha^\beta e^{-cT}\big(\Im_T(e^{ct}s)\big)^2\,dx$$
$$-\frac{1}{2}\big(c(c_0 + \eta c_6) - \big(c_3 + 2c_4^2 + 2c_5^2 + c_7\big)\big)\int_\Omega e^{-ct}\big(\Im_t(e^{c\tau}s)\big)^2\,dx\,dt.$$

Thanks to (4.25), we conclude that

$$\int_\Omega e^{ct}(\Im_x s)^2\,dx\,dt \leqslant 0,$$

and thus s vanishes almost everywhere in Ω, from which we have $\omega = 0$ almost everywhere in Ω. □

Now return back to the proof of Theorem 4.1. Let the element $W = (f, u_0)$ be orthogonal to $L_0 B$ so that

$$(\mathcal{L}_0 u, f)_{L^2(0,T;B_2^1(\alpha,\beta))} + (\ell u, u_0)_{L^2(\alpha,\beta)} = 0, \quad \forall u \in D(L_0). \tag{4.29}$$

Assuming in (4.29) that u is any element of $D_0(L_0)$, we obtain

$$(\mathcal{L}_0 u, f)_{L^2(0,T;B_2^1(\alpha,\beta))} = 0.$$

By virtue of Proposition 4.1 we conclude that $f = 0$; thus (4.29) implies that

$$(\ell u, u_0)_{L^2(\alpha,\beta)} = 0, \quad \forall u \in D(L_0). \tag{4.30}$$

Since ℓB is dense in $L^2(\alpha, \beta)$, equality (4.30) implies that $u_0 = 0$. Consequently $\overline{L_0 B} = F$. In other words, the equation

$$L_0 u = (f, u_0)$$

has a strong solution in B.

Now consider the general case. Since $L_0 B$ is dense in F and $L - L_0 = (\mathcal{L}_0 - \mathcal{L}_0, \ell)$ maps continuously B in F, the density of LB in F can be proved by using the method of continuation of the parameter. We will not describe it here, because it is similar to that in [3]. □

5. Generalization

In this section, we state results concerning some extensions of the studied problem.

(1) Our results remain valid, except for some technical differences, if we replace condition (1.3) for $i = 1$, by the Neumann condition

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=\Gamma_2(\tau)} = \mu(\tau).$$

(2) Let us consider the following $2m$-pseudoparabolic equation with initial-boundary conditions:

$$\begin{aligned}\mathcal{L}\theta = \frac{\partial\theta}{\partial\tau} - \frac{\partial^{2m-1}}{\partial\xi^{2m-1}}\left(a(\xi,\tau)\frac{\partial\theta}{\partial\xi}\right)\\ - \eta\frac{\partial^{2m}}{\partial\tau\,\partial\xi^{2m-1}}\left(a(\xi,\tau)\frac{\partial\theta}{\partial\xi}\right) + b(\xi,\tau)\theta = h(\xi,\tau),\end{aligned} \tag{5.1}$$

$$\ell\theta = \theta(\xi, 0) = \theta_0(\xi), \tag{5.2}$$

$$\int_{\Gamma_1(\tau)}^{\Gamma_2(\tau)} \xi^i \theta(\xi,\tau)\, d\xi = M_i(\tau) \quad (i = 0, 1, \ldots, 2m-1). \tag{5.3}$$

After doing similar reformulations of the problem, as in Section 2, we have the following results.

Theorem 5.1. *Let B be the Banach space of all functions $u \in L^2(0, T; B_2^m(\alpha, \beta))$ having the finite norm*

$$\|u\|_B = \left(\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;B_2^m(\alpha,\beta))} + \|u\|^2_{C(0,T;L^2(\alpha,\beta))}\right)^{1/2}.$$

Then, there exists a positive constant C such that

$$\|u\|_B \leqslant C\left(\|f\|^2_{L^2(0,T;B_2^m(\alpha,\beta))} + \|u_0\|^2_{L^2(\alpha,\beta)}\right)^{1/2}.$$

Proof. Taking the scalar product in $L^2(\alpha,\beta)$ of Eq. (5.1) and $\Im_x^{2m}(\partial u/\partial t)$ (or in $B_2^m(\alpha,\beta)$ of Eq. (5.1) and $\partial u/\partial t$), and making a similar calculation to that in Section 3, we establish estimate (5.4). □

Theorem 5.2. *Problem* (5.1)–(5.3) *admits a unique strong solution* $u = \bar{L}^{-1}(f,u_0) = \overline{L^{-1}}(f,u_0)$, *that depends continuously upon the data, for* $f \in L^2(0,T;B_2^m(\alpha,\beta))$, *and* $u_0 \in L^2(\alpha,\beta)$.

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