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Inversive meadows and divisive meadows

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ABSTRACT

Inversive meadows are commutative rings with a multiplicative identity element and a total multiplicative inverse operation satisfying $0^{-1} = 0$. Divisive meadows are inversive meadows with the multiplicative inverse operation replaced by a division operation. We give finite equational specifications of the class of all inversive meadows and the class of all divisive meadows. It depends on the angle from which they are viewed whether inversive meadows or divisive meadows must be considered more basic. We show that inversive and divisive meadows of rational numbers can be obtained as initial algebras of finite equational specifications. In the spirit of Peacock's arithmetical algebra, we study variants of inversive and divisive meadows without an additive identity element and/or an additive inverse operation. We propose simple constructions of variants of inversive and divisive meadows with a partial multiplicative inverse or division operation from inversive and divisive meadows. Divisive meadows are more basic if these variants are considered as well. We give a simple account of how mathematicians deal with $1/0$, in which meadows and a customary convention among mathematicians play prominent parts, and we make plausible that a convincing account, starting from the popular computer science viewpoint that $1/0$ is undefined, by means of some logic of partial functions is not attainable.

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1. Introduction

The primary mathematical structure for measurement and computation is unquestionably a field. In [16], meadows are proposed as alternatives for fields with a purely equational axiomatization. A meadow is a commutative ring with a multiplicative identity element and a total multiplicative inverse operation satisfying two equations which imply that the multiplicative inverse of zero is zero.¹ Thus, meadows are total algebras. Recently, we found in [34] that meadows were already introduced by Komori [28] in a report from 1975, where they go by the name of *desirable pseudo-fields*. This finding induced us to propose the name *Komori field* for a meadow satisfying $0 \neq 1$ and $x \neq 0 \Rightarrow x \cdot x^{-1} = 1$. The prime example of Komori fields is the field of rational numbers with the multiplicative inverse operation made total by imposing that the multiplicative inverse of zero is zero.

As usual in field theory, the convention to consider p/q as an abbreviation for $p \cdot (q^{-1})$ was used in subsequent work on meadows (see e.g. [5,12]). This convention is no longer satisfactory if partial variants of meadows are considered too, as will be demonstrated in this paper. That is why we rename meadows into inversive meadows and introduce divisive meadows. A divisive meadow is an inversive meadow with the multiplicative inverse operation replaced by the division operation suggested by the above-mentioned abbreviation convention. We give finite equational specifications of the class

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E-mail addresses: J.A.Bergstra@uva.nl (J.A. Bergstra), C.A.Middelburg@uva.nl (C.A. Middelburg).URLs: <http://www.science.uva.nl/~janb> (J.A. Bergstra), <http://www.science.uva.nl/~kmiddelb> (C.A. Middelburg).¹ This structure is called a meadow because a meadow is similar to a field outside of mathematics: a meadow is an open grassland and a field is a wide and open grassy area.

of all inversive meadows and the class of all divisive meadows and demonstrate that it depends on the angle from which they are viewed whether inversive meadows or divisive meadows must be considered more basic. Henceforth, we will use the name meadow whenever the distinction between inversive meadows and divisive meadows is not important.

Peacock introduced in [36] arithmetical algebra as algebra of numbers where an additive identity element and an additive inverse operation are not involved. That is, arithmetical algebra is algebra of positive numbers instead of algebra of numbers in general (see also [27]). In the spirit of Peacock, we use the name *arithmetical meadow* for a meadow without an additive identity element and an additive inverse operation. Moreover, we use the name *arithmetical meadow with zero* for a meadow without an additive inverse operation, but with an additive identity element. Arithmetical meadows of rational numbers are reminiscent of Peacock's arithmetical algebra. We give finite equational specifications of the class of all inversive arithmetical meadows, the class of all divisive arithmetical meadows, the class of all inversive arithmetical meadows with zero and the class of all divisive arithmetical meadows with zero.

The main inversive meadow that we are interested in is the zero-totalized field of rational numbers, which differs from the field of rational numbers only in that the multiplicative inverse of zero is zero. The main divisive meadow that we are interested in is the zero-totalized field of rational numbers with the multiplicative inverse operation replaced by the division operation suggested by the abbreviation p/q for $p \cdot (q^{-1})$. We show that these meadows can be obtained as initial algebras of finite equational specifications. We also show that arithmetical meadows of rational numbers and arithmetical meadows of rational numbers with zero can be obtained as initial algebras of finite equational specifications. Arithmetical meadows of rational numbers and arithmetical meadows of rational numbers with zero provide additional insight in what is yielded by the presence of an operator for multiplicative inverse (or division) in a signature.

Partial variants of meadows can be obtained by turning the total multiplicative inverse or division operation into a partial one. There is one way in which the total multiplicative inverse operation can be turned into a partial one, whereas there are two conceivable ways in which the total division operation can be turned into a partial one. Therefore, we propose one construction of variants of inversive meadows with a partial multiplicative inverse operation from inversive meadows and two constructions of variants of divisive meadows with a partial division operation from divisive meadows. We demonstrate that divisive meadows are more basic if these partial variants of meadows are considered as well.

We can obtain interesting partial versions of the above-mentioned meadows of rational numbers, each of which is the initial algebra of a finite equational specification, by means of the proposed constructions of partial versions. This approach fits in with our position that partial algebras should be made of total ones. Thus, we can obtain total and partial algebras requiring only equational logic for total algebras as a tool for their construction.

It is quite usual that neither the division operator nor the multiplicative inverse operator is included in the signature of number systems such as the field of rational numbers and the field of real numbers. However, the abundant use of the division operator in mathematical practice makes it very reasonable to include the division operator, or alternatively the multiplicative inverse operator, in the signature. It appears that excluding both of them creates more difficulties than that it solves. At the least, the problem of division by zero cannot be avoided by excluding $1/0$ from being written. We give a simple account of how mathematicians deal with $1/0$ in mathematical works. Dominating in this account is the concept of an imperative meadow, a concept in which a customary convention among mathematicians plays a prominent part. We also make plausible that a convincing account, starting from the usual viewpoint of theoretical computer scientists that $1/0$ is undefined, by means of some logic of partial functions is not attainable.

This paper is organized as follows. First, we go into the background of the work presented in this paper with the intention to clarify and motivate this work (Section 2) and discuss the main prevailing viewpoints on the status of $1/0$ in mathematics and theoretical computer science (Section 3). Next, we give equational specifications of the class of all inversive meadows and the class of all divisive meadows (Section 4). After that, we give equational specifications of the arithmetical variants of these classes (Section 5) and connect one of these variants with an arithmetical version of von Neumann regular rings (Section 6). Then, we give equational specifications whose initial algebras are inversive and divisive meadows of rational numbers (Section 7). After that, we give equational specifications whose initial algebras are the arithmetical variants of these meadows of rational numbers (Section 8). Following this, we introduce and discuss constructions of partial variants of meadows from total ones (Section 9) and constructions of partial variants of arithmetical meadows from total ones (Section 10). Next, we introduce imperative meadows of rational numbers (Section 11) and discuss the convention that is involved in them (Section 12). After that, we make plausible the inadequacy of logics of partial functions for a convincing account of how mathematicians deal with $1/0$ (Section 13). Finally, we make some concluding remarks (Section 14).

This paper consolidates material from [10,8,11].

2. Background on the theory of meadows

In this section, we go into the background of the work presented in this paper with the intention to clarify and motivate this work.

The theory of meadows, see e.g. [5,12], constitutes a hybrid between the theory of abstract data type and the theory of rings and fields, more specifically the theory of von Neumann regular rings [31,22] (all fields are von Neumann regular rings).

It is easy to see that each meadow can be reduced to a commutative von Neumann regular ring with a multiplicative identity element. Moreover, we know from [5] that each commutative von Neumann regular ring with a multiplicative

identity element can be expanded to a meadow, and that this expansion is unique. It is easy to show that, if $\phi : X \rightarrow Y$ is an epimorphism between commutative rings with a multiplicative identity element and X is a commutative von Neumann regular ring with a multiplicative identity element, then: (i) Y is a commutative von Neumann regular ring with a multiplicative identity element; (ii) ϕ is also an epimorphism between meadows for the meadows X' and Y' found by means of the unique expansions for X and Y , respectively.

However, there is a difference between commutative von Neumann regular rings with a multiplicative identity element and meadows: the class of all meadows is a variety and the class of all commutative von Neumann regular rings with a multiplicative identity element is not. In particular, the class of commutative von Neumann regular rings with a multiplicative identity element is not closed under taking subalgebras (a property shared by all varieties). Let \mathcal{Q} be the ring of rational numbers, and let \mathcal{Z} be its subalgebra of integers. Then \mathcal{Q} is a field and for that reason a commutative von Neumann regular ring with a multiplicative identity element, but its subalgebra \mathcal{Z} is not a commutative von Neumann regular ring with a multiplicative identity element.

In spite of the fact that meadows and commutative von Neumann regular rings with a multiplicative identity element are so close that no new mathematics can be expected, there is a difference which matters very much from the perspective of abstract data type specification. \mathcal{Q} , the ring of rational numbers, is not a minimal algebra, whereas \mathcal{Q}_0^i , the inversive meadow of rational numbers is a minimal algebra. As such, \mathcal{Q}_0^i is amenable to initial algebra specification. The first initial algebra specification of \mathcal{Q}_0^i is given in [16] and an improvement due to Hirshfeld is given in the current paper. When looking for an initial algebra specification of \mathcal{Q} , adding a total multiplicative inverse operation satisfying $0^{-1} = 0$ as an auxiliary function is the most reasonable solution, assuming that a proper constructor as an auxiliary function is acceptable.

We see a theory of meadows having two roles: (i) a starting-point of a theory of mathematical data types; (ii) an intermediate between algebra and logic.

On investigation of mathematical data types, known countable mathematical structures will be equipped with operations to obtain minimal algebras and specification properties of these minimal algebras will be investigated. If countable minimal algebras can be classified as either computable, semi-computable or co-semi-computable, known specification techniques may be applied (see [15] for a survey of this matter). Otherwise data type specification in its original forms cannot be applied. Further, one may study ω -completeness of specifications and term rewriting system related properties.

It is not a common viewpoint in algebra or in mathematics at large that giving a name to an operation, which is included in a signature, is a very significant step by itself. However, the answer to the notorious question “what is $1/0$ ” is very sensitive to exactly this matter. Von Neumann regular rings provide a classical mathematical perspective on rings and fields, where multiplicative inverse (or division) is only used when its use is clearly justified and puzzling uses are rejected as a matter of principle. Meadows provide a more logical perspective to von Neumann regular rings in which justified and unjustified use of multiplicative inverse cannot be easily distinguished beforehand.

3. Viewpoints on the status of $1/0$

In this section, we shortly discuss two prevailing viewpoints on the status of $1/0$ in mathematics and one prevailing viewpoint on the status of $1/0$ in theoretical computer science. To our knowledge, the viewpoints in question are the main prevailing viewpoints. We take the case of the rational numbers, the case of the real numbers being essentially the same.

One prevailing viewpoint in mathematics is that $1/0$ has no meaning because 1 cannot be divided by 0. The argumentation for this viewpoint rests on the fact that there is no rational number z such that $0 \cdot z = 1$. Moreover, in mathematics, syntax is not prior to semantics and posing the question “what is $1/0$ ” is not justified by the mere existence of $1/0$ as a syntactic object. Given the fact that there is no rational number that mathematicians intend to denote by $1/0$, this means that there is no need to assign a meaning to $1/0$.

Another prevailing viewpoint in mathematics is that the use of $1/0$ is simply disallowed because the intention to divide 1 by 0 is non-existent in mathematical practice. This viewpoint can be regarded as a liberal form of the previous one: the rejection of the possibility that $1/0$ has a meaning is circumvented by disallowing the use of $1/0$. Admitting that $1/0$ has a meaning, such as 0 or “undefined”, is consistent with this viewpoint.

The prevailing viewpoint in theoretical computer science is that the meaning of $1/0$ is “undefined” because division is a partial function. Division is identified as a partial function because there is no rational number z such that $0 \cdot z = 1$. This viewpoint presupposes that the use of $1/0$ should be allowed, for otherwise assigning a meaning to $1/0$ does not make sense. Although this viewpoint is more liberal than the previous one, it is remote from ordinary mathematical practice.

The first of the two prevailing viewpoints in mathematics discussed above only leaves room for very informal concepts of expression, calculation, proof, substitution, etc. For that reason, we refrain from considering that viewpoint any further in the current paper. The prevailing viewpoint in mathematics considered further in this paper corresponds to the inversive and divisive meadows of rational numbers together with an imperative about the use of the multiplicative inverse operator and division operator, respectively. The prevailing viewpoint in theoretical computer science corresponds to two of the partial meadows of rational numbers obtained from the inversive and divisive meadows of rational numbers by constructions proposed in the current paper.

4. Inversive meadows and divisive meadows

In this section, we give finite equational specifications of the class of all inversive meadows and the class of all divisive meadows. In [16], inversive meadows were introduced for the first time. They are further investigated in e.g. [5,12,17,18].

It appears that, in the sphere of groups, rings and fields, the qualifications inversive and divisive have only been used by Yamada [40] and Verloren van Themaat [38], respectively. Our use of these qualifications is in line with theirs.

An inversive meadow is a commutative ring with a multiplicative identity element and a total multiplicative inverse operation satisfying two equations which imply that the multiplicative inverse of zero is zero. A divisive meadow is a commutative ring with a multiplicative identity element and a total division operation satisfying three equations which imply that division by zero always yields zero. Hence, the signature of both inversive and divisive meadows include the signature of a commutative ring with a multiplicative identity element.

The signature of commutative rings with a multiplicative identity element consists of the following constants and operators:

- the *additive identity* constant 0;
- the *multiplicative identity* constant 1;
- the binary *addition* operator +;
- the binary *multiplication* operator ·;
- the unary *additive inverse* operator −.

The signature of inversive meadows consists of the constants and operators from the signature of commutative rings with a multiplicative identity element and in addition:

- the unary *multiplicative inverse* operator $^{-1}$.

The signature of divisive meadows consists of the constants and operators from the signature of commutative rings with a multiplicative identity element and in addition:

- the binary *division* operator /.

We write:

$$\begin{aligned}\Sigma_{\text{CR}} & \text{ for } \{0, 1, +, \cdot, -\}, \\ \Sigma_{\text{Md}}^i & \text{ for } \Sigma_{\text{CR}} \cup \{-^1\}, \\ \Sigma_{\text{Md}}^d & \text{ for } \Sigma_{\text{CR}} \cup \{/ \}.\end{aligned}$$

We assume that there are infinitely many variables, including x , y and z . Terms are build as usual. We use infix notation for the binary operators, prefix notation for the unary operator $-$, and postfix notation for the unary operator $^{-1}$. We use the usual precedence convention to reduce the need for parentheses. We introduce subtraction as an abbreviation: $p - q$ abbreviates $p + (-q)$. We denote the numerals $0, 1, 1 + 1, (1 + 1) + 1, \dots$ by $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \dots$ and we use the notation p^n for exponentiation with a natural number as exponent. Formally, we define \underline{n} inductively by $\underline{0} = 0$, $\underline{1} = 1$ and $\underline{n+2} = \underline{n} + 1$ and we define, for each term p over the signature of inversive meadows or the signature of divisive meadows, p^n inductively by $p^0 = 1$ and $p^{n+1} = p^n \cdot p$.

The constants and operators from the signatures of inversive meadows and divisive meadows are adopted from rational arithmetic, which gives an appropriate intuition about these constants and operators. The set of all terms over the signature of inversive meadows constitutes the *inversive meadow notation* and the set of all terms over the signature of divisive meadows constitutes the *divisive meadow notation*.

A commutative ring with a multiplicative identity element is an algebra over the signature Σ_{CR} that satisfies the equations given in Table 1. An inversive meadow is an algebra over the signature Σ_{Md}^i that satisfies the equations given in Tables 1 and 2. A divisive meadow is an algebra over the signature Σ_{Md}^d that satisfies the equations given in Tables 1 and 3. We write:

$$\begin{aligned}E_{\text{CR}} & \text{ for the set of all equations in Table 1,} \\ E_{\text{inv}} & \text{ for the set of all equations in Table 2,} \\ E_{\text{div}} & \text{ for the set of all equations in Table 3,} \\ E_{\text{Md}}^i & \text{ for } E_{\text{CR}} \cup E_{\text{inv}}, \\ E_{\text{Md}}^d & \text{ for } E_{\text{CR}} \cup E_{\text{div}}.\end{aligned}$$

Table 1

Axioms of a commutative ring with a multiplicative identity element.

$(x + y) + z = x + (y + z)$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x + y = y + x$	$x \cdot y = y \cdot x$
$x + 0 = x$	$x \cdot 1 = x$
$x + (-x) = 0$	$x \cdot (y + z) = x \cdot y + x \cdot z$

Table 2

Additional axioms for an inversive meadow.

$(x^{-1})^{-1} = x$
$x \cdot (x \cdot x^{-1}) = x$

Table 3

Additional axioms for a divisive meadow.

$1/(1/x) = x$
$(x \cdot x)/x = x$
$x/y = x \cdot (1/y)$

The equation $(x^{-1})^{-1} = x$ is called the reflexivity equation and the equation $x \cdot (x \cdot x^{-1}) = x$ is called the restricted inverse equation. The first two equations in Table 3 are the obvious counterparts of the reflexivity equation and restricted inverse equation in divisive meadows. The equation $0^{-1} = 0$ is derivable from the equations E_{Md}^i . The equation $x/0 = 0$ is derivable from the equations E_{Md}^d . The equation $1/0 = 0$ can be derived without using the equation $x/y = x \cdot (1/y)$, and then the latter equation can be applied to derive the equation $x/0 = 0$.

The advantage of working with a total multiplicative inverse operation or total division operation lies in the fact that conditions like $x \neq 0$ in $x \neq 0 \Rightarrow x \cdot x^{-1} = 1$ or $x \neq 0 \Rightarrow x \cdot (1/x) = 1$ are not needed to guarantee meaning.

In [7], projection semantics is proposed as an approach to define the meaning of programs. Projection semantics explains the meaning of programs in terms of known programs instead of in terms of more or less sophisticated mathematical objects. Here, we transpose this approach to the current setting to demonstrate that it depends on the angle from which they are viewed whether inversive meadows or divisive meadows must be considered more basic.

We can explain the meaning of the terms over the signature of divisive meadows by means of a projection dmn2imn from the divisive meadow notation to the inversive meadow notation. This projection is defined as follows:

$$\begin{aligned}
 \text{dmn2imn}(x) &= x, \\
 \text{dmn2imn}(0) &= 0, \\
 \text{dmn2imn}(1) &= 1, \\
 \text{dmn2imn}(p + q) &= \text{dmn2imn}(p) + \text{dmn2imn}(q), \\
 \text{dmn2imn}(p \cdot q) &= \text{dmn2imn}(p) \cdot \text{dmn2imn}(q), \\
 \text{dmn2imn}(-p) &= -\text{dmn2imn}(p), \\
 \text{dmn2imn}(p/q) &= \text{dmn2imn}(p) \cdot (\text{dmn2imn}(q)^{-1}).
 \end{aligned}$$

The projection dmn2imn supports an interpretation of the theory of divisive meadows in the theory of inversive meadows: for each equation $p = q$ derivable from the equations E_{Md}^d , the equation $\text{dmn2imn}(p) = \text{dmn2imn}(q)$ is derivable from the equations E_{Md}^i .² Therefore the projection dmn2imn determines a mapping from divisive meadows to inversive meadows.

We can also explain the meaning of the terms over the signature of inversive meadows by means of a projection imn2dmn from the inversive meadow notation to the divisive meadow notation. This projection is defined as follows:

$$\begin{aligned}
 \text{imn2dmn}(x) &= x, \\
 \text{imn2dmn}(0) &= 0, \\
 \text{imn2dmn}(1) &= 1, \\
 \text{imn2dmn}(p + q) &= \text{imn2dmn}(p) + \text{imn2dmn}(q), \\
 \text{imn2dmn}(p \cdot q) &= \text{imn2dmn}(p) \cdot \text{imn2dmn}(q),
 \end{aligned}$$

² For the notion of a translation that supports a theory interpretation, see e.g. [39].

Table 4

Axioms of a reduced divisive meadow.

$(x - ((1 - 1) - y)) - ((1 - 1) - z) = x - ((1 - 1) - (y - ((1 - 1) - z)))$
$x - ((1 - 1) - y) = y - ((1 - 1) - x)$
$x - (1 - 1) = x$
$x - x = 1 - 1$
$(x/(1/y))/(1/z) = x/(1/(y/(1/z)))$
$x/(1/y) = y/(1/x)$
$x/1 = x$
$x/(1/(y - ((1 - 1) - z))) = x/(1/y) - ((1 - 1) - (x/(1/z)))$
$(x/(1/x))/x = x$

$$\text{imn2dmn}(-p) = -\text{imn2dmn}(p),$$

$$\text{imn2dmn}(p^{-1}) = 1/\text{imn2dmn}(p).$$

The projection imn2dmn supports an interpretation of the theory of inversive meadows in the theory of divisive meadows: for each equation $p = q$ derivable from the equations E_{Md}^i , the equation $\text{imn2dmn}(p) = \text{imn2dmn}(q)$ is derivable from the equations E_{Md}^d . Therefore the projection imn2dmn determines a mapping from inversive meadows to divisive meadows.

Given the finite equational specification of the class of all inversive meadows, we can easily give a modular specification of the class of all divisive meadows using module algebra [4]. In Appendix A, we give the modular specification in question and show that the equational theory associated with it is the same as the equational theory associated with the equational specification of the class of all divisive meadows.

A *non-trivial inversive meadow* is an inversive meadow that satisfies the *separation axiom* $0 \neq 1$. An *inversive cancellation meadow* is an inversive meadow that satisfies the *cancellation axiom* $x \neq 0 \wedge x \cdot y = x \cdot z \Rightarrow y = z$, or equivalently, the *general inverse law* $x \neq 0 \Rightarrow x \cdot x^{-1} = 1$. An *inversive Komori field* is an inversive meadow that satisfies the separation axiom and the cancellation axiom. A *non-trivial divisive meadow* is a divisive meadow that satisfies the separation axiom. A *divisive cancellation meadow* is a divisive meadow that satisfies the cancellation axiom. A *divisive Komori field* is a divisive meadow that satisfies the separation axiom and the cancellation axiom.

An important property of inversive Komori fields is the following: $0 \cdot (0^{-1}) = 0$, whereas $x \cdot (x^{-1}) = 1$ for $x \neq 0$. An important property of divisive Komori fields is the following: $0/0 = 0$, whereas $x/x = 1$ for $x \neq 0$.

The inversive Komori field that we are most interested in is \mathcal{Q}_0^i , the inversive Komori field of rational numbers. The divisive Komori field that we are most interested in is \mathcal{Q}_0^d , the divisive Komori field of rational numbers. In Section 7, both \mathcal{Q}_0^i and \mathcal{Q}_0^d will be obtained by means of the well-known initial algebra construction. \mathcal{Q}_0^i differs from the field of rational numbers only in that the multiplicative inverse of zero is zero. \mathcal{Q}_0^d differs from \mathcal{Q}_0^i only in that the multiplicative inverse operation is replaced by a division operation such that $x/y = x \cdot y^{-1}$.

A reduced divisive meadow is an algebra over the signature $\{1, -, /\}$ that satisfies the equations given in Table 4. We can explain the meaning of the terms over the signature of inversive meadows by means of a projection imn2rdmn to terms over the signature of reduced divisive meadows. This projection is defined as follows:

$$\text{imn2rdmn}(x) = x,$$

$$\text{imn2rdmn}(0) = 1 - 1,$$

$$\text{imn2rdmn}(1) = 1,$$

$$\text{imn2rdmn}(p + q) = \text{imn2rdmn}(p) - ((1 - 1) - \text{imn2rdmn}(q)),$$

$$\text{imn2rdmn}(p \cdot q) = \text{imn2rdmn}(p) / (1 / \text{imn2rdmn}(q)),$$

$$\text{imn2rdmn}(-p) = (1 - 1) - \text{imn2rdmn}(p),$$

$$\text{imn2rdmn}(p^{-1}) = 1 / \text{imn2rdmn}(p).$$

The projection imn2rdmn supports an interpretation of the theory of inversive meadows in the theory of reduced divisive meadows.

The following are some outstanding questions with regard to inversive meadows, divisive meadows, and reduced divisive meadows:

1. Do there exist equational specifications of the class of all inversive meadows, the class of all divisive meadows, and the class of all reduced divisive meadows with less than 10 equations, 11 equations, and 9 equations, respectively?
2. Can the number of binary operators needed to explain the meaning of the terms over the signature of inversive meadows be reduced from two to one?

5. Arithmetical meadows

In this section, we give finite equational specifications of the class of all inversive arithmetical meadows, the class of all divisive arithmetical meadows, the class of all inversive arithmetical meadows with zero and the class of all divisive arithmetical meadows with zero.

The signatures of inversive and divisive arithmetical meadows with zero are the signatures of inversive and divisive meadows with the additive inverse operator $-$ removed. The signatures of inversive and divisive arithmetical meadows are the signatures of inversive and divisive arithmetical meadows with zero with the additive identity constant 0 removed. We write:

$$\begin{aligned}\Sigma_{\text{AMd}}^{\text{iz}} & \text{ for } \Sigma_{\text{Md}}^{\text{i}} \setminus \{-\}, \\ \Sigma_{\text{AMd}}^{\text{dz}} & \text{ for } \Sigma_{\text{Md}}^{\text{d}} \setminus \{-\}, \\ \Sigma_{\text{AMd}}^{\text{i}} & \text{ for } \Sigma_{\text{AMd}}^{\text{iz}} \setminus \{0\}, \\ \Sigma_{\text{AMd}}^{\text{d}} & \text{ for } \Sigma_{\text{AMd}}^{\text{dz}} \setminus \{0\}.\end{aligned}$$

Moreover, we write:

$$\begin{aligned}E_{\text{CR}_{\text{az}}} & \text{ for } E_{\text{CR}} \setminus \{x + (-x) = 0\}, \\ E_{\text{CR}_a} & \text{ for } E_{\text{CR}_{\text{az}}} \setminus \{x + 0 = x\}.\end{aligned}$$

The equations in $E_{\text{CR}_{\text{az}}}$ are the equations from E_{CR} in which the additive inverse operator $-$ does not occur. The equations in E_{CR_a} are the equations from $E_{\text{CR}_{\text{az}}}$ in which the additive identity constant 0 does not occur.

An *inversive arithmetical meadow* is an algebra over the signature $\Sigma_{\text{AMd}}^{\text{i}}$ that satisfies the equations E_{CR_a} and the equation $x \cdot x^{-1} = 1$. A *divisive arithmetical meadow* is an algebra over the signature $\Sigma_{\text{AMd}}^{\text{d}}$ that satisfies the equations E_{CR_a} and the equation $x/x = 1$. An *inversive arithmetical meadow with zero* is an algebra over the signature $\Sigma_{\text{AMd}}^{\text{iz}}$ that satisfies the equations $E_{\text{CR}_{\text{az}}}$ and the equations E_{inv} . A *divisive arithmetical meadow with zero* is an algebra over the signature $\Sigma_{\text{AMd}}^{\text{dz}}$ that satisfies the equations $E_{\text{CR}_{\text{az}}}$ and the equations E_{div} . We write:

$$\begin{aligned}E_{\text{AMd}}^{\text{i}} & \text{ for } E_{\text{CR}_a} \cup \{x \cdot x^{-1} = 1\}, \\ E_{\text{AMd}}^{\text{d}} & \text{ for } E_{\text{CR}_a} \cup \{x/x = 1\}, \\ E_{\text{AMd}}^{\text{iz}} & \text{ for } E_{\text{CR}_{\text{az}}} \cup E_{\text{inv}}, \\ E_{\text{AMd}}^{\text{dz}} & \text{ for } E_{\text{CR}_{\text{az}}} \cup E_{\text{div}}.\end{aligned}$$

The arithmetical meadows that we are most interested in are the arithmetical meadows of rational numbers and the arithmetical meadow of rational numbers with zero. In Section 8, these arithmetical meadows will be obtained by means of the well-known initial algebra construction. The following lemmas about arithmetical meadows and arithmetical meadows with zero will be used in Section 8.

Lemma 1. For all $n, m \in \mathbb{N} \setminus \{0\}$, we have that $E_{\text{CR}_a} \vdash \underline{n} + \underline{m} = \underline{n} + \underline{m}$ and $E_{\text{CR}_a} \vdash \underline{n} \cdot \underline{m} = \underline{n} \cdot \underline{m}$.

Proof. The fact that $\underline{n} + \underline{m} = \underline{n} + \underline{m}$ is derivable from E_{CR_a} is easily proved by induction on n . The basis step is trivial. The inductive step goes as follows: $(\underline{n} + 1) + \underline{m} = (\underline{n} + \underline{m}) + 1 = \underline{n} + \underline{m} + 1 = \underline{n} + \underline{m} + 1 = \underline{n} + 1 + \underline{m} = \underline{n} + 1 + \underline{m}$. The fact that $\underline{n} \cdot \underline{m} = \underline{n} \cdot \underline{m}$ is derivable from E_{CR_a} is easily proved by induction on n , using that $\underline{n} + \underline{m} = \underline{n} + \underline{m}$ is derivable from E_{CR_a} . The basis step is trivial. The inductive step goes as follows: $(\underline{n} + 1) \cdot \underline{m} = \underline{n} \cdot \underline{m} + 1 \cdot \underline{m} = \underline{n} \cdot \underline{m} + 1 \cdot \underline{m} = \underline{n} \cdot \underline{m} + 1 \cdot \underline{m} = (\underline{n} + 1) \cdot \underline{m} = \underline{n} + 1 \cdot \underline{m}$. \square

Lemma 2. We have $E_{\text{AMd}}^{\text{i}} \vdash (x^{-1})^{-1} = x$ and $E_{\text{AMd}}^{\text{i}} \vdash (x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.

Proof. We derive $(x^{-1})^{-1} = x$ from $E_{\text{AMd}}^{\text{i}}$ as follows: $(x^{-1})^{-1} = 1 \cdot (x^{-1})^{-1} = (x \cdot x^{-1}) \cdot (x^{-1})^{-1} = x \cdot (x^{-1} \cdot (x^{-1})^{-1}) = x \cdot 1 = x$. We derive $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ from $E_{\text{AMd}}^{\text{i}}$ as follows: $(x \cdot y)^{-1} = 1 \cdot (x \cdot y)^{-1} = (x \cdot x^{-1}) \cdot ((y \cdot y^{-1}) \cdot (x \cdot y)^{-1}) = (x^{-1} \cdot y^{-1}) \cdot ((x \cdot y) \cdot (x \cdot y)^{-1}) = (x^{-1} \cdot y^{-1}) \cdot 1 = x^{-1} \cdot y^{-1}$. \square

Lemma 3. We have $E_{\text{AMd}}^{\text{iz}} \vdash 0 \cdot x = 0$ and $E_{\text{AMd}}^{\text{iz}} \vdash 0^{-1} = 0$.

Proof. Firstly, we derive $x + y = x \Rightarrow y = 0$ from $E_{\text{AMd}}^{\text{iz}}$ as follows: $x + y = x \Rightarrow 0 + y = 0 \Rightarrow y + 0 = 0 \Rightarrow y = 0$. Secondly, we derive $x + 0 \cdot x = x$ from $E_{\text{AMd}}^{\text{iz}}$ as follows: $x + 0 \cdot x = x \cdot 1 + 0 \cdot x = 1 \cdot x + 0 \cdot x = (1 + 0) \cdot x = 1 \cdot x = x \cdot 1 = x$. From

$x + y = x \Rightarrow y = 0$ and $x + 0 \cdot x = x$, it follows that $0 \cdot x = 0$. We derive $0^{-1} = 0$ from $E_{\text{AMd}}^{\text{iz}}$ as follows: $0^{-1} = 0^{-1} \cdot (0^{-1} \cdot (0^{-1})^{-1}) = (0^{-1})^{-1} \cdot (0^{-1} \cdot 0^{-1}) = 0 \cdot (0^{-1} \cdot 0^{-1}) = 0$. \square

Lemma 4. We have $E_{\text{AMd}}^{\text{iz}} \vdash (x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.

Proof. Proposition 2.8 from [5] states that $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ is derivable from $E_{\text{AMd}}^{\text{iz}} \cup \{x + 0 = x, x + (-x) = 0\}$. The proof of this proposition given in [5] goes through because no use is made of the equations $x + 0 = x$ and $x + (-x) = 0$. \square

Lemma 5. For each $\Sigma_{\text{AMd}}^{\text{iz}}$ -term t , either $E_{\text{AMd}}^{\text{iz}} \vdash t = 0$ or there exists a $\Sigma_{\text{AMd}}^{\text{i}}$ -term t' such that $E_{\text{AMd}}^{\text{iz}} \vdash t = t'$.

Proof. The proof is easy by induction on the structure of t , using Lemma 3. \square

6. Arithmetical meadows and regular arithmetical rings

We can define commutative arithmetical rings with a multiplicative identity element in the same vein as arithmetical meadows. Moreover, we can define commutative von Neumann regular arithmetical rings with a multiplicative identity element as commutative arithmetical rings with a multiplicative identity element satisfying the regularity condition $\forall x \cdot \exists y \cdot x \cdot (x \cdot y) = x$.

The following theorem states that commutative von Neumann regular arithmetical rings with a multiplicative identity element are related to inversive arithmetical meadows like commutative von Neumann regular rings with a multiplicative identity element are related to inversive meadows.

Theorem 6. Each commutative von Neumann regular arithmetical ring with a multiplicative identity element can be expanded to an inversive arithmetical meadow, and this expansion is unique.

Proof. Lemma 2.11 from [5] states that each commutative von Neumann regular ring with a multiplicative identity element can be expanded to an inversive meadow, and this expansion is unique. The only use that is made of the equations $x + 0 = x$ and $x + (-x) = 0$ in the proof of this lemma given in [5] originates from the proof of another lemma that is used in the proof. However, the latter lemma, Lemma 2.12 from [5], concerns the same property as Proposition 2.3 from [18] and in the proof of this proposition given in [18] no use is made of the equations $x + 0 = x$ and $x + (-x) = 0$. Hence, there is an alternative proof of Lemma 2.11 from [5] that goes through for the arithmetical case. \square

We can also define commutative arithmetical rings with additive and multiplicative identities and commutative von Neumann regular arithmetical rings with additive and multiplicative identities in the obvious way. We also have that commutative von Neumann regular arithmetical rings with additive and multiplicative identities are related to inversive arithmetical meadows with zero like commutative von Neumann regular rings with a multiplicative identity element are related to inversive meadows.

7. Meadows of rational numbers

In this section, we obtain inversive and divisive meadows of rational numbers as initial algebras of finite equational specifications. Moreover, we prove that the inversive meadow in question differs from the field of rational numbers only in that the multiplicative inverse of zero is zero. As usual, we write $I(\Sigma, E)$ for the initial algebra among the algebras over the signature Σ that satisfy the equations E (see e.g. [14]).

The inversive meadow that we are interested in is \mathcal{Q}_0^{i} , the inversive meadow of rational numbers:

$$\mathcal{Q}_0^{\text{i}} = I(\Sigma_{\text{Md}}^{\text{i}}, E_{\text{Md}}^{\text{i}} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\}).$$

The divisive meadow that we are interested in is \mathcal{Q}_0^{d} , the divisive meadow of rational numbers:

$$\mathcal{Q}_0^{\text{d}} = I(\Sigma_{\text{Md}}^{\text{d}}, E_{\text{Md}}^{\text{d}} \cup \{(1 + x^2 + y^2)/(1 + x^2 + y^2) = 1\}).$$

\mathcal{Q}_0^{d} differs from \mathcal{Q}_0^{i} only in that the multiplicative inverse operation is replaced by a division operation in conformity with the projection imn2dmn defined in Section 4.

To prove that \mathcal{Q}_0^{i} differs from the field of rational numbers only in that the multiplicative inverse of zero is zero, we need some auxiliary results.

Lemma 7. Let p be a prime number. Then for each $u \in \mathbb{Z}_p$, there exists $v, w \in \mathbb{Z}_p$ such that $u = v^2 + w^2$.

Proof. The case where $p = 2$ is trivial. In the case where $p \neq 2$, p is odd, say $2 \cdot n + 1$. Let S be the set $\{u \in \mathbb{Z}_p \mid \exists v \in \mathbb{Z}_p \cdot u = v^2\}$, and let $c \in \mathbb{Z}_p$ be such that $c \notin S$. Because $0 \in \mathbb{Z}_p$ and each element of S has at most two roots, we have $|S| \geq n + 1$. For each $u \in c \cdot S$, $u = 0$ or $u \notin S$, as $u \neq 0$ and $u \in S$ only if $c \in S$. Because $c \cdot u \neq c \cdot v$ for each $u, v \in S$ with $u \neq v$, we have $|c \cdot S| \geq n + 1$. It follows that $S \cup c \cdot S = \mathbb{Z}_p$ and $S \cap c \cdot S = \{0\}$. This implies that $c \cdot S = \{u \in \mathbb{Z}_p \mid \forall v \in \mathbb{Z}_p \cdot u \neq v^2\} \cup \{0\}$. Hence, for each $u \in \mathbb{Z}_p$ with $u \notin S$, there exists an $v \in \mathbb{Z}_p$ such that $u = c \cdot v^2$. The set S is not closed under sums, as $1 \in S$, and every element of \mathbb{Z}_p is a sum of ones. This implies that there exist $u, v \in \mathbb{Z}_p$ such that $u^2 + v^2 \notin S$. Let $a, b \in \mathbb{Z}_p$ be such that $a^2 + b^2 \notin S$, and take $a^2 + b^2$ for c . Then for each $u \in \mathbb{Z}_p$ with $u \notin S$, there exists an $v \in \mathbb{Z}$ such that $u = (a^2 + b^2) \cdot v^2$. Because $(a^2 + b^2) \cdot v^2 = (a \cdot v)^2 + (b \cdot v)^2$, we have that, for each $u \in \mathbb{Z}_p$ with $u \notin S$, there exist $v, w \in \mathbb{Z}$ such that $u = v^2 + w^2$. Because $u \in S$ iff $u = v^2 + 0^2$ for some $v \in \mathbb{Z}_p$, we have that, for each $u \in \mathbb{Z}_p$ with $u \in S$, there exist $v, w \in \mathbb{Z}$ such that $u = v^2 + w^2$. \square

Corollary 8. Let p be a prime number. Then there exists $u, v, w \in \mathbb{N}$ such that $w \cdot p = u^2 + v^2 + 1$.

Proof. By Lemma 7, there exist $u, v \in \mathbb{Z}_p$ such that $-1 = u^2 + v^2$. Let $a, b \in \mathbb{Z}_p$ be such that $-1 = a^2 + b^2$. Then $a^2 + b^2 + 1$ is a multiple of p in \mathbb{N} . Hence, there exists $u, v, w \in \mathbb{N}$ such that $w \cdot p = u^2 + v^2 + 1$. \square

Theorem 9. \mathcal{Q}_0^i is the zero-totalized field of rational numbers, i.e. the Σ_{Md}^i -algebra that differs from the field of rational numbers only in that $0^{-1} = 0$.

Proof. From the proof of Theorem 3.6 from [16], we already know that, for each set E' of Σ_{Md}^i -equations valid in the zero-totalized field of rational numbers, $I(\Sigma_{\text{Md}}^i, E_{\text{Md}}^i \cup E')$ is the zero-totalized field of rational numbers if it follows from $E_{\text{Md}}^i \cup E'$ that \underline{u} has a multiplicative inverse for each $u \in \mathbb{N} \setminus \{0\}$. Because $1 + x^2 + y^2 \neq 0$, we have that $(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1$ is valid in the zero-totalized field of rational numbers. So it remains to be proved that \underline{u} has a multiplicative inverse for each $u \in \mathbb{N} \setminus \{0\}$.

Let p be a prime number. Then, by Corollary 8, there exist $u, v, w \in \mathbb{N}$ such that $w \cdot p = u^2 + v^2 + 1$. Let $m, a, b \in \mathbb{N}$ be such that $m \cdot p = a^2 + b^2 + 1$. As a corollary of Lemma 1, we have $\underline{u + v} = \underline{u} + \underline{v}$ and $\underline{u \cdot v} = \underline{u} \cdot \underline{v}$ for all $u, v \in \mathbb{N}$. It follows that $\underline{m \cdot p} = \underline{a^2 + b^2 + 1}$. Because $(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1$, we have $(\underline{m \cdot p}) \cdot (\underline{m \cdot p})^{-1} = 1$. This implies that $\underline{m} \cdot (\underline{m \cdot p})^{-1}$ is the multiplicative inverse of \underline{p} . Hence, \underline{u} has a multiplicative inverse for each $u \in \mathbb{N} \setminus \{0\}$ that is a prime number. Let $c \in \mathbb{N} \setminus \{0\}$. Then c is the product of finitely many prime numbers, say $p_1 \cdot \dots \cdot p_n$. Because $(\underline{p_1} \cdot \dots \cdot \underline{p_n})^{-1} = \underline{p_1}^{-1} \cdot \dots \cdot \underline{p_n}^{-1}$ (see e.g. Proposition 2.8 in [5]) and $\underline{c} = \underline{p_1} \cdot \dots \cdot \underline{p_n}$, we have that $\underline{p_1}^{-1} \cdot \dots \cdot \underline{p_n}^{-1}$ is the multiplicative inverse of \underline{c} . Hence, \underline{u} has a multiplicative inverse for each $u \in \mathbb{N} \setminus \{0\}$. \square

Lemma 7, Corollary 8, and Theorem 9 come from Hirshfeld (personal communication, 31 January 2009). Lemma 7 is a folk theorem in the area of field theory, but we could not find a proof of it in the literature.

We remark that in [16], the initial algebra specification of \mathcal{Q}_0^i is obtained by adding the equation $(1 + x^2 + y^2 + z^2 + w^2) \cdot (1 + x^2 + y^2 + z^2 + w^2)^{-1} = 1$ instead of the equation $(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1$ to E_{Md}^i . In other words, in the current paper, we have reduced the number of squares needed in the equation added to E_{Md}^i from 4 to 2. In [6], it is shown that the number of squares cannot be reduced to 1.

8. Arithmetical meadows of rational numbers

In this section, we obtain inverse and divisive arithmetical meadows of rational numbers and inverse and divisive arithmetical meadows of rational numbers with zero as initial algebras of finite equational specifications. Moreover, we prove that the inversive meadows in question are subalgebras of reducts of the inversive meadow of rational numbers and some results concerning the decidability of derivability from the equational specifications concerned.

\mathcal{Q}^{ia} , the inversive arithmetical meadow of rational numbers, is defined as follows:

$$\mathcal{Q}^{\text{ia}} = I(\Sigma_{\text{AMd}}^i, E_{\text{AMd}}^i).$$

\mathcal{Q}^{da} , the divisive arithmetical meadow of rational numbers, is defined as follows:

$$\mathcal{Q}^{\text{da}} = I(\Sigma_{\text{AMd}}^d, E_{\text{AMd}}^d).$$

Notice that \mathcal{Q}^{ia} and \mathcal{Q}^{da} are the initial algebras in the class of inversive arithmetical meadows and the class of divisive arithmetical meadows, respectively.

\mathcal{Q}^{ia} is a subalgebra of a reduct of \mathcal{Q}_0^i .

Theorem 10. \mathcal{Q}^{ia} is the subalgebra of the Σ_{AMd}^i -reduct of \mathcal{Q}_0^i whose domain is the set of all positive rational numbers.

Proof. Like in the case of Theorem 3.1 from [16], it is sufficient to prove that, for each closed term t over the signature Σ_{AMd}^i , there exists a unique term t' in the set

$$\{\underline{n} \cdot \underline{m}^{-1} \mid n, m \in \mathbb{N} \setminus \{0\} \wedge \gcd(n, m) = 1\}$$

such that $E_{\text{AMd}}^i \vdash t = t'$. Like in the case of Theorem 3.1 from [16], this is proved by induction on the structure of t , using Lemmas 1 and 2. The proof is similar, but simpler owing to: (i) the absence of terms of the forms 0 and $-t'$; (ii) the absence of terms of the forms 0 and $-(\underline{n} \cdot \underline{m}^{-1})$ among the terms that exist by the induction hypothesis; (iii) the presence of the axiom $x \cdot x^{-1} = 1$. \square

The fact that \mathcal{Q}^{da} is a subalgebra of a reduct of \mathcal{Q}_0^{d} is proved similarly.

Derivability of equations from the equations of the initial algebra specification of \mathcal{Q}^{ia} is decidable.

Theorem 11. For all Σ_{AMd}^i -terms t and t' , it is decidable whether $E_{\text{AMd}}^i \vdash t = t'$.

Proof. For each Σ_{AMd}^i -term r , there exist Σ_{AMd}^i -terms r_1 and r_2 in which the multiplicative inverse operator does not occur such that $E_{\text{AMd}}^i \vdash r = r_1 \cdot r_2^{-1}$. The proof of this fact is easy by induction on the structure of r , using Lemma 2. Inspection of the proof yields that there is an effective way to find witnessing terms.

For each closed Σ_{AMd}^i -term r in which the multiplicative inverse operator does not occur there exists a $k \in \mathbb{N} \setminus \{0\}$, such that $E_{\text{AMd}}^i \vdash r = \underline{k}$. The proof of this fact is easy by induction on the structure of r . Moreover, for each Σ_{AMd}^i -term r in which the multiplicative inverse operator does not occur there exists a Σ_{AMd}^i -term r' of the form $\sum_{i_1=1}^{n_1} \dots \sum_{i_m=1}^{n_m} k_{i_1 \dots i_m} \cdot x_1^{i_1} \cdot \dots \cdot x_m^{i_m}$, where $k_{i_1 \dots i_m} \in \mathbb{N} \setminus \{0\}$ for each $i_1 \in [1, n_1], \dots, i_m \in [1, n_m]$ and x_1, \dots, x_m are variables, such that $E_{\text{AMd}}^i \vdash r = r'$. The proof of this fact is easy by induction on the structure of r , using the previous fact. Inspection of the proof yields that there is an effective way to find a witnessing term. Terms of the form described above are polynomials in several variables with positive integer coefficients.

Let t_1, t_2, t'_1, t'_2 be Σ_{AMd}^i -terms in which the multiplicative inverse operator does not occur such that $E_{\text{AMd}}^i \vdash t = t_1 \cdot t_2^{-1}$ and $E_{\text{AMd}}^i \vdash t' = t'_1 \cdot t'_2^{-1}$. Moreover, let s and s' be Σ_{AMd}^i -terms of the form $\sum_{i_1=1}^{n_1} \dots \sum_{i_m=1}^{n_m} k_{i_1 \dots i_m} \cdot x_1^{i_1} \cdot \dots \cdot x_m^{i_m}$, where $k_{i_1 \dots i_m} \in \mathbb{N} \setminus \{0\}$ for each $i_1 \in [1, n_1], \dots, i_m \in [1, n_m]$ and x_1, \dots, x_m are variables, such that $E_{\text{AMd}}^i \vdash t_1 \cdot t'_2 = s$ and $E_{\text{AMd}}^i \vdash t'_1 \cdot t_2 = s'$. We have that $E_{\text{AMd}}^i \vdash t = t'$ iff $E_{\text{AMd}}^i \vdash t_1 \cdot t_2^{-1} = t'_1 \cdot t'_2^{-1}$ iff $E_{\text{AMd}}^i \vdash t_1 \cdot t'_2 = t'_1 \cdot t_2$ iff $E_{\text{AMd}}^i \vdash s = s'$. Moreover, we have that $E_{\text{AMd}}^i \vdash s = s'$ only if s and s' denote the same function on positive real numbers in the inversive arithmetical meadow of positive real numbers. The latter is decidable because polynomials in several variables with positive integer coefficients denote the same function on positive real numbers in the inversive arithmetical meadow of positive real numbers only if they are syntactically equal. \square

The fact that derivability of equations from the equations of the initial algebra specification of \mathcal{Q}^{da} is decidable is proved similarly.

$\mathcal{Q}_0^{\text{iaz}}$, the inversive arithmetical meadow of rational numbers with zero, is defined as follows:

$$\mathcal{Q}_0^{\text{iaz}} = I(\Sigma_{\text{AMd}}^{\text{iaz}}, E_{\text{AMd}}^{\text{iaz}} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\}).$$

$\mathcal{Q}_0^{\text{daz}}$, the divisive arithmetical meadow of rational numbers with zero, is defined as follows:

$$\mathcal{Q}_0^{\text{daz}} = I(\Sigma_{\text{AMd}}^{\text{daz}}, E_{\text{AMd}}^{\text{daz}} \cup \{(1 + x^2 + y^2)/(1 + x^2 + y^2) = 1\}).$$

$\mathcal{Q}_0^{\text{iaz}}$ is a subalgebra of a reduct of \mathcal{Q}_0^{i} . First we prove a fact that is useful in the proving this result.

Lemma 12. It follows from $E_{\text{AMd}}^{\text{iaz}} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\}$ that \underline{n} has a multiplicative inverse for each $n \in \mathbb{N} \setminus \{0\}$.

Proof. In the proof of Theorem 9, it is among other things proved that it follows from $E_{\text{AMd}}^{\text{iaz}} \cup \{x + (-x) = 0\} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\}$ that \underline{n} has a multiplicative inverse for each $n \in \mathbb{N} \setminus \{0\}$. The proof concerned goes through because no use is made of the equation $x + (-x) = 0$. \square

Theorem 13. $\mathcal{Q}_0^{\text{iaz}}$ is the subalgebra of the $\Sigma_{\text{AMd}}^{\text{iaz}}$ -reduct of \mathcal{Q}_0^{i} whose domain is the set of all non-negative rational numbers.

Proof. Like in the case of Theorem 10, it is sufficient to prove that, for each closed term t over the signature Σ_{AMd}^i , there exists a unique term t' in the set

$$\{0\} \cup \{\underline{n} \cdot \underline{m}^{-1} \mid n, m \in \mathbb{N} \setminus \{0\} \wedge \gcd(n, m) = 1\}$$

such that $E_{\text{AMd}}^{\text{iz}} \cup \{(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1\} \vdash t = t'$. Like in the case of [Theorem 10](#), this is proved by induction on the structure of t , now using [Lemmas 1, 3, and 4](#). The proof is similar, but more complicated owing to: (i) the presence of terms of the form 0 ; (ii) the presence of terms of the form Q among the terms that exist by the induction hypothesis; (iii) the absence of the axiom $x \cdot x^{-1} = 1$. Because of the last point, use is made of [Lemma 12](#). \square

The fact that Q_0^{daz} is a subalgebra of a reduct of Q_0^{d} is proved similarly.

An alternative initial algebra specification of Q_0^{iaz} is obtained if the equation $(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1$ is replaced by $(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}$.

Theorem 14. $Q_0^{\text{iaz}} \cong I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\})$.

Proof. It is sufficient to prove that $(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}$ is valid in Q_0^{iaz} and $(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1$ is valid in $I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\})$. It follows from [Lemma 4](#), and the associativity and commutativity of \cdot , that $(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1} \Leftrightarrow (x \cdot x^{-1}) \cdot ((x+y) \cdot (x+y)^{-1}) = x \cdot x^{-1}$ is derivable from $E_{\text{AMd}}^{\text{iz}}$. This implies that $(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}$ is valid in Q_0^{iaz} iff $(x \cdot x^{-1}) \cdot ((x+y) \cdot (x+y)^{-1}) = x \cdot x^{-1}$ is valid in Q_0^{iaz} . The latter is easily established by distinction between the cases $x=0$ and $x \neq 0$. To show that $(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1$ is valid in $I(\Sigma_{\text{AMd}}^{\text{iz}}, E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\})$, it is sufficient to derive $(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1$ from $E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot x^{-1}) \cdot ((x+y) \cdot (x+y)^{-1}) = x \cdot x^{-1}\}$. The derivation is fully trivial with the exception of the first step, viz. substituting 1 for x and x^2+y^2 for y in $(x \cdot x^{-1}) \cdot ((x+y) \cdot (x+y)^{-1}) = x \cdot x^{-1}$. \square

An alternative initial algebra specification of Q_0^{daz} is obtained in the same vein.

In Q_0^{iaz} , the *general inverse law* $x \neq 0 \Rightarrow x \cdot x^{-1} = 1$ is valid. Derivability of equations from the equations of the alternative initial algebra specification of Q_0^{iaz} and the general inverse law is decidable. First we prove a fact that is useful in proving this decidability result.

Lemma 15. For all $\Sigma_{\text{AMd}}^{\text{iz}}$ -terms t in which no other variables than x_1, \dots, x_n occur, $E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\} \cup \{x_1 \cdot x_1^{-1} = 1, \dots, x_n \cdot x_n^{-1} = 1\} \vdash_{x_1, \dots, x_n} t \cdot t^{-1} = 1$.

Proof. The proof is easy by induction on the structure of t , using [Lemma 4](#). \square

Theorem 16. For all $\Sigma_{\text{AMd}}^{\text{iz}}$ -terms t and t' , it is decidable whether $E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\} \cup \{x \neq 0 \Rightarrow x \cdot x^{-1} = 1\} \vdash t = t'$.

Proof. Let $E_{\text{AMd}}^{\text{iz}+} = E_{\text{AMd}}^{\text{iz}} \cup \{(x \cdot (x+y)) \cdot (x \cdot (x+y))^{-1} = x \cdot x^{-1}\} \cup \{x \neq 0 \Rightarrow x \cdot x^{-1} = 1\}$. We prove that $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ is decidable by induction on the number of variables occurring in $t = t'$. In the case where the number of variables is 0 , we have that $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ iff $Q_0^{\text{iaz}} \models t = t'$ iff $E_{\text{AMd}}^{\text{iz}} \cup \{(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1\} \vdash t = t'$. The last is decidable because, by the proof of [Theorem 13](#), there exist unique terms s and s' in the set $\{Q\} \cup \{\underline{n} \cdot \underline{m}^{-1} \mid n, m \in \mathbb{N} \setminus \{0\} \wedge \gcd(n, m) = 1\}$ such that $E_{\text{AMd}}^{\text{iz}} \cup \{(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1\} \vdash t = s$ and $E_{\text{AMd}}^{\text{iz}} \cup \{(1+x^2+y^2) \cdot (1+x^2+y^2)^{-1} = 1\} \vdash t' = s'$, and inspection of that proof yields that there is an effective way to find s and s' . Hence, in the case where the number of variables is 0 , $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ is decidable. In the case where the number of variables is $n+1$, suppose that the variables are x_1, \dots, x_{n+1} . Let s be such that $E_{\text{AMd}}^{\text{iz}} \vdash t = s$ and s is either a $\Sigma_{\text{AMd}}^{\text{iz}}$ -term or the constant 0 and let s' be such that $E_{\text{AMd}}^{\text{iz}} \vdash t' = s'$ and s' is either a $\Sigma_{\text{AMd}}^{\text{iz}}$ -term or the constant 0 . Such s and s' exist by [Lemma 5](#), and inspection of the proof of that lemma yields that there is an effective way to find s and s' . We have that $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ iff $E_{\text{AMd}}^{\text{iz}+} \vdash s = s'$. In the case where not both s and s' are $\Sigma_{\text{AMd}}^{\text{iz}}$ -terms, $E_{\text{AMd}}^{\text{iz}+} \vdash s = s'$ only if s and s' are syntactically equal. Hence, in this case, $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ is decidable. In the case where both s and s' are $\Sigma_{\text{AMd}}^{\text{iz}}$ -terms, by the general inverse law, we have that $E_{\text{AMd}}^{\text{iz}+} \vdash s = s'$ iff $E_{\text{AMd}}^{\text{iz}+} \vdash s[0/x_i] = s'[0/x_i]$ for all $i \in [1, n+1]$ and $E_{\text{AMd}}^{\text{iz}+} \cup \{x_1 \cdot x_1^{-1} = 1, \dots, x_{n+1} \cdot x_{n+1}^{-1} = 1\} \vdash_{x_1, \dots, x_{n+1}} s = s'$. By [Lemma 15](#), we have that $E_{\text{AMd}}^{\text{iz}+} \cup \{x_1 \cdot x_1^{-1} = 1, \dots, x_{n+1} \cdot x_{n+1}^{-1} = 1\} \vdash_{x_1, \dots, x_{n+1}} s = s'$ iff $E_{\text{AMd}}^{\text{iz}} \vdash s = s'$. For each $i \in [1, n+1]$, $E_{\text{AMd}}^{\text{iz}+} \vdash s[0/x_i] = s'[0/x_i]$ is decidable because the number of variables occurring in $s[0/x_i] = s'[0/x_i]$ is n . Moreover, we know from [Theorem 11](#) that $E_{\text{AMd}}^{\text{iz}} \vdash s = s'$ is decidable. Hence, in the case where both s and s' are $\Sigma_{\text{AMd}}^{\text{iz}}$ -terms, $E_{\text{AMd}}^{\text{iz}+} \vdash t = t'$ is decidable as well. \square

The fact that derivability of equations from the equations of the alternative initial algebra specification of Q_0^{daz} and $x \neq 0 \Rightarrow x/x = 1$ is decidable is proved similarly. It is an open problem whether derivability of equations from the equations of the alternative initial algebra specifications of Q_0^{iaz} and Q_0^{daz} is decidable.

The following are some outstanding questions with regard to arithmetical meadows:

1. Is the initial algebra specification of \mathcal{Q}_0^i a conservative extension of the initial algebra specifications of \mathcal{Q}^{ia} and \mathcal{Q}_0^{iaz} ?
2. Do \mathcal{Q}^{ia} and \mathcal{Q}_0^{iaz} have initial algebra specifications that constitute complete term rewriting systems (modulo associativity and commutativity of $+$ and \cdot)?
3. Do \mathcal{Q}^{ia} and \mathcal{Q}_0^{iaz} have ω -complete initial algebra specifications?
4. What are the complexities of derivability of equations from E_{AMd}^i and $E_{AMd}^{iz} \cup \{(x \cdot (x + y)) \cdot (x \cdot (x + y))^{-1} = x \cdot x^{-1}, x \neq 0 \Rightarrow x \cdot x^{-1} = 1\}$?
5. Is derivability of equations from $E_{AMd}^{iz} \cup \{(x \cdot (x + y)) \cdot (x \cdot (x + y))^{-1} = x \cdot x^{-1}\} \vdash t = t'$ decidable?
6. Do we have $\mathcal{Q}_0^{iaz} \cong I(\Sigma_{AMd}^{iz}, E_{AMd}^{iz} \cup \{(1 + x^2) \cdot (1 + x^2)^{-1} = 1\})$?

These questions are formulated for the inversive case, but they have counterparts for the divisive case of which some might lead to different answers.

9. Partial meadows

In this section, we introduce simple constructions of partial inversive and divisive meadows from total ones. Divisive meadows are more basic than inversive meadows if the partial ones are considered as well.

We take the position that partial algebras should be made from total ones. For the particular case of meadows, this implies that relevant partial meadows are obtained by making operations undefined for certain arguments.

Let \mathcal{M}_i be an inversive meadow. Then it makes sense to construct one partial inversive meadow from \mathcal{M}_i :

- $0^{-1} \uparrow \mathcal{M}_i$: the partial algebra that is obtained from \mathcal{M}_i by making 0^{-1} undefined.

Let \mathcal{M}_d be a divisive meadow. Then it makes sense to construct two partial divisive meadows from \mathcal{M}_d :

- $Q/0 \uparrow \mathcal{M}_d$: the partial algebra that is obtained from \mathcal{M}_d by making $q/0$ undefined for all q in the domain of \mathcal{M}_d ;
- $(Q \setminus \{0\})/0 \uparrow \mathcal{M}_d$: the partial algebra that is obtained from \mathcal{M}_d by making $q/0$ undefined for all q in the domain of \mathcal{M}_d different from 0.

Clearly, the partial meadow constructions are special cases of a more general partial algebra construction for which we have coined the term *punching*. Presenting the details of the general construction is outside the scope of the current paper.

Let \mathcal{M}_i be an inversive meadow and let \mathcal{M}_d be a divisive meadow. It happens that the projection imn2dmn recovers $0^{-1} \uparrow \mathcal{M}_i$ from $Q/0 \uparrow \mathcal{M}_d$ as well as $(Q \setminus \{0\})/0 \uparrow \mathcal{M}_d$, the projection dmn2imn recovers $Q/0 \uparrow \mathcal{M}_d$ from $0^{-1} \uparrow \mathcal{M}_i$, and the projection dmn2imn does not recover $(Q \setminus \{0\})/0 \uparrow \mathcal{M}_d$ from $0^{-1} \uparrow \mathcal{M}_i$:

- 0^{-1} is undefined in $0^{-1} \uparrow \mathcal{M}_i$, $\text{imn2dmn}(0^{-1}) = 1/0$, and $1/0$ is undefined in $Q/0 \uparrow \mathcal{M}_d$ and $(Q \setminus \{0\})/0 \uparrow \mathcal{M}_d$;
- $x/0$ is undefined in $Q/0 \uparrow \mathcal{M}_d$, $\text{dmn2imn}(x/0) = x \cdot (0^{-1})$, and $x \cdot (0^{-1})$ is undefined in $0^{-1} \uparrow \mathcal{M}_i$;
- $0/0 = 0$ in $(Q \setminus \{0\})/0 \uparrow \mathcal{M}_d$, $\text{dmn2imn}(0/0) = 0 \cdot (0^{-1})$, but $0 \cdot (0^{-1})$ is undefined in $0^{-1} \uparrow \mathcal{M}_i$.

This uncovers that $(Q \setminus \{0\})/0 \uparrow \mathcal{M}_d$ expresses a view on the partiality of division by zero that cannot be expressed if only multiplicative inverse is available. Therefore, we take divisive meadows as more basic than inversive meadows if their partial variants are considered as well. Otherwise, we might take inversive meadows for more basic instead, e.g. because of supposed notational simplicity (see Section 4). Thus, the move from a total algebra to a partial algebra may imply a reversal of the preferred direction of projection from dmn2imn to imn2dmn . This shows that projection semantics is a tool within a setting: if the setting changes, the tool, or rather its way of application, changes as well.

Returning to $(Q \setminus \{0\})/0 \uparrow \mathcal{M}_d$, the question remains whether the equation $0/0 = 0$ is natural. The total cost C_n of producing n items of some product is often viewed as the sum of a fixed cost FC and a variable cost VC_n . Moreover, for $n \geq 1$, the variable cost VC_n of producing n items is usually viewed as n times the marginal cost per item, taking VC_n/n as the marginal cost per item. For $n = 0$, the variable cost of producing n items and the marginal cost per item are both 0. This makes the equation $VC_0/0 = 0$ natural.

The partial meadows that we are most interested in are the three partial meadows of rational numbers that can be obtained from \mathcal{Q}_0^i and \mathcal{Q}_0^d by means of the partial meadow constructions introduced above:

$$0^{-1} \uparrow \mathcal{Q}_0^i, \quad Q/0 \uparrow \mathcal{Q}_0^d, \quad (Q \setminus \{0\})/0 \uparrow \mathcal{Q}_0^d.$$

Notice that these partial algebras have been obtained by means of the well-known initial algebra construction and a straightforward partial algebra construction. This implies that only equational logic for total algebras has been used as a logical tool for their construction, like in case of \mathcal{Q}_0^i and \mathcal{Q}_0^d . The approach followed here contrasts with the usual approach where a special logic for partial algebras would be used for the construction of partial algebras (see e.g. [20]).

We believe that many complications and obscurities in the development of the theories of the partial algebras are avoided by not using some logic of partial functions as a logical tool for their construction. Having constructed $0^{-1} \uparrow \mathcal{Q}_0^i$ in the way described above, the question whether it satisfies the equation $0^{-1} = 0^{-1}$ and related questions are still open

because the logic of partial functions to be used when working with $0^{-1} \uparrow \mathcal{Q}_0^i$ has not been fixed yet. This means that it is still a matter of design which logic of partial functions will be used when working with this partial algebra.³ As soon as the logic is fixed, the above-mentioned questions are no longer open: it is anchored in the logic whether $0^{-1} = 0^{-1}$ is satisfied, $0^{-1} \neq 0^{-1}$ is satisfied, or neither of the two is satisfied. Similar remarks apply to the other two partial algebras introduced above.

Many people prefer $0^{-1} \uparrow \mathcal{Q}_0^i$ to any other inversive algebra of rational numbers. It is likely that this is because $x \cdot x^{-1} = 1$ serves as an implicit definition of $^{-1}$ in $0^{-1} \uparrow \mathcal{Q}_0^i$.

From the partial meadows of rational numbers introduced above, $0^{-1} \uparrow \mathcal{Q}_0^i$ and $Q/0 \uparrow \mathcal{Q}_0^d$ correspond most closely to the prevailing viewpoint on the status of $1/0$ in theoretical computer science that is mentioned in Section 3. In the sequel, we will focus on $Q/0 \uparrow \mathcal{Q}_0^d$ because the divisive notation is used more often than the inversive notation.

10. Partial arithmetical meadows with zero

In this section, we introduce simple constructions of partial inversive and divisive arithmetical meadows with zero from total ones. The constructions in question are variants of the constructions of partial inversive and divisive meadows introduced in Section 9.

Let $\mathcal{M}_0^{\text{iaz}}$ be an inversive arithmetical meadow with zero. Then it makes sense to construct one partial inversive arithmetical meadow with zero from $\mathcal{M}_0^{\text{iaz}}$:

- $0^{-1} \uparrow \mathcal{M}_0^{\text{iaz}}$: the partial algebra that is obtained from $\mathcal{M}_0^{\text{iaz}}$ by making 0^{-1} undefined.

Let $\mathcal{M}_0^{\text{daz}}$ be a divisive arithmetical meadow with zero. Then it makes sense to construct two partial divisive arithmetical meadows with zero from $\mathcal{M}_0^{\text{daz}}$:

- $Q/0 \uparrow \mathcal{M}_0^{\text{daz}}$: the partial algebra that is obtained from $\mathcal{M}_0^{\text{daz}}$ by making $q/0$ undefined for all q in the domain of $\mathcal{M}_0^{\text{daz}}$;
- $(Q \setminus \{0\})/0 \uparrow \mathcal{M}_0^{\text{daz}}$: the partial algebra that is obtained from $\mathcal{M}_0^{\text{daz}}$ by making $q/0$ undefined for all q in the domain of $\mathcal{M}_0^{\text{daz}}$ different from 0.

The following partial arithmetical meadows of rational numbers with zero can be obtained from \mathcal{Q}_0^i and \mathcal{Q}_0^d by means of the partial meadow constructions introduced above:

$$0^{-1} \uparrow \mathcal{Q}_0^{\text{iaz}}, \quad Q/0 \uparrow \mathcal{Q}_0^{\text{daz}}, \quad (Q \setminus \{0\})/0 \uparrow \mathcal{Q}_0^{\text{daz}}.$$

At first sight, the absence of the additive inverse operator does not seem to add anything new to the treatment of partial meadows in Section 9. However, this is not quite the case. Consider $0^{-1} \uparrow \mathcal{Q}_0^{\text{iaz}}$. In the case of this algebra, there is a useful syntactic criterion for “being defined”. The set *Def* of defined terms and the auxiliary set *Nz* of non-zero terms can be inductively defined by:

- $1 \in \text{Nz}$;
- if $x \in \text{Nz}$, then $x + y \in \text{Nz}$ and $y + x \in \text{Nz}$;
- if $x \in \text{Nz}$ and $y \in \text{Nz}$, then $x \cdot y \in \text{Nz}$;
- if $x \in \text{Nz}$, then $x^{-1} \in \text{Nz}$;
- $0 \in \text{Def}$;
- if $x \in \text{Nz}$, then $x \in \text{Def}$;
- if $x \in \text{Def}$ and $y \in \text{Def}$, then $x + y \in \text{Def}$ and $x \cdot y \in \text{Def}$.

This indicates that the absence of the additive inverse operator allows a typing based solution to problems related to “division by zero” in elementary school mathematics. So there may be a point in dealing first and thoroughly with non-negative rational numbers in a setting where division by zero is not defined.

Working in \mathcal{Q}^{ia} simplifies matters even more because there is no distinction between terms and defined terms. Again, this may be of use in the teaching of mathematics at elementary school.

11. Imperative meadows of rational numbers

In this section, we introduce imperative inversive and divisive meadows of rational numbers.

³ A relevant survey and discussion of logics of partial functions can be found in Sections 7–9 of [10]. The rest of that paper is fully included in the current paper.

An imperative meadow of rational numbers is a meadow of rational numbers together with an imperative to comply with a very strong convention with regard to the use of the multiplicative inverse or division operator.

Like with the partial meadows of rational numbers, we introduce three imperative meadows of rational numbers:

- $0^{-1} \uparrow \mathcal{Q}_0^i$: \mathcal{Q}_0^i together with the imperative to comply with the convention that q^{-1} is not used with $q = 0$;
- $Q/0 \uparrow \mathcal{Q}_0^d$: \mathcal{Q}_0^d together with the imperative to comply with the convention that p/q is not used with $q = 0$;
- $(Q \setminus \{0\})/0 \uparrow \mathcal{Q}_0^d$: \mathcal{Q}_0^d together with the imperative to comply with the convention that p/q is not used with $q = 0$ if $p \neq 0$.

The conventions are called the *relevant inversive convention*, the *relevant division convention* and the *liberal relevant division convention*, respectively.

The conventions are very strong in the settings in which they must be complied with. For example, the relevant division convention is not complied with if the question “what is $1/0$ ” is posed. Using $1/0$ is disallowed, although we know that $1/0 = 0$ in \mathcal{Q}_0^d .

The first two of the imperative meadows of rational numbers introduced above correspond most closely to the second of the two prevailing viewpoints on the status of $1/0$ in mathematics that are mentioned in Section 3. In the sequel, we will focus on $Q/0 \uparrow \mathcal{Q}_0^d$ because the divisive notation is used more often than the inversive notation.

12. Discussion on the relevant division convention

In this section, we discuss the relevant division convention, i.e. the convention that plays a prominent part in imperative meadows.

The existence of the relevant division convention can be explained by assuming a context in which two phases are distinguished: a definition phase and a working phase. A mathematician experiences these phases in this order. In the definition phase, the status of $1/0$ is dealt with thoroughly so as to do away with the necessity of reflection upon it later on. As a result, \mathcal{Q}_0^d and the relevant division convention come up. In the working phase, \mathcal{Q}_0^d is simply used in compliance with the relevant division convention when producing mathematical texts. Questions relating to $1/0$ are understood as being part of the definition phase, and thus taken out of mathematical practice. This corresponds to a large extent with how mathematicians work.

In the two phase context outlined above, the definition phase can be made formal and logical whereas the results of this can be kept out of the working phase. Indeed, in mathematical practice, we find a world where logic does not apply and where validity of work is not determined by the intricate details of a very specific formal definition but rather by the consensus obtained by a group of readers and writers.

Whether a mathematical text, including definitions, questions, answers, conjectures and proofs, complies with the relevant division convention is a judgment that depends on the mathematical knowledge of the reader and writer. For example, $\forall x \cdot (x^2 + 1)/(x^2 + 1) = 1$ complies with the relevant division convention because the reader and writer of it both know that $\forall x \cdot x^2 + 1 \neq 0$.

Whether a mathematical text complies with the relevant division convention may be judged differently even with sufficient mathematical knowledge. This is illustrated by the following mathematical text, where $>$ is the usual ordering on the set of rational numbers:

Theorem. If $p/q = 7$ then $\frac{q^2 + p/q - 7}{q^4 + 1} > 0$.

Proof. Because $q^4 + 1 > 0$, it is sufficient to show that $q^2 + p/q - 7 > 0$. It follows from $p/q = 7$ that $q^2 + p/q - 7 = q^2$, and $q^2 > 0$ because $q \neq 0$ (as $p/q = 7$). \square

Reading from left to right, it cannot be that first p/q is used while knowing that $q \neq 0$ and that later on $q \neq 0$ is inferred from the earlier use of p/q . However, it might be said that the first occurrence of the text fragment $p/q = 7$ introduces the knowledge that $q \neq 0$ at the right time, i.e. only after it has been entirely read.

The possibility of different judgments with sufficient mathematical knowledge looks to be attributable to the lack of a structure theory of mathematical text. However, with a formal structure theory of mathematical text, we still have to deal with the fact that compliance with the relevant division convention is undecidable.

The imperative to comply with the relevant division conventions boils down to the disallowance of the use of $1/0$, $1/(1 + (-1))$, etc. in mathematical text. The usual explanation for this is the non-existence of a z such that $0 \cdot z = 1$. This makes the legality of $1/0$ comparable to the legality of $\sum_{m=1}^{\infty} 1/m$, because of the non-existence of the limit of $(\sum_{m=1}^{n+1} 1/m)_{n \in \mathbb{N}}$. However, a mathematical text may contain the statement “ $\sum_{m=1}^{\infty} 1/m$ is divergent”. That is, the use of $\sum_{m=1}^{\infty} 1/m$ is not disallowed. So the fact that there is no rational number that mathematicians intend to denote by an expression does not always lead to the disallowance of its use.

In the case of $1/0$, there is no rational number that mathematicians intend to denote by $1/0$, there is no real number that mathematicians intend to denote by $1/0$, there is no complex number that mathematicians intend to denote by $1/0$,

etc. A slightly different situation arises with $\sqrt{2}$: there is no rational number that mathematicians intend to denote by $\sqrt{2}$, but there is a real number that mathematicians intend to denote by $\sqrt{2}$. It is plausible that the relevant division convention has emerged because there is no well-known extension of the field of rational numbers with a number that mathematicians intend to denote by $1/0$.

13. Partial meadows and logics of partial functions

In this section, we bring forward arguments in support of the statement that partial meadows together with logics of partial functions do not quite explain how mathematicians deal with $1/0$ in mathematical works. It needs no explaining that a real proof of this statement is out of the question. However, we do not preclude the possibility that more solid arguments exist. Moreover, as it stands, it is possible that our argumentation leaves room for controversy.

In the setting of a logic of partial functions, there may be terms whose value is undefined. Such terms are called non-denoting terms. Moreover, often three truth values, corresponding to true, false and neither-true-nor-false, are considered. These truth values are denoted by T, F, and *, respectively.

In logics of partial functions, three different kinds of equality are found (see e.g. [33]). They only differ in their treatment of non-denoting terms:

- *weak equality*: if either t or t' is non-denoting, then the truth value of $t = t'$ is *;
- *strong equality*: if either t or t' is non-denoting, then the truth value of $t = t'$ is T whenever both t and t' are non-denoting and F otherwise;
- *existential equality*: if either t or t' is non-denoting, then the truth value of $t = t'$ is F.

With strong equality, the truth value of $1/0 = 1/0 + 1$ is T. This does not at all fit in with mathematical practice. With existential equality, the truth value of $1/0 = 1/0$ is F. This does not at all fit in with mathematical practice as well. Weak equality is close to mathematical practice: the truth value of an equation is neither T nor F if a term of the form p/q with $q = 0$ occurs in it.

This means that the classical logical connectives and quantifiers must be extended to the three-valued case. Many ways of extending them must be considered uninteresting for a logic of partial functions because they lack an interpretation of the third truth value that fits in with its origin: dealing with non-denoting terms. If these ways are excluded, only four ways to extend the classical logical connectives to the three-valued case remain (see e.g. [3]). Three of them are well-known: they lead to Bochvar's strict connectives [19], McCarthy's sequential connectives [30], and Kleene's monotonic connectives [26]. The fourth way leads to McCarthy's sequential connectives with the role of the operands of the binary connectives reversed.

In mathematical practice, the truth value of $\forall x \cdot x \neq 0 \Rightarrow x/x = 1$ is considered T. Therefore, the truth value of $0 \neq 0 \Rightarrow 0/0 = 1$ is T as well. With Bochvar's connectives, the truth value of this formula is *. With McCarthy's or Kleene's connectives the truth value of this formula is T. However, unlike with Kleene's connectives, the truth value of the seemingly equivalent $0/0 = 1 \vee 0 = 0$ is * with McCarthy's connectives. Because this agrees with mathematical practice, McCarthy's connectives are closest to mathematical practice.

The conjunction and disjunction connectives of Bochvar and the conjunction and disjunction connectives of Kleene have natural generalizations to quantifiers, which are called Bochvar's quantifiers and Kleene's quantifiers, respectively. Both Bochvar's quantifiers and Kleene's quantifiers can be considered generalizations of the conjunction and disjunction connectives of McCarthy.⁴

With Kleene's quantifiers, the truth value of $\forall x \cdot x/x = 1$ is * and the truth value of $\exists x \cdot x/x = 1$ is T. The latter does not at all fit in with mathematical practice. Bochvar's quantifiers are close to mathematical practice: the truth value of a quantified formula is neither T nor F if it contains a term of the form p/q where q has a closed substitution instance q' with $q' = 0$.

The preceding arguments suggest that mathematical practice is best approximated by a logic of partial functions with weak equality, McCarthy's connectives and Bochvar's quantifiers. We call this logic the *logic of partial meadows*, abbreviated \mathcal{L}_{PMd} .

In order to explain how mathematicians deal with $1/0$ in mathematical works, we still need the convention that a sentence is not used if its truth value is neither T nor F. We call this convention the *two-valued logic convention*.

\mathcal{L}_{PMd} together with the imperative to comply with the two-valued logic convention gets us quite far in explaining how mathematicians deal with $1/0$ in mathematical works. However, in this setting, not only the truth value of $0 \neq 0 \Rightarrow 0/0 = 1$ is T, but also the truth value of $0 = 0 \vee 0/0 = 1$ is T. In our view, the latter does not fit in with how mathematicians deal with $1/0$ in mathematical works. Hence, we conclude that \mathcal{L}_{PMd} , even together with the imperative to comply with the two-valued logic convention, fails to provide a convincing account of how mathematicians deal with $1/0$ in mathematical works.

⁴ In [29], Bochvar's quantifiers are called McCarthy's quantifiers, but McCarthy combines his connectives with Kleene's quantifiers (see e.g. [26]).

14. Concluding remarks

We have made a formal distinction between inversive meadows and divisive meadows. We have given finite equational specifications of the class of all inversive meadows, the class of all divisive meadows, and arithmetical variants of them. We have also given finite equational specifications whose initial algebras are inversive meadows of rational numbers, divisive meadows of rational numbers, and arithmetical variants of them. We have introduced and discussed constructions of variants of inversive meadows, divisive meadows, and arithmetical variants of them with a partial multiplicative inverse or division operation from the total ones. Moreover, we have given an account on how mathematicians deal with $1/0$ in mathematical work, using the concept of an imperative meadow, and have made plausible that a convincing account of how mathematicians deal with $1/0$ by means of some logic of partial functions is not attainable.

We have obtained various algebras of rational numbers by means of the well-known initial algebra construction and, in some cases, the above-mentioned partial algebra constructions. This implies that in all cases only equational logic for total algebras has been used as a logical tool for their construction. In this way, we have avoided choosing or developing an appropriate logic, which we consider a design problem of logics, not of data types. We claim that, viewed from the theory of abstract data types, the way in which partial algebras are constructed in this paper is the preferred way. Its main advantage is that no decision need to be taken in the course of the construction about matters concerning the logic to be used when working with the partial algebras in question. For that reason, we consider it useful to generalize the partial algebra constructions on inversive and divisive meadows to a partial algebra construction that can be applied to any total algebra.

Our account on how mathematicians deal with $1/0$ in mathematical work makes use of the concept of an imperative meadow. This concept is a special case of the more general concept of an imperative algebra, i.e. an algebra together with the imperative to comply with one or more conventions about its use. An example of an imperative algebra is imperative stack algebra: stack algebra, whose signature consists of *empty*, *push*, *pop* and *top*, together with the imperative to comply with the convention that *top*(*s*) is not used with *s* = *empty*. In [9], this idea is successfully used in work on the autosolvability requirement inherent in Turing's result regarding the undecidability of the halting problem.

We have argued that a logic of partial functions with weak equality, McCarthy's connectives and Bochvar's quantifiers, together with the imperative to comply with the convention that sentences whose truth value is neither T nor F are not used, approximates mathematical practice best, but after all fails to provide a convincing account of how mathematicians deal with $1/0$ in mathematical works. To our knowledge, there are no published elaborations on such a logic of partial functions. In most logics of partial functions that have been proposed by computer scientists, including PPC [25], LPF [1], PFOL [21] and WS [35], weak equality, Kleene's connectives and Kleene's quantifiers are taken as basic.

The axioms of an inversive meadow forces that the equation $0^{-1} = 0$ holds. It happens that this equation is used for technical convenience in several other places, see e.g. [24,23]. The axioms of a divisive meadow forces that the equation $x/0 = 0$ holds. One of the few published pieces of writing about this equation that we have been able to trace is [32].

We have answered a number of questions about arithmetical meadows of rational numbers, and stated a number of outstanding questions about them. We remark that the name arithmetical algebra is not always used in the same way as Peacock [36] used it. It is sometimes difficult to establish whether the notion in question is related to Peacock's notion of arithmetical algebra. For example, it is not clear to us whether the notion of arithmetical algebra defined in [37] is related to Peacock's notion of arithmetical algebra.

The theory of meadows has among other things been applied in [13,2].

Appendix A. Modular specification of divisive meadows

In this appendix, we give a modular specification of divisive meadows using basic module algebra [4].

BMA[fol] (Basic Module Algebra for first-order logic specifications) is a many-sorted equational theory of modules which covers the concepts on which the key modularization mechanisms found in existing specification formalisms are based. The signature of *BMA[fol]* includes among other things:

- the sorts *ATSIG* of atomic signatures, *ATREN* of atomic renamings, *SIG* of signatures, and *M* of modules;
- the binary deletion operator $\Delta : \text{ATSIG} \times \text{SIG} \rightarrow \text{SIG}$;
- the unary signature operator $\Sigma : M \rightarrow \text{SIG}$;
- for each first-order sentence ϕ over some signature, the constant $\langle \phi \rangle : M$;
- the binary renaming application operator $\cdot : \text{ATREN} \times M \rightarrow M$;
- the binary combination operator $+$: $M \times M \rightarrow M$;
- the binary export operator $\square : \text{SIG} \times M \rightarrow M$.

The axioms of *BMA[fol]* as well as four different models for *BMA[fol]* can be found in [4]. A useful derived operator is the *hiding* operator $\Delta : \text{ATSIG} \times M \rightarrow M$ defined by $a\Delta X = (a\Delta \Sigma(X))\square X$. Below, we will use the notational conventions introduced in Section 3.5 of [4].

Let Md_i be the closed module expression corresponding to the equations E_{Md}^i , i.e. $Md_i = \langle (x + y) + z = x + (y + z) \rangle + \dots + \langle x \cdot (x \cdot x^{-1}) = x \rangle$. We give a modular specification of divisive meadows using *BMA[fol]* as follows:

$$Md_d = \mathbf{F} : ^{-1} : Q \rightarrow Q \Delta (Md_i + \langle x/y = x \cdot (y^{-1}) \rangle).$$

In [4], a semantic mapping *EqTh* is defined that gives, for each closed module expression, its equational theory. We have the following theorem:

Theorem 17. *EqTh(Md_d) is the equational theory associated with the equational specification of divisive meadows given in Section 4.*

Proof. In [4], a semantic mapping *Mod* is defined that gives, for each closed module expression, its model class. *Mod* and *EqTh* are defined such that *EqTh*(*m*) is the equational theory of *Mod*(*m*) for each closed module expression *m*. Hence, it is sufficient to show that *Mod*(Md_d) is the class of models of the equational specification of divisive meadows. By the definition of *Mod*, we have to show that: (i) the reduct to the signature of divisive meadows of each model of the equational specification of inversive meadows extended with the equation $x/y = x \cdot (y^{-1})$ is a model of the equational specification of divisive meadows; (ii) each model of the equational specification of divisive meadows can be expanded with a multiplicative inverse operation satisfying $(x^{-1})^{-1} = x$ and $x \cdot (x \cdot x^{-1}) = x$. Using the equations from the equational specification of inversive meadows and the equation $x/y = x \cdot (y^{-1})$, it can easily be proved by equational reasoning that all equations from the equational specification of divisive meadows are satisfied by the reducts in question. Let $^{-1}$ be defined by $x^{-1} = 1/x$. Then, using the equations from the equational specification of divisive meadows and the equation $x^{-1} = 1/x$, it can easily be proved by equational reasoning that the equations $(x^{-1})^{-1} = x$ and $x \cdot (x \cdot x^{-1}) = x$ are satisfied by the expansions in question. \square

We give the following modular specification of reduced divisive meadows:

$$Md_{rd1} = \mathbf{F} : \cdot : Q \times Q \rightarrow Q \Delta Md_d,$$

$$Md_{rd2} = \mathbf{F} : - : Q \rightarrow Q \Delta (Md_{rd1} + \langle x - y = x + (-y) \rangle),$$

$$Md_{rd3} = \mathbf{F} : + : Q \times Q \rightarrow Q \Delta Md_{rd2},$$

$$Md_{rd} = \mathbf{F} : 0 : Q \Delta Md_{rd3}.$$

We have the following theorem:

Theorem 18. *EqTh(Md_{rd}) is the equational theory associated with the equational specification of reduced divisive meadows given in Section 4.*

Proof. The proof follows the same line as the proof of Theorem 17. For the expansion, we define zero, addition, multiplication, and additive inverse as follows: $0 = 1 - 1$, $x + y = x - ((1 - 1) - y)$, $x \cdot y = x/(1/y)$, and $-x = (1 - 1) - x$. \square

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