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A family of invariants of rooted forests

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Abstract

Let A be a commutative k-algebra over a field of k and Ξ a linear operator defined on A. We define a family of A-valued invariants Ψ for finite rooted forests by a recurrent algorithm using the operator Ξ and show that the invariant Ψ distinguishes rooted forests if (and only if) it distinguishes rooted trees T, and if (and only if) it is *finer* than the quantity $\alpha(T) = |\mathrm{Aut}(T)|$ of rooted trees T. We also consider the generating function $U(q) = \sum_{n=1}^{\infty} U_n q^n$ with $U_n = \sum_{T\in\mathbb{T}_n} (1/\alpha(T))\Psi(T)$, where $\mathbb{T}_n$ is the set of rooted trees with n vertices. We show that the generating function $U(q)$ satisfies the equation $\Xi \exp U(q) = q^{-1}U(q)$. Consequently, we get a recurrent formula for U_n ($n \geqslant 1$), namely, $U_1 = \Xi(1)$ and $U_n = \Xi S_{n-1}(U_1, U_2, \ldots, U_{n-1})$ for any $n \geqslant 2$, where $S_n(x_1, x_2, \ldots)$ ($n \in \mathbb{N}$) are the elementary Schur polynomials. We also show that the (strict) order polynomials and two well-known quasi-symmetric function invariants of rooted forests are in the family of invariants Ψ and derive some consequences about these well-known invariants from our general results on Ψ. Finally, we generalize the invariant Ψ to labeled planar forests and discuss its certain relations with the Hopf algebra $\mathscr{H}^{D}_{P,R}$ in Foissy (Bull. Sci. Math. 126 (2002) 193) spanned by labeled planar forests.

MSC: 05C05; 05A15

1. Introduction

By a *rooted tree* we mean a finite 1-connected graph with one vertex designated as its *root*. Rooted trees not only form a family of important objects in combinatorics, they are also closely related with many other mathematical areas. For the connection with the inversion problem and the Jacobian problem, see [1,13]. For the connection with D-log

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and formal flow of analytic maps, see [14]. For the connection with renormalization of quantum field theory, see [6,2].

In this paper, motivated by certain properties of the (strict) order polynomials encountered in [14], we define a family of A-valued invariants Ψ for rooted forests by a recurrent algorithm (see Algorithm 3.1) starting with an arbitrary commutative k-algebra A over a field of k and a fixed linear operator Ξ defined on A. We show in Proposition 4.2 and Theorem 4.3 that the invariant Ψ distinguishes rooted forests if (and only if) it distinguishes rooted trees T, and if (and only if) it is *finer* than the quantity $\alpha(T)=|\mathrm{Aut}(T)|$ of rooted trees T. In Section 5, we consider the generating function $U(q)=\sum_{n=1}^{\infty} U_n q^n$, where $U_n=\sum_{T\in\mathbb{T}_n}(1/\alpha(T))\Psi(T)$ with $\mathbb{T}_n$ the set of rooted trees with n vertices and $\alpha(T)=|\mathrm{Aut}(T)|$. We show that the generating function $U(q)$ satisfies the equation $\Xi \exp U(q)=q^{-1}U(q)$. Consequently, we get the recurrent formula $U_1=\Xi(1)$ and $U_n=\Xi S_{n-1}(U_1,U_2,\dots,U_{n-1})$ for any $n\geqslant 2$, where $S_n(x_1,x_2,\dots)$ are the elementary Schur polynomials. In Sections 6 and 7, we show that, with properly chosen A and the linear operator Ξ, the (strict) order polynomials of rooted trees and two families of quasi-symmetric functions for rooted forests (see (7.1) and (7.2) for the definitions) are in the family of invariants Ψ defined by Algorithm 3.1. We also derive some consequences on these well-known invariants from our general results on the invariant Ψ. Finally, in Section 8, We generalize our invariants to labeled planar forests and discuss certain relations of our invariants with the Hopf algebra $\mathscr{H}^D_{P,R}$ in [3] spanned by labeled planar forests.

2. Notation and an operation for rooted forests

Notation. In a rooted tree there are natural ancestral relations between vertices. We say a vertex w is a child of vertex v if the two are connected by an edge and w lies further from the root than v. We define the *degree* of a vertex v of T to be the number of its children. A vertex is called a *leaf* if it has no children. By a rooted forest we mean a disjoint union of finitely many rooted trees. When we speak of isomorphisms between rooted forests, we will always mean root-preserving isomorphisms, i.e. the image of a root of a connected component which is always a rooted tree must be a root.

Once for all, we fix the following notation for the rest of this paper.

(1) We let $\mathbb{T}$ be the set of isomorphism classes of all rooted trees and $\mathbb{F}$ the set of isomorphism classes of all rooted forests. For $m\geqslant 1$ an integer, we let $\mathbb{T}_m$ (resp $\mathbb{F}_m$) the set of isomorphism classes of all rooted trees (resp. forests) with m vertices.

(2) For any rooted tree T, we set the following notation:
- rt_T denotes the root vertex of T,
- $V(T)$ (resp. $L(T)$) denotes the set of vertices (resp. leaves) of T,
- $v(T)$ (resp. $l(T)$) denotes the number of the elements of $V(T)$ (resp. $L(T)$),
- for $v\in V(T)$ we define the *height* of v to be the number of edges in the (unique) geodesic connecting v to rt_T,

- $h(T)$ denotes the height of T,
- $\alpha(T)$ denotes the number of the elements of the automorphism group $\mathrm{Aut}(T)$,
- for $v_1,\dots,v_r \in V(T)$, we write $T\setminus\{v_1,\dots,v_r\}$ for the graph obtained by deleting each of these vertices and all edges adjacent to these vertices.

(3) A *rooted subtree* of a rooted tree T is defined as a connected subgraph of T containing rt_T, with $\mathrm{rt}_{T'} = \mathrm{rt}_T$.

(4) We call the rooted tree with one vertex the *singleton*, denoted by $\circ$.

We define the operation B_+ for rooted forests as follows. Let S be a rooted forest which is disjoint union of rooted trees T_i $(i = 1,2,\dots,d)$. We define $B_+(S) = B_+(T_1,\dots,T_d)$ to be the rooted tree obtained by connecting all roots of T_i $(i=1,2,\dots,d)$ to a single new vertex, which is set to the root of the new rooted tree $B_+(T_1,\dots,T_d)$. If a forest S is the disjoint union of k_1 copies of rooted tree T_1, k_2 copies of $T_2,\dots$, k_d copies of T_d, we also use the notation $B_+(T_1^{k_1},\dots T_r^{k_d})$ for the new rooted tree $B_+(S)$.

Lemma 2.1. *For any $k_i \in \mathbb{N}$, $T_i \in \mathbb{T}$ $(i = 1,2,\dots,d)$ with $T_i \ncong T_j$ $(i \neq j)$, we have*

$$\alpha(B_+(T_1^{k_1},\dots T_r^{k_d})) = (k_1)!\dots(k_d)!\alpha(T_1)^{k_1}\dots\alpha(T_d)^{k_d}. \tag{2.1}$$

Proof. Set $T = B_+(T_1^{k_1},\dots T_r^{k_d})$ and R the rooted subtree of T consisting of the root rt_T of T and all its children in T. Let ϕ: $\mathrm{Aut}(T) \to \mathrm{Aut}(R)$ be the restriction map which clearly is a homomorphism of groups. Let $H \leqslant \mathrm{Aut}(R)$ be the image of ϕ. Since $T_i \ncong T_j$ for any $i \neq j$, It is easy to see that $|H| = k_1!k_2!\dots k_d!$. Let K be the kernal of ϕ. Note that an element $\alpha \in \mathrm{Aut}(T)$ is in K if and only if it fixes all the vertices of R. Hence the order $|K|$ is equals to $\prod_{i=1}^{d} \alpha(T_i)^{k_i}$. Therefore, we have

$$\alpha(T) = |K||H| = (k_1)!\dots(k_d)!\alpha(T_1)^{k_1}\dots\alpha(T_d)^{k_d}. \qquad \square$$

3. A family of invariants Ψ for rooted forests

Let A be a commutative k-algebra over a field k and Ξ a k-linear map from A to A. Set $a = \Xi(1)$. We first define an A-valued invariant Ψ for rooted forests by the following algorithm:

Algorithm 3.1.

(1) For any rooted tree $T \in \mathbb{T}$, we define $\Psi(T)$ as follows.
  (i) For each leaf v of T, set $N_v = \Xi(1) = a$.
  (ii) For any other vertex v of T, define N_v inductively starting from the highest level by setting $N_v = \Xi\ (N_{v_1}N_{v_2}\dots N_{v_k})$, where v_j $(j = 1,2,\dots,k)$, are the distinct children of v.
  (iii) Set $\Psi(T) = N_{\mathrm{rt}_T}$.

(2) For any rooted forest $S \in \mathbb{F}$, we set

$$\Psi(S) = \prod_{i=1}^{m} \Psi(T_i), \tag{3.1}$$

where T_i $(i = 1, 2, \ldots, m)$ are connected components of S.

From Algorithm 3.1, the following two lemmas are obvious.

Lemma 3.2. (a) $B = \{\Psi(S) \mid S \in \mathbb{F}\}$ *is a multiplicative subset of* A, *i.e. it is closed under the multiplication of the algebra* A.

(b) $\Xi(B) \subset B$.

Lemma 3.3. *Let* Γ *be an* A*-valued invariant for rooted forests.* Γ *can be re-defined and calculated by Algorithm* 3.1 *for some* k*-linear map* Ξ *if and only if*

(1) *It satisfies Eq.* (3.1) *for any rooted forest* $S \in \mathbb{F}$.

(2) *For any rooted tree* $T \in \mathbb{T}$,

$$\Gamma(T) = \Xi \prod_{i=1}^{d} \Gamma(T_i), \tag{3.2}$$

where T_i $(i = 1, 2, \ldots, d)$ *are the connected components of* $T \setminus \mathrm{rt}_T$.

In Sections 6 and 7, we will show that the strict order polynomials, order polynomials as well as two well-known quasi-symmetric functions of rooted forests (see (7.1) and (7.2) for the definitions) are in this family of invariants Ψ.

4. When the invariants Ψ distinguish rooted forests

Definition 4.1. We say an invariant Ψ distinguishes rooted forests (resp. trees) if, for any $S_1, S_2 \in \mathbb{F}$ (resp. $S_1, S_2 \in \mathbb{T}$), $\Psi(S_1) = \Psi(S_2)$ if and only if $S_1 \simeq S_2$. We say an invariant Ψ is finer than the quantity $\alpha(T)$ if, for any $T_1, T_2 \in \mathbb{T}$, $\alpha(T_1) \neq \alpha(T_2)$ implies $\Psi(T_1) \neq \Psi(T_2)$.

In combinatorics, it is very desirable to find an invariant which can distinguish rooted trees or rooted forests. For the invariant Ψ defined by Algorithm 3.1, we have the following results.

Proposition 4.2. *An* A*-valued invariant* Ψ *defined by Algorithm* 3.1 *distinguishes rooted forests if* (*and only if*) *it distinguishes rooted trees.*

Proof. Suppose Ψ distinguishes rooted trees. Let $S_1, S_2 \in \mathbb{F}$ with $\Psi(S_1) = \Psi(S_2)$. We need show that $S_1 \simeq S_2$. First, by Lemma 3.3, we have $\Psi(B_+(S_i)) = \Xi\Psi(S_i)$ for $i = 1, 2$. Hence, $\Psi(B_+(S_1)) = \Psi(B_+(S_2))$. Therefore, by our assumption, we have $B_+(S_1) \simeq B_+(S_2)$, which clearly implies $S_1 \simeq S_2$. $\square$

Theorem 4.3. *An A-valued invariant Ψ defined by Algorithm* 3.1 *distinguishes rooted trees if* (*and only if*) *it is finer than* $\alpha(T)$.

Proof. Suppose Ψ is finer than $\alpha(T)$. Let $T_1, T_2 \in \mathbb{T}$ such that $\Psi(T_1) = \Psi(T_2)$. Hence $\alpha(T_1) = \alpha(T_2)$. We need show that $T_1 \simeq T_2$.

Suppose T_1 and T_2 are not isomorphic to each other. Let $T = B_+(T_1, T_1)$ and $T' = B_+(T_1, T_2)$. By Lemma 3.3, we have

$$\Psi(T) = \Xi(\Psi(T_1)^2),$$

$$\Psi(T') = \Xi(\Psi(T_1)\Psi(T_2)).$$

Since $\Psi(T_1) = \Psi(T_2)$, we have $\Psi(T) = \Psi(T')$. On the other hand, by Lemma 2.1, we have

$$\alpha(T) = 2\alpha(T_1)^2,$$

$$\alpha(T') = \alpha(T_1)\alpha(T_2) = \alpha(T_1)^2.$$

Therefore, $\alpha(T) \neq \alpha(T')$. So Ψ is not finer than $\alpha(T)$, which is a contradiction. □

5. A generating function for the invariant Ψ of rooted trees

In this section, we fix an invariant Ψ defined by Algorithm 3.1 and consider the generating function

$$U(q) = \sum_{T \in \mathbb{T}} \frac{1}{\alpha(T)} \Psi(T) q^{v(T)}. \tag{5.1}$$

For any $n \geqslant 1$, set $U_n = \sum_{T \in \mathbb{T}_n} (1/\alpha(T))\Psi(T)$. Hence, we have $U(q) = \sum_{n=1}^{\infty} U_n q^n$. We will derive an equation satisfied by the generating function $U(q)$, from which $\{U_n \mid n \in \mathbb{N}\}$ $(n \geqslant 2)$ can be calculated recursively by using the elementary Schur polynomials.

Theorem 5.1. *The generating function $U(q)$ satisfies the equation*

$$\Xi\, e^{U(q)} = q^{-1} U(q). \tag{5.2}$$

Proof. Consider

$$\begin{aligned} \Xi\, e^{U(q)} &= \Xi(1) + \Xi \sum_{k=1}^{\infty} \frac{U^k(q)}{k!}, \\ &= a + \sum_{k=1}^{\infty} \frac{\Xi}{k!} \left(\sum_{T \in \mathbb{T}} \frac{1}{\alpha(T)} (\Psi(T) q^{v(T)}) \right)^k. \end{aligned}$$

While, for the general term of the right-hand side of the equation above, we have

$$\frac{\Xi}{k!}\left(\sum_{T\in\mathbb{T}}\frac{1}{\alpha(T)}\,\Psi(T)q^{v(T)}\right)^k$$

$$=\sum_{r=1}^{k}\sum_{\substack{T_1,\dots,T_r\in\mathbb{T}\\ T_i\neq T_j(i\neq j)}}\sum_{\substack{k_1+\dots+k_r=k\\ k_1,\dots,k_r\geqslant 1}}\frac{1}{(k_1)!\dots(k_r)!}\,\frac{\Xi\left(\prod_{i=1}^r\Psi(T_i)^{k_i}\right)}{\prod_{i=1}^r\alpha(T_i)^{k_i}}\,q^{\sum_{i=1}^r k_i v(T_i)}.$$

By Lemmas 2.1 and 3.3, we have

$$=\sum_{r=1}^{k}\sum_{\substack{T_1,\dots,T_r\in\mathbb{T}\\ T_i\neq T_j(i\neq j)}}\sum_{\substack{k_1+\dots+k_r=k\\ k_1,\dots,k_r\geqslant 1}}\frac{\Psi(B_+(T_1^{k_1},\dots,T_r^{k_r}))}{\alpha(B_+(T_1^{k_1},\dots,T_r^{k_r}))}\,q^{v(B_+(T_1^{k_1},\dots,T_r^{k_r}))-1}$$

$$=q^{-1}\sum_{r=1}^{k}\sum_{T\in T_{r,k+1}}\frac{\Psi(T)}{\alpha(T)}\,q^{v(T)},$$

where $T_{r,k+1}$ is the set of equivalence classes of rooted trees with $k+1$ vertices and the degree of the root being exactly r. Therefore, we have

$$\Xi\, e^{U(q)}=\Xi(1)+q^{-1}\sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{T\in T_{r,k+1}}\frac{\Psi(T)}{\alpha(T)}\,q^{v(T)}$$

$$=\Xi(1)+q^{-1}\sum_{\substack{T\in\mathbb{T}\\ T\neq\circ}}\frac{\Psi(T)}{\alpha(T)}\,q^{v(T)}$$

$$=q^{-1}U(q)\quad(\text{since } U_1=\Psi(\circ)=\Xi(1)).\qquad\square$$

Recall that the elementary Schur polynomials $S_n(x)$ $(n\in\mathbb{N})$ in $x=(x_1,x_2,\dots,x_k\dots)$ are defined by the generating function:

$$e^{\sum_{k=1}^{\infty}x_kq^k}=\sum_{n=0}^{\infty}S_n(x)q^n=1+\sum_{n=1}^{\infty}S_n(x)q^n. \tag{5.3}$$

Note that, if we sign the *weight* of the variable x_k to be k for any $k\in\mathbb{N}^+$ and set

$$\mathrm{wt}(x_{i_1}^{a_1}x_{i_2}^{a_2}\dots x_{i_d}^{a_d})=\sum_{k=1}^{d}a_ki_k$$

for any $i_k,a_k\in\mathbb{N}^+$ $(k=1,2,\dots,d)$. Then, for any $n\in\mathbb{N}$, $S_n(x)$ is a polynomial which is homogeneous with respect to weight with $\mathrm{wt}\,S_n(x)=n$. In particular, $S_n(x)$ depends only on the variables x_i $(i = 1,2,\dots,n)$. For more properties of the elementary Schur polynomials $S_n(x)$ and their relationship with Schur symmetric functions, see [5,7].

Proposition 5.2. *For any $n \geqslant 1$, we have*

$$U_1 = \Xi(1) = a, \tag{5.4}$$

$$U_n = \Xi\, S_{n-1}(U_1, U_2, \dots, U_{n-1}). \tag{5.5}$$

Proof. Set $U = (U_1, U_2, \dots, U_k, \dots)$. From (5.2) and (5.3), we have

$$\sum_{n=1}^{\infty} \Xi S_n(U) q^n = q^{-1} \sum_{n=2}^{\infty} U_n(t) q^{n-1}.$$

By comparing the coefficient of q^{n-1} $(n \geqslant 2)$, we have

$$U_n(t) = \Xi\, S_{n-1}(U_1, U_2, \dots, U_{n-1}).$$

Hence, we get (5.5). □

Remark 5.3. One interesting aspect of the invariant Ψ and its generating function $U(q)$ is as follows. From Proposition 5.2, we see that $U(q)$ is the unique solution of Eq. (5.2) in the power series algebra $A[[q]]$. Therefore, any equation of the form (5.2) can be solved by looking at the invariant Ψ defined by Algorithm 3.1 for rooted trees and its generating function $U(q)$ defined by (5.1).

6. (Strict) order polynomials

Let $T \in \mathbb{T}$ be a rooted tree. Note that T with the natural partial order induced from rooted tree structure forms a finite poset (partially ordered set), in which the root of T serves the unique minimum element. Similarly, any rooted forest also forms a finite poset. In the rest of this paper, we will always view rooted forests as finite posets in this way. Recall the *strict order polynomial* $\bar{\Omega}(P)$ for a finite poset P is defined to be the unique polynomial $\bar{\Omega}(P)$ such that $\bar{\Omega}(P)(n)$ equals to the number of strict order preserving maps ϕ from P to the totally ordered set $[n] = \{1,2,\dots,n\}$ for any $n \geqslant 1$. Here a map $\phi : P \to [n]$ is said to be *strict order preserving* if, for any elements $x, y \in P$ with $x > y$ in P, then $\phi(x) > \phi(y)$ in $[n]$. Also recall that the *order polynomial* $\Omega(P)$ for a finite poset P is defined to be the unique polynomial $\Omega(P)$ such that $\Omega(P)(n)$ equals to the number of order preserving maps $\phi : P \to [n]$ for any $n \geqslant 1$. Here a map $\phi : P \to [n]$ is said to be *order preserving* if, for any elements $x, y \in P$ with $x > y$ in P, then $\phi(x) \geqslant \phi(y)$ in $[n]$. For general studies of these two invariants, see [10].

In this section, we show that the strict order polynomials $\bar{\Omega}(T)$ and order polynomials $\Omega(T)$ are both in the family of the invariants Ψ defined by Algorithm 3.1. We also derive some consequences from our general results on the invariants Ψ.

Consider the polynomial ring $\mathbb{C}[t]$ in one variable t over $\mathbb{C}$ and the difference operator Δ, which is defined by

$$\Delta : \mathbb{C}[t] \to \mathbb{C}[t], \tag{6.1}$$

$$f(t) \to f(t+1) - f(t). \tag{6.2}$$

We define the operator $\Delta^{-1} : \mathbb{C}[t] \to t\mathbb{C}[t]$ by setting $\Delta^{-1}(g)$ to be the unique polynomial $f \in t\mathbb{C}[t]$ such that $\Delta(f) = g$ for any $g \in \mathbb{C}[t]$. Note that $\Delta^{-1} : \mathbb{C}[t] \to t\mathbb{C}[t]$ is well defined because that, for any polynomial $f \in \mathbb{C}[t]$, $\Delta(f) = 0$ if and only if f is a constant.

We also define the operator ∇ by

$$\nabla : \mathbb{C}[t] \to \mathbb{C}[t],$$

$$f(t) \to f(t) - f(t-1)$$

and ∇^{-1} by setting $\nabla^{-1}(g)$ to be the unique polynomial $f \in t\mathbb{C}[t]$ such that $\nabla(f) = g$ for any $g \in \mathbb{C}[t]$.

Proposition 6.1. *Let $A = \mathbb{C}[t]$, then the strict order polynomials $\bar{\Omega}$ (resp. order polynomials Ω) of rooted forests can be re-defined and calculated by Algorithm* 3.1 *with $\Xi = \Delta^{-1}$ (resp. $\Xi = \nabla^{-1}$).*

The proof of the proposition above immediately follows from Lemma 3.3, the fact that $\bar{\Omega}$ and Ω also satisfy Eq. (3.1) and the following lemma due to John Shareshian.

Lemma 6.2 (J. Shareshian). *For any rooted trees T_i $(i = 1, 2, \ldots, r)$, we have*

$$\Delta\bar{\Omega}(B_+(T_1, T_2, \ldots, T_r)) = \bar{\Omega}(T_1)\bar{\Omega}(T_2)\ldots\bar{\Omega}(T_r), \tag{6.3}$$

$$\nabla\Omega(B_+(T_1, T_2, \ldots, T_r)) = \Omega(T_1)\Omega(T_2)\ldots\Omega(T_r). \tag{6.4}$$

For the proof of Eq. (6.3), see the proof of Theorem 4.5 in [14]. Eq. (6.4) can be proved similarly. Actually, Proposition 6.1 has been proved in [14] for the strict order polynomials $\bar{\Omega}$.

Now we consider the corresponding generating functions $\bar{U}(t,q) = \sum_{T \in \mathbb{T}} (\bar{\Omega}(T)/\alpha(T))q^{v(T)}$ and $U(t,q) = \sum_{T \in \mathbb{T}} (\Omega(T)/\alpha(T))q^{v(T)}$. By Theorem 5.1, we have

Proposition 6.3. *The generating functions satisfy the equations*

$$e^{\bar{U}(t,q)} = q^{-1}\Delta\bar{U}(t,q), \tag{6.5}$$

$$e^{U(t,q)} = q^{-1}\nabla U(t,q). \tag{6.6}$$

For any $n \geqslant 1$, we set $\bar{U}_n(t) = \sum_{T \in \mathbb{T}_n} \bar{\Omega}(T)/\alpha(T)$ and $U_n(t) = \sum_{T \in \mathbb{T}_n} \Omega(T)/\alpha(T)$. By Proposition 5.2, we have

Proposition 6.4. (a) *For any* $n \geqslant 2$, *we have*

$$\bar{U}_1 = \Delta^{-1}(1) = t, \tag{6.7}$$

$$\bar{U}_n = \Delta^{-1} S_{n-1}(\bar{U}_1, \bar{U}_2, \ldots, \bar{U}_{n-1}). \tag{6.8}$$

(b) *For any* $n \geqslant 2$, *we have*

$$U_1 = \nabla^{-1}(1) = t, \tag{6.9}$$

$$U_n = \nabla^{-1} S_{n-1}(U_1, U_2, \ldots, U_{n-1}). \tag{6.10}$$

Set $u(q) = U(t,1)$ and write $u(q) = \sum_{n=1}^{\infty} u_n q^n$. Since $\Omega(T)(1) = 1$ and $\Omega(T)(0) = 0$ for any rooted tree T, we see that $u_n = \sum_{T \in \mathbb{T}_n} \frac{1}{\alpha(T)}$ and $U(0,q) = 0$. Therefore,

$$(\nabla U(t,q))(1) = U(1,q) - U(0,q) = u(q).$$

Combining with Eq. (6.6), we see that the generating function $u(q)$ satisfies the equation

$$\mathrm{e}^{u(q)} = q^{-1} u(q). \tag{6.11}$$

But, on the other hand, it is well known that there is another generating function related with rooted trees satisfying Eq. (6.11) which is defined as follows. Let $r(n)$ be the number of rooted trees on the labeled set $[n] = \{1, 2, \ldots, n\}$. Let $R(q) = \sum_{n \geqslant 1} (r(n)/n!) q^n$. Then, by Proposition 5.3.1 in [11], $R(q)$ also satisfies Eq. (6.11) and by Proposition 5.3.2 in [11], we know that $r(n) = n^{n-1}$. Therefore, we have

Corollary 6.5. $u(q) = R(q)$. *In particular*, *for any* $n \geqslant 1$, *we have the identities*

$$n! \sum_{T \in \mathbb{T}_n} \frac{1}{\alpha(T)} = r(n), \tag{6.12}$$

$$\sum_{T \in \mathbb{T}_n} \frac{1}{\alpha(T)} = \frac{n^{n-1}}{n!}. \tag{6.13}$$

For the corollary above, we see that Eq. (6.6) can be viewed as a natural generalization of Eq. (6.11).

7. Two quasi-symmetric function invariants for rooted forests

Let us first recall the following well-known quasi-symmetric functions $\bar{K}(P)$ and $K(P)$ defined in [11] for finite posets P. For more general studies on quasi-symmetric functions, see [4,12,8,11]

Let $x = (x_1, x_2, \ldots,)$ be a sequence of commutative variables and $\mathbb{C}[[x]]$ the formal power series algebra in x_k ($k \geqslant 1$) over $\mathbb{C}$. For any finite poset P and any map $\sigma : P \to$

$\mathbb{N}^+$ of sets, we set $x^\sigma := \prod_{i=1}^{\infty} x_i^{|\sigma^{-1}(i)|}$ and define

$$\bar{K}(P)(x) = \sum_{\sigma} x^\sigma, \tag{7.1}$$

where the sum runs over the set of all strict order preserving maps $\sigma : P \to \mathbb{N}^+$. Similarly, we define

$$K(P)(x) = \sum_{\sigma} x^\sigma, \tag{7.2}$$

where the sum runs over the set of all order preserving maps $\sigma : P \to \mathbb{N}^+$. Note that $\bar{K}(P)(x)$ and $K(P)(x)$ are always in $\mathbb{C}[[x]]$ and satisfy Eq. (3.1) for rooted forests.

Recall that an element $f \in \mathbb{C}[[x]]$ is said to be *quasi-symmetric* if the degree of f is bounded, and for any $a_1, a_2, \ldots, a_k \in \mathbb{N}^+$, $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$, the coefficient of the monomial $x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_k}^{a_k}$ is always same as the coefficient of the monomial $x_{j_1}^{a_1} x_{j_2}^{a_2} \ldots x_{j_k}^{a_k}$. From definitions (7.1) and (7.2), it is easy to check that, for any finite poset P, $\bar{K}(P)$ and $K(P)$ are quasi-symmetric.

In this section, we will show that the quasi-symmetric functions $\bar{K}$ and K for rooted forests are also in the family of the invariants Ψ defined by Algorithm 3.1.

We define the shift operator $S : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ by first setting

$$S(1) = 1$$

$$S(x_m) = x_{m+1}$$

and then extending it to $\mathbb{C}[[x]]$ to be the unique $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[[x]]$ to $\mathbb{C}[[x]]$. For any $m \in \mathbb{N}^+$, we denote by the abusing notation x_m the $\mathbb{C}$-linear map from $\mathbb{C}[[x]]$ to $\mathbb{C}[[x]]$ induced by the multiplication by x_m.

The following lemma follows immediately from the definition of the linear operator S.

Lemma 7.1. *As the linear maps from $\mathbb{C}[[x]]$ to $\mathbb{C}[[x]]$, $x_m S^k = S^k x_{m-k}$ for any $k, m \in \mathbb{N}^+$ with $k < m$.*

We define the linear maps $\bar{\Lambda}$ and Λ from $\mathbb{C}[[x]]$ to $\mathbb{C}[[x]]$ by setting

$$\bar{\Lambda} = \sum_{k=1}^{\infty} x_k S^k = \left(\sum_{k=1}^{\infty} S^k \right) x_1 S, \tag{7.3}$$

$$\Lambda = \sum_{k=1}^{\infty} x_k S^{k-1} = \left(\sum_{k=1}^{\infty} S^k \right) x_1, \tag{7.4}$$

where the last equalities of the equations above follow from Lemma 7.1. It is easy to see that $\bar{\Lambda}$ and Λ are well defined.

Lemma 7.2. (a) *The linear maps $\bar{\Lambda}$ and Λ from $\mathbb{C}[[x]]$ to $\mathbb{C}[[x]]$ are injective.*

(b)

$$\bar{\Lambda}(1) = \Lambda(1) = \sum_{k=1}^{\infty} x_k. \tag{7.5}$$

Proof. (b) follows immediately from Eqs. (7.3) and (7.4).

To prove (a), let $f \in \mathbb{C}[[x]]$ such that $\bar{\Lambda} f = 0$. By Eq. (7.3), we have $(1-S)\bar{\Lambda} = x_1 S$. Hence, $x_1 S f = 0$ and $S f = 0$. Therefore, we must have $f = 0$. The injectivity of Λ can be proved similarly. □

Lemma 7.3. *For any rooted tree T, we have*

$$\bar{K}(T) = \bar{\Lambda} \prod_{i=1}^{d} \bar{K}(T_i), \tag{7.6}$$

$$K(T) = \Lambda \prod_{i=1}^{d} K(T_i), \tag{7.7}$$

where T_i $(i = 1, 2, \ldots, T_d)$ are the connected components of $T \setminus \mathrm{rt}_T$.

Proof. Here we only prove Eq. (7.6). For Eq. (7.7), the ideas of the proof are similar.

Let W be the set of all strict order preserving maps $\sigma : P \to \mathbb{N}^+$ and W_k $(k \geqslant 1)$ the set of $\sigma \in W$ such that $\sigma(\mathrm{rt}_T) = k$. Clearly, W equals to the disjoint union of W_k $(k \geqslant 1)$. By the definition of $\bar{K}$, see (7.1), we see that $\sum_{\sigma \in W_k} x^\sigma \in \mathbb{C}[[x_k, x_{k+1}, \ldots,]]$. Since $\bar{K}$ satisfies Eq. (3.1) for rooted forests, we have

$$\sum_{\sigma \in W_k} x^\sigma = x_k S^k \bar{K}(T \setminus \mathrm{rt}_T) = x_k S^k \prod_{i=1}^{d} \bar{K}(T_i). \tag{7.8}$$

Therefore,

$$\begin{aligned}
\bar{K}(T) &= \sum_{k=1}^{\infty} \sum_{\sigma \in W_k} x^\sigma, \\
&= \sum_{k=1}^{\infty} x_k S^k \prod_{i=1}^{d} K(T_i), \\
&= \Lambda \left(\prod_{i=1}^{d} K(T_i) \right). \qquad \square
\end{aligned}$$

From the lemma above and Lemma 3.3 and the fact that $\bar{K}$ and K satisfy Eq. (3.1) for rooted forests, we immediately have

Proposition 7.4. *The quasi-symmetric functions $\bar{K}$ (resp. K) for rooted forests can be re-defined and calculated by Algorithm* 3.1 *with $A = \mathbb{C}[[x]]$ and $\Xi = \bar{\Lambda}$ (resp. $\Xi = \Lambda$).*

Now we consider the generating functions

$$\bar{Q}(x,q) = \sum_{T \in \mathbb{T}} \frac{\bar{K}(P)(x)}{\alpha(T)} q^{v(T)} = \sum_{n=1}^{\infty} Q_n(x) q^n,$$

$$Q(x,q) = \sum_{T \in \mathbb{T}_n} \frac{K(P)(x)}{\alpha(T)} q^{v(T)} = \sum_{n=1}^{\infty} Q_n(x) q^n,$$

where $\bar{Q}_n(x) = \sum_{T \in \mathbb{T}_n} \bar{K}(P)(x)/\alpha(T)$ and $Q_n(x) = \sum_{T \in \mathbb{T}_n} K(P)(x)/\alpha(T)$ for any $n \geqslant 1$. By Theorem 5.1, Lemma 7.2 and Proposition 7.4, we have

Proposition 7.5. (a) *The generating functions* $Q(x,q)$ *and* $Q(x,q)$ *satisfy the equations*

$$\bar{\Lambda} e^{\bar{Q}(x,t)} = q^{-1} Q(x,t), \tag{7.9}$$

$$\Lambda e^{Q(x,t)} = q^{-1} Q(x,t). \tag{7.10}$$

(b) *Consequently, we have the recurrent formula for* $\bar{Q}_n(x)$ *and* $Q_n(x)$ $(n \in \mathbb{N}^+)$

$$\bar{Q}_1(x) = \sum_{k=1}^{\infty} x_k, \tag{7.11}$$

$$\bar{Q}_n(x) = \bar{\Lambda}(S_{n-1}(\bar{Q}_1(x), \bar{Q}_2(x), \ldots, \bar{Q}_{n-1}(x))) \tag{7.12}$$

and

$$Q_1(x) = \sum_{k=1}^{\infty} x_k, \tag{7.13}$$

$$Q_n(x,t) = \Lambda(S_{n-1}(Q_1(x), Q_2(x), \ldots, Q_{n-1}(x))), \tag{7.14}$$

respectively.

One natural question one may ask is whether or not the invariants $\bar{\Omega}(T)$, $\Omega(T)$, $\bar{K}(T)$ and $K(T)$ distinguish rooted forests. The answers for the strict order polynomials $\bar{\Omega}$ and order polynomial Ω are well known to be negative. (See, for example, Exercise 3.60 in [10].) For the quasi-symmetric polynomial invariants $\bar{K}$ and K, the answers seem to be positive, but we do not know any proof in literature.

One remark is that the invariant Ψ defined by Algorithm 3.1 can also be extended to the set of finite posets by a more general recurrent procedure. This will be done in the appearing paper [9]. But for the corresponding generating function

$$V(q) = \sum_{P} \frac{\Psi(P)}{\alpha(P)} q^{v(P)}, \tag{7.15}$$

where the sum runs over the set of all finite posets P, it is not clear what the generalization of Eq. (5.2) satisfied by $V(q)$ should be. This is unknown even for the case of (strict) order polynomials.

8. Generalization to labeled planar forests

In this section, we first generalize the construction of the invariant Ψ defined by Algorithm 3.1 for rooted forests to *labeled planar forests* and then consider its certain relationships with the Hopf algebra $\mathscr{H}^D_{P,R}$ in [3] spanned by labeled planar forests.

Once for all, we fix a non-empty finite or countable set D. By a labeled planar rooted tree T, we always mean in this section a rooted tree T such that each vertex of T is assigned a unique element of D and set of all children of any single vertex of T is an ordered set. A labeled planar rooted forest F is an ordered set of finitely many labeled planar rooted trees. We let $\mathbb{T}^D_{P,R}$ denote the set of all labeled planar rooted trees and $\mathbb{F}^D_{P,R}$ the set of all labeled planar rooted forests. For any labeled planar rooted forest $F = T_1T_2\dots T_d$, with $T_i \in \mathbb{T}^D_{P,R}$ $(1 \leqslant i \leqslant d)$ and $\alpha \in D$, we define $B^\alpha_+(F) = B^\alpha_+(T_1T_2\dots T_d)$ to be the labeled planar rooted tree obtained by connecting the root of each T_i to a α-labeled vertex v by an edge and set the new vertex v to be the root of this new labeled planar rooted tree.

We also fix an associative (not necessarily commutative) algebra A over a field k and $\{\Xi_\alpha \mid \alpha \in D\}$ a sequence of linear operators of A. Now we define an A-valued invariant $\Psi(F)$ for labeled planar forests F by the following algorithm.

Algorithm 8.1.

(1) For any labeled planar rooted tree $T \in \mathbb{T}$, we define $\Psi(T)$ as follows.
  (i) For each α-labeled leaf v of T, set $N_v = \Xi_\alpha(1)$.
  (ii) For any other vertex v of T, define N_v inductively starting from the highest level by setting $N_v = \Xi_\alpha(N_{v_1}N_{v_2}\dots N_{v_k})$, where α is the label of v and $(v_1, v_2, \dots, v_k)$ are the ordered children of v.
  (iii) Set $\Psi(T) = N_{\mathrm{rt}_T}$.
(2) For any labeled planar rooted forest $F = T_1T_2\dots T_m$, where T_i $(i = 1, 2, \dots, m)$ are connected components of F, we set

$$\Psi(F) = \Psi(T_1)\Psi(T_2)\dots\Psi(T_m). \tag{8.1}$$

Note that the order in the product in Eq. (8.1) must be same as the one in the expression $F = T_1T_2\dots T_m$.

From Algorithm 8.1, the following lemma is obvious.

Lemma 8.2. *Let Γ be an A-valued invariant for labeled planar rooted forests $\mathbb{F}^D_P$. Then Γ can be re-defined and calculated by Algorithm* 3.1 *for some k-linear map Ξ if and only if*

(1) *It satisfies Eq.* (8.1) *for any $F \in \mathbb{F}^D_{P,R}$.*
(2) *For any $T \in \mathbb{T}^D_{P,R}$ with $T = B^\alpha_+(T_1T_2\dots T_d)$, we have*

$$\Gamma(T) = \Xi_\alpha(\Gamma(T_1)\Gamma(T_2)\dots\Gamma(T_d)). \tag{8.2}$$

Remark 8.3. Let $\mathscr{H}^D_{P,R}$ be the vector spaces spanned by labeled planar forests. In [3], a Hopf algebra structure in $\mathscr{H}^D_{P,R}$ is given, which is a labeled planar version of Kreimer's Hopf algebra (see [6,2]) spanned by rooted forests. The product of the Hopf algebra $\mathscr{H}^D_{P,R}$ is given by the ordered disjoint union operation. We extend the map Γ defined by Algorithm 8.1 to $\mathscr{H}^D_{P,R}$ linearly and still denote it by Γ. Then it is easy to see that condition (1) in the lemma above is equivalent to saying that the map Γ is a homomorphism of algebras from $\mathscr{H}^D_{P,R}$ to A, while condition (2) is equivalent to the following equation.

$$\Gamma \circ B_+^\alpha = \Xi_\alpha \circ \Gamma. \tag{8.3}$$

Now let us consider the corresponding generating functions $U(q)$ for the invariants of Ψ defined by Algorithm 8.1.

First, for each $\alpha \in D$, we set

$$U_\alpha(q) = \sum_{T \in \mathbb{T}^D_{P,R,\alpha}} \Psi(T) q^{v(T)}, \tag{8.4}$$

where $\mathbb{T}^D_{P,R,\alpha}$ is the set of all labeled planar rooted trees with α-labeled roots. We also set

$$U(q) = \sum_{\alpha \in D} U_\alpha(q), \tag{8.5}$$

$$\Xi = \sum_{\alpha \in D} \Xi_\alpha. \tag{8.6}$$

Theorem 8.4. *The generating functions $U_\alpha(q)$ $(\alpha \in D)$ and $U(q)$ satisfy the following equations*:

$$\Xi_\alpha \frac{1}{1 - U(q)} = q^{-1} U_\alpha(q), \tag{8.7}$$

$$\Xi \frac{1}{1 - U(q)} = q^{-1} U(q). \tag{8.8}$$

First, note that the second equation follows from the first one by taking sum over the set D. The proof of the first equation is parallel to the proof of Eq. (5.2) in Theorem 5.1 but a little easier, since automorphism groups of planar labeled rooted trees are trivial. So we omit the proof here.

Remark 8.5. (1) Note that, when $|D| = 1$, $\mathbb{F}^D_{P,R}$ is same as the set $\mathbb{F}_{P,R}$ of unlabeled planar rooted forests. Hence Algorithm 8.1 gives an invariant for planar rooted forests in this case. Since the solution of Eq. (8.8) in $A[[q]]$ is unique, any equation of the form Eq. (8.8) can be solved by looking at the invariant Ψ defined by Algorithm 8.1 for planar rooted trees and its generating function $U(q)$ defined by Eq. (8.5).

(2) When $|D| = 1$ and the algebra A is commutative, for any planar rooted forest F, the invariant $\Psi(F)$ defined by Algorithm 8.1 coincides with the one defined by

Algorithm 3.1 for the underlying rooted forest of F, which is obtained by simply ignoring the planar structure of F.

Next, we discuss certain relationships of the invariants Ψ defined by Algorithm 8.1 with the Hopf algebra $\mathscr{H}^D_{P,R}$ defined and studied in [3]. Even though, the links present here have no obvious logical implication one way or the other, they provide a new point of view to the invariants Ψ defined by Algorithm 8.1. Besides the Hopf algebra $\mathscr{H}^D_{P,R}$ and its certain universal property studied in [3], we also need the Hopf algebra structure defined in [3] on the tensor algebra $T(V)$ for any vector space V. Since definitions of various operations of the Hopf algebras $\mathscr{H}^D_{P,R}$ and $T(V)$ are quite involved, we will follow the notation in [3] closely and quote necessary results directly from [3]. We refer readers to [3] and references there for more details.

First, let us assume that our fixed associate algebra (A, m, η) also has a co-algebra structure with which it forms a bi-algebra $(A, m, \eta, \Delta, \varepsilon)$. We further assume that the linear operators Ξ_α $(\alpha \in D)$ are 1-cocycles, i.e. they satisfy the following equation:

$$\Delta \circ \Xi_\alpha = \Xi_\alpha \otimes 1 + (\mathrm{id} \otimes \Xi_\alpha) \circ \Delta. \tag{8.9}$$

By the universal property of the Hopf algebra $\mathscr{H}^D_{P,R}$ given in Theorem 24 in [3], there exists a unique homomorphism of bi-algebras $\varphi : \mathscr{H}^D_{P,R} \to A$ such that

$$\varphi \circ B^+_\alpha = L_\alpha \circ \varphi. \tag{8.10}$$

Note that the map $\varphi : \mathscr{H}^D_{P,R} \to A$ gives an A-valued invariant for labeled planar forests.

Proposition 8.6. *The A-valued invariant $\varphi(F)$ defined above belongs to the family of invariants of labeled planar forests defined by Algorithm* 8.1 *with the linear operators L_α $(\alpha \in D)$.*

In other words, in this special situation, the invariant Ψ by Algorithm 8.1 coincides with the unique map φ guaranteed by the universal property of the Hopf algebra $\mathscr{H}^D_{P,R}$.

Proof. Since the homomorphism φ preserves the algebra products and satisfies Eq. (8.3), the proposition follows immediately from Remark 8.3 and Lemma 8.2. □

One remark is that Algorithm 8.1 does not depends on whether the algebra A has a bi-algebra structure. It only depends on the associate algebra structure of A. But, on the other hand, it is shown in [3] that the tensor algebra $T(V)$ of any vector space V has a Hopf algebra structure. In particular, we have a Hopf algebra structure on the tensor algebra $T(A)$. Next we show that, by using the linear operators Ξ_α $(\alpha \in D)$ and the associate algebra structure of A, we can construct a family of 1-cocycles L_α $(\alpha \in D)$ of the Hopf algebra $T(A)$. Therefore, the corresponding unique map $\varphi : \mathscr{H}^D_{P,R} \to T(A)$ does give us a family of $T(A)$-valued invariants for labeled planar forests.

First, for any $\alpha \in D$, we define a linear map from $\tilde{\Xi}_\alpha : T(A) \to A$ by setting

$$\tilde{\Xi}_\alpha : k \to A,$$

$$a \to a\Xi(1_A)$$

and, for any $n \geqslant 1$,

$$\tilde{\Xi}_\alpha : A^{\otimes n} \to A,$$

$$v_1 \otimes v_2 \cdots \otimes v_n \to \Xi_\alpha(v_1 \cdot v_2 \cdots v_n)$$

and extend it linearly to $T(A)$. Note that here we use 1_A for the identity element of the algebra A to distinguish the identity element 1_k in the ground field k.

Next, we define a sequence linear maps $\{L_\alpha : T(A) \to T(A) \,|\, \alpha \in D\}$ by setting

$$L_\alpha(a) = a\,\Xi(1_A) \quad \text{for any } a \in k \text{ and} \tag{8.11}$$

$$\begin{aligned} &L_\alpha(v_1 \otimes v_2 \cdots \otimes v_n) \\ &\quad = \sum_{j=1}^{n-1} v_1 \otimes v_2 \cdots \otimes v_j \otimes \tilde{\Xi}_\alpha(v_{j+1} \otimes \cdots \otimes v_n) \\ &\quad\quad + \tilde{\Xi}_\alpha(v_1 \otimes \cdots \otimes v_n) + v_1 \otimes v_2 \cdots \otimes v_n \otimes \tilde{\Xi}_\alpha(1), \\ &\quad = \sum_{j=1}^{n-1} v_1 \otimes v_2 \cdots \otimes v_j \otimes \Xi_\alpha(v_{j+1} \cdot v_{j+2} \cdots v_n) \\ &\quad\quad + \Xi_\alpha(v_1 \cdot v_2 \cdots v_n) + v_1 \otimes v_2 \cdots \otimes v_n \otimes \Xi_\alpha(1) \end{aligned} \tag{8.12}$$

and extend it linearly to $T(A)$.

By Proposition 72 in [3], the linear maps $L_\alpha : T(A) \to T(A)$ are 1-cocycles of the Hopf algebra $T(A)$. By the universal property of the Hopf algebra $\mathscr{H}_{P,R}^D$ given in Theorem 24 in [3], there exists a unique homomorphism of Hopf algebras $\varphi : \mathscr{H}_{P,R}^D \to T(A)$ such that

$$\varphi \circ B_\alpha^+ = L_\alpha \circ \varphi. \tag{8.13}$$

Note that the map $\varphi : \mathscr{H}_{P,R}^D \to T(A)$ gives a $T(A)$-valued invariant for labeled planar forests, which, by Proposition 8.6, is same as the one defined by Algorithm 8.1 with the linear operators L_α $(\alpha \in D)$.

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