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On degenerations between preprojective modules over wild quivers

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Abstract

We study minimal degenerations between preprojective modules over wild quivers. Asymptotic properties of such degenerations are studied, with respect to codimension and numbers of indecomposable direct summands. We provide families of minimal disjoint degenerations of arbitrary codimension for almost all wild quivers and show that no such examples exist in the remaining cases.

Keywords: Preprojective component; Degenerations; Wild quivers

1. Introduction

Given a finite connected quiver Q and a fixed dimension vector, all representations of Q of this dimension vector are naturally parametrised by tuples of matrices representing the action of the arrows of Q. They form a vector space, on which a product of general linear groups acts in such a way that the orbits correspond to the isomorphism classes of the representations. If the orbit closure of M contains the orbit of N then we say that M degenerates into N or equivalently that N deforms into M. This relation induces an interesting partial order on the

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set of isomorphism classes which plays an important role in representation theory. While there have been substantial advances like Zwara's module theoretic characterisation of degenerations [10], there is still no efficient algorithm available to compute the closures of the orbit or the degenerations of a module.

In general, a consequence of the degeneration of M into N is that for any representation U the dimension of the homomorphism space from U to M is at most the dimension of the homomorphism space from U to N. But for certain special classes of representations, called preprojective representations (to be defined in Section 2) we have an equivalence by [3]. Hence, we can assume without loss of generality that M and N are disjoint, i.e. have no direct summands in common.

It is shown in [2] that if N is a minimal degeneration of M, with M and N disjoint, then N is the direct sum of exactly two indecomposables U and V. Here minimal means that there is no proper degeneration of M of which N is a proper degeneration. Therefore, to investigate all minimal disjoint degenerations, it suffices to investigate all deformations of $U \oplus V$ for any pair of indecomposables U and V.

For a quiver of finite representation type this is a finite problem. It has been solved with the aid of a computer in [7]. In the case of tame representation type one can still classify the degenerations by combining a periodicity theorem with computer calculations (see [4]). This cannot be expected for wild representation type. Instead, we turn our attention in this paper to phenomena which occur only for wild quivers.

Theorem 3.1 reveals such a phenomenon: We consider the preprojective indecomposable modules U and V as vertices in the preprojective component of the Auslander–Reiten quiver (see [1]) as a graph. In Section 2 we introduce a notion of distance for preprojective indecomposable modules. For any wild quiver and any number i there is a natural number $K(i)$ such that for any pair U, V of preprojective indecomposable modules of a distance greater than $K(i)$ in the Auslander–Reiten quiver the number of direct summands of each deformation M of $U \oplus V$ is at least $i + 1$.

Theorem 3.2 is concerned with the codimensions of minimal disjoint degenerations: while there is a bound of one for quivers of finite type and a bound of two for quivers of tame type, there exist minimal disjoint degenerations of arbitrarily high codimension for wild quivers except for linear quivers with exactly one arrow doubled (denoted by $K_{m,n}$ in Section 3). For those the codimension is at most two.

The proof of Theorem 3.1 is brief and conceptual. It is based on the fact that the kernel of the generalised Cartan matrix of any wild quiver does not contain any componentwise nonnegative vector different from zero. In contrast, even the statement of Theorem 3.2 already consists of several cases. The proof requires distinguishing between even more cases which all need different treatment. But most of the proofs adhere to the same idea so that we only discuss the case of an extended $\tilde{D}_n$-quiver.

In Section 2 we recall some facts about representations and their deformations. We prove Lemma 2.1 about vector subspaces in which all vectors have both positive and negative components, and we formalise a concept already used by Riedtmann in [9] which we denote by deformation shape. Both theorems mentioned above and the proof of Theorem 3.1 are the content of Section 3, while the whole of the fourth section is about the proof of Theorem 3.2.

The behaviour of the deformations gives rise to further interesting questions which are still open: All minimal deformation shapes we find are bounded by 6 but there is no obvious reason why they should be bounded at all. Also any minimal deformation shape seems to be already determined by its support.

2. Basic notions and notation

Throughout this paper k denotes an algebraically closed field of arbitrary characteristic. For basic notions and notation we refer to [1]. By *quiver* we always mean a connected directed graph $\mathcal{Q}$ without oriented cycles that consists of finite sets of vertices $\mathcal{Q}_0$ and arrows $\mathcal{Q}_1$. This implies that the path algebra $k\mathcal{Q}$ has finite k-dimension. We identify representations of the quiver and $k\mathcal{Q}$-modules and refer to both as *$\mathcal{Q}$-modules*. The *preprojective indecomposables* are exactly the indecomposable $\mathcal{Q}$-modules we obtain by applying the transpose of the dual finitely often to a projective indecomposable $\mathcal{Q}$-module, or equivalently, all the $\mathcal{Q}$-modules in the *preprojective component* $\mathcal{Z}$ of the Auslander–Reiten quiver. On the preprojective indecomposables a partial order is given by setting $U \preccurlyeq V$ if and only if there is a path from U to V inside $\mathcal{Z}$. The length of the shortest path from U to V taken as points in the Auslander–Reiten quiver is called the *distance* of U and V. The set $[U, V]$ is the set of all W such that $U \preccurlyeq W \preccurlyeq V$. In the whole paper we only consider the full subcategory $\mathcal{P}(\mathcal{Q})$ of mod $k\mathcal{Q}$ which we call *preprojective modules*: These are all finite direct sums of the preprojective indecomposable $\mathcal{Q}$-modules. We abbreviate the dual of the transpose of a $\mathcal{Q}$-module M by τM. Then $\tau^{-1}M$ is the transpose of the dual of the $\mathcal{Q}$-module M. Let P_p be the projective cover of the simple $\mathcal{Q}$-module associated to a vertex p of $\mathcal{Q}$.

We introduce the abbreviation $\operatorname{pre}(U)$ for a preprojective indecomposable $\mathcal{Q}$-module U: If U is nonprojective then we set $\operatorname{pre}(U)$ equal to the middle term of the Auslander–Reiten sequence (see [1]) ending in U. If U is projective then we set $\operatorname{pre}(U)$ equal to $\operatorname{rad} U$.

The matrix $C \in \mathbb{Z}^{\mathcal{Q}_0} \times \mathbb{Z}^{\mathcal{Q}_0}$ defined by

$$C_{pq} := \begin{cases} 2, & p = q, \\ -n(p,q) - n(q,p), & \text{otherwise.} \end{cases}$$

where we use $n(x, y)$ to refer to the number of arrows from x to y in $\mathcal{Q}$, is called *generalised Cartan matrix of* $\mathcal{Q}$. For example, if $\mathcal{Q}$ is of type A, D or E, this is the Cartan matrix as in [6]. For any $\mathcal{Q}$ the definition matches the more general one in [5].

It is a well-known fact stated e.g. in [5] that the kernel of C for any wild quiver is a *mixed subspace* of $\mathbb{R}^{\mathcal{Q}_0}$, meaning that every nonzero element of the kernel has positive as well as negative components. We will need the following remark:

Lemma 2.1. *Let $U \subset \mathbb{R}^n$ be a mixed subspace. For every $x \in \mathbb{R}^n$ the set $M = \{x + u \mid u \in U, x_k + u_k \geqslant 0 \text{ for all } k \in \{1, \ldots, n\}\}$ is bounded.*

Proof. Assume that $(v(i))_{i\in\mathbb{N}_0}$ is a sequence in M satisfying $\lim_{i\to\infty} \|v(i)\| \to \infty$ where we take $\|\cdot\|$ to be the maximum norm of $\mathbb{R}^n$ for simplicity. Set $u(i) := v(i) - x$ and observe that for any component $u(i)_j$ of an $u(i)$ the component $-x_j$ is a lower bound. Hence the bounded sequence $\frac{1}{\|u(i)\|}u(i)$ in U has a convergent subsequence which converges to a nonzero element of U containing only nonnegative components. $\square$

For any nonnegative integer i there is a wild quiver whose Cartan matrix has a kernel of dimension i. The situation is better in the tame case: the kernel of the generalised Cartan matrix of a tame quiver $\mathcal{Q}$ is always one-dimensional and can be generated by a vector $\underline{n}_{\mathcal{Q}} \in \mathbb{N}_0^{\mathcal{Q}_0}$ which has 1 as a component. We call $\underline{n}_{\mathcal{Q}}$ the *null-root of* $\mathcal{Q}$. The Coxeter transformation of $\mathcal{Q}$ does not change $\underline{n}_{\mathcal{Q}}$ and has finite order, the so-called *Coxeter number of* $\mathcal{Q}$, on the quotient $\mathbb{R}^{\mathcal{Q}_0}/\mathbb{R}\underline{n}_{\mathcal{Q}}$.

A *slice of* $\mathcal{Z}$ is a connected full subquiver of $\mathcal{Z}$ which contains exactly one indecomposable of the τ-orbit of any projective indecomposable. We introduce some abbreviations for those types of slices we frequently use: By $R^q(\tau^{-i}P_j)$ we denote the uniquely determined slice with $\tau^{-i}P_j$ as its only source. The slice with the only sink $\tau^{-i}P_j$ is called $R^s(\tau^{-i}P_j)$, provided it exists. For a function $\delta : \mathcal{Z} \to \mathbb{Z}$, we abbreviate by $\delta_R(p)$ the value $\delta(\tau^{-i}P_p)$ of the uniquely determined indecomposable of the τ-orbit of P_p that belongs to R.

Given a slice R with a source $\tau^{-i}P_p$, the slice determined by the points $(R \setminus \{\tau^{-i}P_p\}) \cup \{\tau^{-(i+1)}P_p\}$ is denoted by $\sigma_p R$ and is called the *reflection of R at p*.

For a quiver $\mathcal{Q}$ and a dimension vector $d \in \mathbb{N}_0^{\mathcal{Q}_0}$ we define the representation variety to be the affine k-variety $\mathcal{D}(\mathcal{Q}, d) := \prod_{\alpha \in \mathcal{Q}_1} k^{e(\alpha)\times s(\alpha)}$, where $e(\alpha)$ denotes the ending vertex of the arrow α and $s(\alpha)$ its starting vertex. The points of $\mathcal{D}(\mathcal{Q}, d)$ correspond to $\mathcal{Q}$-modules of dimension vector d together with a basis for each of its vector spaces. The group $G(d) := \prod GL(d_i)$ acts on $\mathcal{D}(\mathcal{Q}, d)$ by conjugation such that there is a bijection between the orbits and the isomorphism classes of $\mathcal{Q}$-modules that have dimension vector d. We observe that the isotropy group of $G(d)$ at each point p is isomorphic to the group of automorphisms of the corresponding $\mathcal{Q}$-module M. If the orbit of N is contained in the closure of the orbit of M then we say that M *degenerates* into N or N *deforms* into M or shortly $M \leqslant_{\deg} N$. As Riedtmann has observed [9] this implies $\langle M, U\rangle \leqslant \langle N, U\rangle$ for all $\mathcal{Q}$-modules U, where $\langle X, Y\rangle$ is defined to be the k-dimension of $\mathrm{Hom}_{\mathcal{Q}}(X, Y)$. We set the *codimension* of the degeneration of M into N equal to the difference of the dimensions of the orbits of M and N. By a standard dimension calculation we get that $\dim G(d)p = \dim G(d) - \langle M, M\rangle$, hence $\langle N, N\rangle - \langle M, M\rangle$ is equal to the codimension of the degeneration M in N. We set $M \leqslant N$ if and only if $\langle M, U\rangle \leqslant \langle N, U\rangle$ for all $\mathcal{Q}$-modules U. By a theorem of Auslander [1], this is a partial order but in general it differs from the degeneration order (see [9] for an example).

But if M and N are preprojective and have the same dimension vector it has been proven in [3] that both orders coincide. Hence to consider minimal degenerations we can safely assume M and N to have no direct summands in common. Furthermore, it is shown in [3] that N can be taken to be $U \oplus V$ for two indecomposables U, V with the property $U \preccurlyeq V$. We develop a notion which is suitable for our calculations:

Definition 2.2. For a quiver $\mathcal{Q}$ and indecomposable preprojective $\mathcal{Q}$-modules U and V with the property $\tau^{-1}U \preccurlyeq V$ we call a function $\delta : \mathcal{P}(\mathcal{Q}) \to \mathbb{N}_0$ a *deformation shape of U and V* if it satisfies the following conditions:

- For each $X, Y \in \mathcal{P}(\mathcal{Q})$ we have $\delta(X \oplus Y) = \delta(X) + \delta(Y)$.
- We have for any indecomposable $\mathcal{Q}$-module W: If $\delta(W) > 0$ then $U \preccurlyeq W \preccurlyeq \tau V$.
- For any nonprojective indecomposable $\mathcal{Q}$-module W isomorphic neither to U nor to V the *subadditivity inequation* $s(\delta, W) := \delta(\mathrm{pre}(W)) - \delta(\tau W) - \delta(W) \geqslant 0$ holds.
- For any projective indecomposable $\mathcal{Q}$-module W isomorphic neither to U nor to V the *subadditivity inequation* $s(\delta, W) := \delta(\mathrm{pre}(W)) - \delta(W) \geqslant 0$ holds.
- We have $\delta(U) = \delta(\tau V) = 1$.

Let δ and δ' be deformation shapes for U and V. We use the term $\delta \leqslant \delta'$ if and only if $\delta(W) \leqslant \delta'(W)$ for any indecomposable W. This induces a partial order which we refer to by *deformation order*. A deformation shape δ is defined to be *subadditive at W* if and only if $s(\delta, W) > 0$ and *strictly additive at W* otherwise.

Lemma 2.3.

(a) *There is a bijection from the set of deformation shapes of U and V to the set of $\mathcal{Q}$-modules M satisfying $M < U \oplus V$ which maps a deformation shape δ onto*

$$M_\delta := \bigoplus_{W \in \mathcal{Z},\, W \ncong U, V} W^{s(\delta, W)}$$

and whose inverse assigns to a $\mathcal{Q}$-module M the function

$$\delta_M := \bigl(N \mapsto \langle U \oplus V, N\rangle - \langle M, N\rangle\bigr).$$

(b) *$M \leqslant M'$ if and only if $\delta_{M'} \leqslant \delta_M$ for δ_M, $\delta_{M'}$ defined as in* (a).

(c) *The codimension of any minimal degeneration of M into $U \oplus V$ is $\delta_M(M) + 1$. It is also called codimension of δ_M.*

Proof. (a) For any δ and any indecomposable W isomorphic to neither U nor V each $s(\delta, W)$ is nonnegative, hence M_δ is well defined. For each M the deformation shape δ_M is well defined because $M < U \oplus V$ implies $\langle U \oplus V, X\rangle \geqslant \langle M, X\rangle$ for any preprojective $\mathcal{Q}$-module X by Theorem 3.3 in [3].

By the defining properties of an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ we have that for an indecomposable $\mathcal{Q}$-module X any nonsplit morphism $X \to C$ factors through B. This implies that $\langle X, A\rangle + \langle X, C\rangle - \langle X, B\rangle = 0$ for $X \ncong C$ and $\langle C, A\rangle + \langle C, C\rangle - \langle C, B\rangle = 1$. Hence $\langle N, A\rangle + \langle N, C\rangle - \langle N, B\rangle$ is equal to the multiplicity of C as a direct summand of N. We conclude that $s(\delta_M, W)$ is equal to the multiplicity of W as a direct summand of M, hence $M_{\delta_M} \cong M$.

To show $\delta_{M_\delta} = \delta$ assume $\delta_{M_\delta} \neq \delta$ and take a $\preccurlyeq$-minimal indecomposable $\mathcal{Q}$-module X with $\delta_{M_\delta}(X) \neq \delta(X)$. Then

$$\begin{aligned}\delta_{M_\delta}(X) &= \delta_{M_\delta}\bigl(\mathrm{pre}(X)\bigr) - \delta_{M_\delta}(\tau X) - s(\delta_{M_\delta}, X)\\ &= \delta\bigl(\mathrm{pre}(X)\bigr) - \delta(\tau X) - s(\delta, X) = \delta(X)\end{aligned}$$

yields the required contradiction.

(b) M degenerates into M' if and only if $\langle M, X\rangle \leqslant \langle M', X\rangle$ holds for any preprojective $\mathcal{Q}$-module X by Theorem 3.3 in [3].

(c) If M degenerates into $U \oplus V$ minimally then there is an exact sequence $0 \to U \to M \to V \to 0$ by Theorem 4.1 in [3]. Applying $\mathrm{Hom}(U, \_)$ we get $\langle U, M\rangle = \langle U, U\rangle + \langle U, V\rangle$ by using that $\mathrm{Ext}^1(U, U) = 0$. Now (a) enables us to compute $\langle U \oplus V, U \oplus V\rangle - \langle M, M\rangle = \langle V, V\rangle + \delta_M(M)$. $\square$

We collect some useful observations about deformation shapes. For this purpose we need some further notation: Consider a quiver $\mathcal{Q}$ and let $\mathcal{Q}'$ be a full subquiver which is not of finite representation type. The preprojective component $\mathcal{Z}'$ of the Auslander–Reiten quiver of $\mathcal{Q}'$ can be embedded naturally as a full subquiver into the preprojective component $\mathcal{Z}$ of the Auslander–Reiten quiver of $\mathcal{Q}$. Given two indecomposables $U' \preccurlyeq V'$ in $\mathcal{Z}'$ which are mapped to $U, V \in \mathcal{Z}$ via the embedding and a deformation shape δ' of U' and V' we define $\delta'_! : \mathcal{Z} \to \mathbb{N}_0$ to be the extension of δ' by zero.

Corollary 2.4.

(a) *Every deformation shape δ of $U \oplus V$ has connected support.*
(b) *Assigning to each δ' the deformation shape $\delta'_!$ yields a bijection from the deformation shapes of U' and V' to the deformation shapes of U and V which have support only in the image of $\mathcal{Z}'$. It preserves the deformation order and the codimension.*
(c) *Let $p \in \mathcal{Q}_0$ be a sink and let $\tilde{\mathcal{Q}}$ be a reflection of $\mathcal{Q}$ at p. Then $M \leqslant U \oplus V$ for $U, V, M \in \mathcal{P}(\tilde{\mathcal{Q}})$ is a degeneration if and only if the images under the reflection functor in $\mathcal{P}(\mathcal{Q})$ form a degeneration. This correspondence preserves the deformation order and codimension.*

Proof. (a) For any indecomposable $\mathcal{Q}$-module X which is $\preccurlyeq$-maximal with $\delta(X) > 0$, we have $s(\delta, \tau^{-1}X) < 0$.

(b) Use Lemma 2.3 for the deformation order and codimension.

(c) We assign to each vertex of the preprojective component $\tilde{\mathcal{Z}}$ of $\tilde{\mathcal{Q}}$ marked by $X \in \mathcal{P}(\tilde{\mathcal{Q}})$ the vertex of $\mathcal{Z}$ marked by the image of X under the reflection functor. This yields an embedding of $\tilde{\mathcal{Z}}$ into $\mathcal{Z}$ as a full subquiver. Now proceed as above. □

3. Some special behaviour of wild quivers

We define for any $\mathcal{Q}$-module M its *number of blocks* $\mu(M)$ to be the number of indecomposable nonzero direct summands it can be decomposed into.

Theorem 3.1. *Let $\mathcal{Q}$ be a wild quiver. We define a function $K : \mathbb{N}_0 \to \mathbb{N}_0$ by setting*

$$K(j) := \min\bigl\{\mu(M) \bigm| M \leqslant U \oplus V,\ U \text{ and } V \text{ have distance at least } j\bigr\}.$$

Then K rises monotonically and is unbounded.

Proof. We denote the set of all degenerations between preprojective $\mathcal{Q}$-modules $\{(M, U, V) \mid M \leqslant U \oplus V\}$ by D. Let D_i be the set $\{(M, U, V) \in D \mid \mu(M) \leqslant i\}$.

We define a function $t : D \to \mathbb{Z}^{\mathcal{Q}_0}$ and a function $v : D \to \mathbb{N}_0^{\mathcal{Q}_0}$ as follows: We extend the notation $s(\delta_M, \_)$ used in Definition 2.2 by setting $s(\delta_M, U) = s(\delta_M, V) = -1$. Now for given U, V, M with $M \leqslant U \oplus V$ we set $t((M, U, V))_p := \sum_{l \in \mathbb{N}_0} s(\delta_M, \tau^{-l} P_p)$. We define $v((M, U, V))_p := \sum_{l \in \mathbb{N}_0} \delta(\tau^{-l} P_p)$.

Now $t(D_i)$ is a finite set for any i: Any component of any $t \in t(D_i)$ is greater or equal than -2 and the sum over all components $\sum_{p \in \mathcal{Q}_0} t((M, U, V))_p$ is equal to $\mu(M) - 2$.

We always get $t((M, U, V)) = -Cv((M, U, V))$ with C the Cartan matrix of $\mathcal{Q}$: By first applying the definition we obtain for each $p \in \mathcal{Q}_0$

$$t\bigl((M, U, V)\bigr)_p = \sum_{l \in \mathbb{N}_0} \delta_M\bigl(\mathrm{pre}\bigl(\tau^{-l} P_p\bigr)\bigr) - 2 \sum_{l \in \mathbb{N}_0} \delta_M\bigl(\tau^{-l} P_p\bigr)$$

$$
\begin{aligned}
&= \delta_M\bigl(P_q^{\sum_{q\in Q_0\setminus\{p\}} n(p,q)}\bigr) \\
&\quad + \sum_{l\in\mathbb{N}_0\setminus\{0\}} \delta_M\bigl(\tau^{-l} P_q^{\sum_{q\in Q_0\setminus\{p\}} n(p,q)} \oplus \tau^{-l+1} P_q^{\sum_{q\in Q_0\setminus\{p\}} n(q,p)}\bigr) \\
&\quad - 2\sum_{l\in\mathbb{N}_0} \delta_M\bigl(\tau^{-l} P_p\bigr) \\
&= \sum_{q\in Q_0\setminus\{p\}} \bigl(n(p,q)+n(q,p)\bigr) \sum_{l\in\mathbb{N}_0} \delta_M\bigl(\tau^{-l} P_q\bigr) - 2\sum_{l\in\mathbb{N}_0} \delta_M\bigl(\tau^{-l} P_p\bigr).
\end{aligned}
$$

We obtain the second equation by using that τ preserves Auslander–Reiten sequences and by using the structure of rad P_p.

We conclude that $v(D_i)$ is a finite set for any i: By using that the kernel of the Cartan matrix of a wild quiver is always a mixed subspace [5] and Lemma 2.1 we obtain that for any given t there are only finitely many $v \in \mathbb{N}_0^{Q_0}$ satisfying $t = -Cv$.

Thus we obtain for each i an upper bound for the possible distance of U and V for all $(M, U, V) \in D_i$: Given a $w \in \mathbb{N}_0^{Q_0}$, for all (M, U, V) fulfilling $v((M, U, V)) = w$ the number $\hat{w} := \sum_{p\in Q_0} w_p + 1$ is an upper bound for the possible distance of U and V. Hence $\max\{\hat{w} \mid w \in v(D_i)\}$ is an upper bound for the distance of any U and V such that $(M, U, V) \in D_i$. □

Theorem 3.2. *For any wild quiver Q we have*:

(a) *There is a global bound on the codimension of any minimal disjoint degeneration of Q if and only if the underlying undirected graph of Q is one of*

$$K_{m,n} := a_m — \ldots — a_1 = b_1 — \ldots — b_n$$

for arbitrary $m, n > 0$. In this case the bound on the codimension is 2.

(b) *Suppose Q contains as a full subquiver a wild quiver in which each pair of vertices is connected by at most two arrows and Q is none of the quivers $K_{m,n}$. Then minimal disjoint degenerations of arbitrary codimension occur in $\mathcal{P}(Q)$.*

Proof. We first give an overview how the proof proceeds: We have to differentiate by three different cases. First we treat any quiver Q such that the underlying undirected graph of Q is a $K_{m,n}$. This proves the "if" of (a).

Then we consider quivers whose underlying unoriented graph is of the form

$$V_m := p \overset{\cdots}{=} q$$

consisting of two vertices and m arrows between them. We show that these admit minimal disjoint degenerations of arbitrarily high codimension. For any quiver Q having a V_m as a full subquiver, Corollary 2.4 then proves that Q admits minimal disjoint degenerations of arbitrarily high codimension.

Finally we give a list of quivers such that any remaining wild quiver contains at least one of the quivers from the list as a full subquiver. Again by Corollary 2.4, it suffices to give minimal disjoint degenerations of arbitrary codimension for the quivers from the list to prove (b) and the "only if" of (a).

To show that the codimension of all minimal deformation shapes of $K_{m,n}$ is bounded by 2 we take a closer look at them: By Corollary 2.4 without loss of generality, we can fix an orientation such that b_1 is the only sink. First fix indecomposables $U = P_{a_p}$ and $V = \tau^{-r} P_{a_q}$ such that $[U, V]$ has nonempty intersection with the τ-orbits of both P_{a_1} and P_{b_1}. Denote by W the $\preccurlyeq$-minimal $\mathcal{Q}$-module in $[U, V]$ which is in one of these τ-orbits and by W' the $\preccurlyeq$-maximal $\mathcal{Q}$-module satisfying these conditions. Then the deformation shape $\delta : \mathcal{P}(\mathcal{Q}) \to \mathbb{N}_0$ defined by

$$\delta : X \mapsto \begin{cases} 1, & U \preccurlyeq X \preccurlyeq W, \\ 1, & W' \preccurlyeq X \preccurlyeq V, \\ 1, & X \in \{\tau^{-i} P_{a_1} \mid i \in \mathbb{N}_0\} \cap [U, V], \\ 1, & X \in \{\tau^{-i} P_{b_1} \mid i \in \mathbb{N}_0\} \cap [U, V], \\ 0, & \text{otherwise}, \end{cases}$$

is minimal by construction. For any deformation shape δ' of U and V we observe that $\delta'(P_{a_{p+j}}) \geqslant \delta'(P_{a_{p+j+1}})$ and by induction using the subadditivity inequation $\delta'(\tau^k P_{a_j}) \geqslant \delta'(\tau^k P_{a_{j+1}})$ for $\tau^k P_{a_j}, \tau^k P_{a_{j+1}} \in [U, \tau V]$. Hence we have $\delta'(X) \geqslant 1$ for any $U \preccurlyeq X \preccurlyeq W$ and we get in a similar way $\delta'(X) \geqslant 1$ for any X such that $\delta(X) = 1$. It follows that any deformation shape δ' of U and V is larger than δ. The cases of pairs $U = P_{a_p}$, $V = P_{b_q}$ or $U = P_{b_p}$, $V = P_{a_q}$ or $U = P_{b_p}$, $V = P_{b_q}$ are treated in a similar way. If $[U, V]$ has empty intersection with the τ-orbits of P_{a_1} or P_{b_1} then the subadditivity inequalities admit only the deformation shape

$$\delta : X \mapsto \begin{cases} 1, & X \in [U, \tau V], \\ 0, & \text{otherwise}. \end{cases}$$

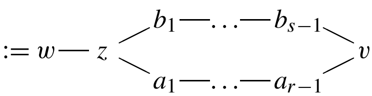
On each of the quivers V_m the only minimal deformation shape δ is given by $\delta(W) = 1$ if and only if $U \preccurlyeq W \preccurlyeq \tau V$ and $\delta(W) = 0$ everywhere else, due to the connectedness of the support of δ. When the distance between U and V increases, the codimension of the δ's increases without bound.

We turn our attention to the list of remaining quivers. Each contains at least one wild quiver as a full subquiver whose underlying unoriented graph is one of the following:

$$S := \begin{tikzcd} c & & d \\ & z \arrow[ul, dash] \arrow[ur, dash] \arrow[r, dash] \arrow[dl, dash] \arrow[dr, dash] & e \\ b & & a \end{tikzcd} \qquad \tilde{\tilde{D}}_n := \begin{tikzcd} b & & & & v \\ & z_1 \arrow[ul, dash] \arrow[dl, dash] \arrow[r, dash] & \ldots \arrow[r, dash] & z_{n-3} \arrow[ur, dash] \arrow[dr, dash] & \\ a & & & & d \arrow[r, dash] & w \end{tikzcd}$$

$$\tilde{\tilde{E}}_6 := \begin{tikzcd} v \arrow[r, dash] & b_1 \arrow[r, dash] & z \arrow[r, dash] \arrow[d, dash] & c_1 \arrow[r, dash] & c_2 \arrow[r, dash] & w \\ & & a_1 \arrow[d, dash] & & & \\ & & a_2 & & & \end{tikzcd} \qquad \tilde{\tilde{E}}_7 := \begin{tikzcd} v \arrow[r, dash] & b_2 \arrow[r, dash] & b_1 \arrow[r, dash] & z \arrow[r, dash] \arrow[d, dash] & c_1 \arrow[r, dash] & c_2 \arrow[r, dash] & c_3 \arrow[r, dash] & w \\ & & & a_1 & & & & \end{tikzcd}$$

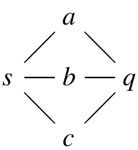
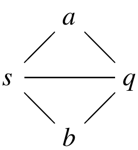
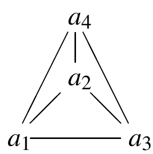
$$\tilde{\tilde{E}}_8 := \begin{tikzcd} b_2 \arrow[r, dash] & b_1 \arrow[r, dash] & z \arrow[r, dash] \arrow[d, dash] & c_1 \arrow[r, dash] & c_2 \arrow[r, dash] & c_3 \arrow[r, dash] & c_4 \arrow[r, dash] & v \arrow[r, dash] & w \\ & & a_1 & & & & & & \end{tikzcd}$$

$\tilde{\tilde{A}}_{r,s} :=$ 

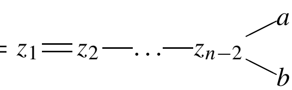
Except in case $\tilde{\tilde{A}}_{r,s}$ the reflecting operation on a quiver is transitive on all possible orientations, so we need to provide families of minimal disjoint degenerations only for one orientation for each quiver. In case $\tilde{\tilde{A}}_{r,s}$ the reflecting operation does not change the total number of arrows in each direction inside the circle, so we have to provide families for each number of arrows in a certain direction.

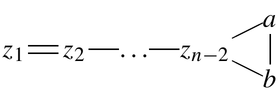
In all these cases each graph consists of the graph of a tame quiver and an additional vertex w. We will use that observation to give a family of deformation shapes, prove their minimality and calculate their codimensions. We will do this explicitly only for the quiver $\tilde{\tilde{D}}_n$ in the following section. For the other quivers in the above list one can construct such deformation shapes in a similar way.

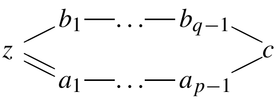
It remains to give minimal disjoint degenerations for quivers of which the underlying unoriented graphs are of the following form.

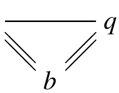
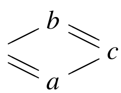
$AA_5 :=$  $AA_4 :=$  $T :=$ 

$KK_n := a_1 = a_2 — \ldots — a_{n-1} = a_n$

$KD_n :=$ 

$KA_n :=$ 

$AK_{p,q} :=$ 

$\widetilde{AK}_3 :=$  $\widetilde{AK}_4 :=$ 

These are easier to find. Details can be found in [8]. We give only one example here: Let Q be the quiver with the unoriented graph KK_3 and the only sink a_2, let m be a positive integer, $U := P_{a_2}$ and $V := \tau^{-m} P_{a_2}$. Then

$$\delta : \tau^{-i} P_j \mapsto \begin{cases} 1, & j = a_1,\ i \in \{2k \mid k \in \mathbb{N}_0,\ 2k < m-1\}, \\ 1, & j = a_2,\ i \in \{0, \ldots, m-1\}, \\ 1, & j = a_3,\ i \in \{2k+1 \mid k \in \mathbb{N}_0,\ 2k+1 < m-1\}, \\ 0, & \text{otherwise} \end{cases}$$

is a deformation shape of codimension m. Its minimality can be proven immediately by the subadditivity equations. $\square$

4. Families of minimal degenerations

We provide in this section a family of minimal deformation shapes of any codimension greater or equal 2 for quivers whose underlying unoriented graph is of the form $\tilde{\tilde{D}}_n$. Due to Corollary 2.4, such a family for some orientation can easily be transformed into a family with the same property for another orientation. Hence we fix an orientation specified by making z_1 the only sink. Let c be the Coxeter number of $\tilde{\tilde{D}}_n$. Now for given m we set $U := P_{z_1}$ and $V := \tau^{-((m-1)(c+1)+n-2)} P_{z_1}$ and define a deformation shape δ_m of U and V.

For the purpose of defining δ_m and proving its minimality we cover $[U, \tau V]$ by segments:

- the initial segment consisting of $\preccurlyeq$-predecessors of any indecomposable in $R^q(B_0^q)$,
- the final segment consisting of all $\preccurlyeq$-successors of $R^s(B_{m-1}^s)$,
- the tame segments between $R^q(B_i^q)$ and $R^s(B_{i+1}^s)$ for $0 \leqslant i \leqslant m-2$, and
- the wild segments between $R^s(B_i^s)$ and $R^q(B_i^q)$ for $1 \leqslant i \leqslant m-1$,

where $B_i^q := \tau^{-(i(c+1)+1)} P_{z_{n-3}}$ and $B_i^s := \tau^{-i(c+1)} P_{z_{n-3}}$. Inside the tame segments the deformation shapes δ_m have support only inside the τ-orbits of $\mathcal{Q}_0 \setminus \{w\}$.

Unfortunately, the proof is intricate. So we start with an overview over the strategy of the proof: The proof will start with the definition of the defect of a deformation shape at a slice. This definition extends the usual definition of the defect for tame quivers onto wild quivers that consist of a tame quiver as a full subquiver and an additional vertex w.

Now the difference between the defects of a deformation shape on the leftmost slice and the rightmost slice of a segment reveals essential properties of the deformation shape: We conclude from Lemma 4.1 that on any tame segment the defect of a deformation shape that has support only inside the τ-orbits of $\mathcal{Q}_0 \setminus \{w\}$ is on the leftmost slice always greater or equal than on the rightmost slice of this segment. Furthermore the deformation shape is strictly additive on the τ-orbits of $\mathcal{Q}_0 \setminus \{w\}$ in the tame segment if and only if these defects are equal.

An essential upper bound for the defect of the leftmost slice of the final segment can now be computed by induction: Lemma 4.2 starts the induction on the initial segment in part (a) and makes it work on the wild segments in part (b). To complete the induction, observe that in any tame segment between $R^q(B_i^q)$ and $R^s(B_{i+1}^s)$ the deformation shape δ is properly subadditive or it is subadditive at $\tau^{-((i+1)(c+1)+1)} P_v$ because $\delta \leqslant \delta_m$ and $\delta_m(\tau^{-((i+1)(c+1)+1)} P_v) = 0$. This yields that the defect of δ on each leftmost slice of a tame segment is at most 0.

To proceed, we will use the symmetry inherent to the conditions for deformation shapes to define for any deformation shape δ of U and V the reflection $\bar{\delta}$. This is again a deformation shape of U and V such that $d_{\overline{R}^\delta}(\bar{\delta}) = -d_R(\delta)$.

The proof of the minimality is completed by Lemma 4.2(c).

We start with the details of the proof. First we construct δ_m by

- setting $\delta_m(U) := 1$,
- setting $\delta_m(\tau^{-i} P_v) := \delta_m(\mathrm{pre}(\tau^{-i} P_v)) - \delta_m(\tau^{-(i-1)} P_v) - 1$ for any $\tau^{-i} P_v$ with $\tau^{-1} B_j^q \not\preccurlyeq \tau^{-i} P_v \not\preccurlyeq B_j^s$ for an arbitrary j or $\tau^{-i} P_v \not\preccurlyeq B_{m-1}^s$ or $\tau^{-1} B_0^q \not\preccurlyeq \tau^{-i} P_v$,
- setting $\delta_m(\tau^{-i} P_w) := \delta_m(\mathrm{pre}(\tau^{-i} P_w)) - \delta_m(\tau^{-(i-1)} P_w) - 1$ for any $\tau^{-i} P_w$ with $\tau^{-1} B_j^q \preccurlyeq \tau^{-i} P_w \preccurlyeq B_{j+1}^s$ for an arbitrary j or $\tau^{-i} P_w \not\preccurlyeq B_{m-1}^s$ or $\tau^{-1} B_0^q \not\preccurlyeq \tau^{-i} P_w$,

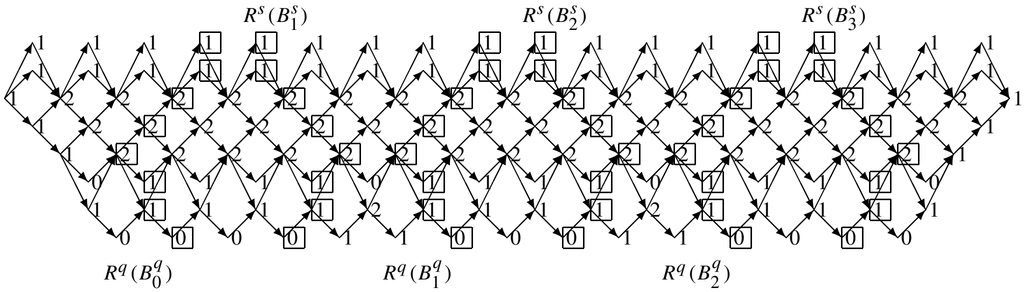


Fig. 1. A minimal deformation shape of $\tilde{\tilde{D}}_6$ having codimension 4.

- making it strictly additive at any other indecomposable $\mathcal{Q}$-module W with $\tau^{-1}W \preccurlyeq V$,
- setting it equal to zero anywhere else.

We illustrate this choice with the case $\tilde{\tilde{D}}_6$ and $m = 4$ in Fig. 1. We observe that the $\mathcal{Q}$-module M associated to δ_m has for each B_j^q one direct summand isomorphic to the $\tau^{-i}P_v$ which lies in B_j^q and that for any other direct summand W of M we get $\delta_m(W) = 0$. Hence, by Lemma 2.3(c), δ_m has codimension m.

We define the *defect* $d_R(\delta)$ *of* δ *at* R for any quiver that consists of a tame quiver plus an additional vertex w: For any slice R of $\mathcal{Z}$ and any function $\delta : \mathcal{Z} \to \mathbb{Z}$ we extend, by abuse of notation, the notion of defect $d_R(\delta)$ of δ at R from the tame quiver determined by $\mathcal{Q}_0 \setminus \{w\}$ to $\mathcal{Q}$: first we assign to each vertex $\tau^{-i}P_j$, $j \in \mathcal{Q}_0 \setminus \{w\}$ in R the weight

$$w_j := \underline{n}_{\mathcal{Q}}(j) - \sum_{k \in \mathcal{Q}_0 \setminus \{w\}} m(j,k)\underline{n}_{\mathcal{Q}}(k),$$

where $m(j,k)$ denotes the number of arrows in the Auslander–Reiten quiver from $\tau^{-i}P_j$ to $\tau^{-i'}P_k$ with $\tau^{-i'}P_k \in R$ for suitable i' for any pair (j,k). Now we set

$$d_R(\delta) := \sum_{j \in \mathcal{Q}_0 \setminus \{w\}} w_j \delta_R(j).$$

We split the proof of the minimality of δ_m into two lemmas. The first one applies to all extended tame quivers:

Lemma 4.1. *Let $\mathcal{Q}$ be a quiver of which the underlying unoriented graph is of one of the forms $\tilde{\tilde{A}}_{p,q}$, $\tilde{\tilde{D}}_n$, $\tilde{\tilde{E}}_6$, $\tilde{\tilde{E}}_7$ or $\tilde{\tilde{E}}_8$, let δ be a deformation shape for $\mathcal{Q}$, let R be a section, $p \in \mathcal{Q}_0 \setminus \{w\}$ a vertex such that $\tau^{-i}P_p$ for an appropriate i is a source inside R.*

(a) *Let C be the generalised Cartan matrix of $\mathcal{Q}$. We have*

$$d_R(\delta) - d_{\sigma_p R}(\delta) = \underline{n}_{\mathcal{Q}}(p)\Bigg(-\delta_R(p) - \delta_{\sigma_p R}(p) - \sum_{q \in \mathcal{Q}_0 \setminus \{p,w\}} C_{pq}\delta_R(q)\Bigg).$$

(b) *If p and w are connected by an arrow it follows that $d_R(\delta) - d_{\sigma_p R}(\delta) \geqslant -\underline{n}_{\mathcal{Q}}(p)\delta_R(w)$. Otherwise we have $d_R(\delta) - d_{\sigma_p R}(\delta) \geqslant 0$.*
(c) *If $d_R(\delta) = d_{\sigma_p R}(\delta)$, then every $\delta|_R$ determines $\delta|_{\sigma_p R}$ uniquely and vice versa.*

Proof. (a)

$$\begin{aligned} & d_R(\delta) - d_{\sigma_p R}(\delta) \\ &\quad = \sum_{q \in \mathcal{Q}_0 \setminus \{p,w\}} m(q,p)\underline{n}_{\mathcal{Q}}(p)\delta_R(q) \\ &\qquad + \left(\underline{n}_{\mathcal{Q}}(p) - \sum_{q \in \mathcal{Q}_0 \setminus \{w\}} m(p,q)\underline{n}_{\mathcal{Q}}(q)\right)\delta_R(p) - \underline{n}_{\mathcal{Q}}(p)\delta_{\sigma_p R}(p) \\ &\quad = C_{pp}\underline{n}_{\mathcal{Q}}(p)\delta_R(p) + \sum_{q \in \mathcal{Q}_0 \setminus \{p,w\}} C_{pq}\underline{n}_{\mathcal{Q}}(q)\delta_R(p) \\ &\qquad + \underline{n}_{\mathcal{Q}}(p)\left(-\delta_R(p) - \delta_{\sigma_p R}(p) - \sum_{q \in \mathcal{Q}_0 \setminus \{p,w\}} C_{pq}\delta_R(q)\right). \end{aligned}$$

(b) The subadditivity inequation on $\tau^{-i}P_p \in \sigma_p R$ yields

$$-\delta_R(p) - \delta_{\sigma_p R}(p) - \sum_{q \in \mathcal{Q}_0 \setminus \{p,w\}} C_{pq}\delta_R(q) - C_{pw}\delta_R(w) \geqslant 0.$$

(c) For a fixed $\delta|_R$, anything in the equation of part (a) except $\delta_{\sigma_p R}(p)$ is fixed. □

We will use the symmetry inherent to the conditions for deformation shapes to define some notation: Whenever we have chosen an orientation of a quiver $\mathcal{Q}$ without cycles such that there is only one sink called p and have a deformation shape δ of P_p and $\tau^{-(i+1)}P_p$, we define $k(q) := \max\{k \mid \tau^{-k}P_q \preccurlyeq \tau^{-i}P_p\}$ for any $q \in \mathcal{Q}_0$. Then let the *reflection of* δ be the deformation shape $\bar{\delta}$ determined by $\bar{\delta}(\tau^{-j}P_q) := \delta(\tau^{-(k(q)-j)}P_q)$. Given a slice R, the slice $\overline{R}^{\delta}$ defined by $\overline{R}^{\delta} := \{\tau^{-(k(q)-j)}P_q \mid \tau^{-j}P_q \in R\}$ will be called the *reflection of* R. Clearly $\bar{\delta}$ is a deformation shape if and only if δ is. Furthermore, we have $d_{\overline{R}^{\delta}}(\bar{\delta}) = -d_R(\delta)$.

Now we can prove the minimality of δ_m for $\mathcal{Q} = \tilde{\tilde{D}}_n$:

Lemma 4.2. *Set $\mathcal{Q}$ equal to $\tilde{\tilde{D}}_n$ with an orientation fixed by making z_1 the only sink and let δ be an arbitrary deformation shape of $U := P_{z_1}$ and $V := \tau^{-((m-1)(c+1)+n-2)}P_{z_1}$ satisfying $\delta \leqslant \delta_m$. Then the following holds*:

(a) *We have $d_{R^q(B_0^q)}(\delta) \leqslant 0$. Moreover, $d_{R^q(B_0^q)}(\delta) = 0$ implies $\delta(\tau^{-1}P_v) = 1$.*
(b) *If $d_{\sigma_v R^s(B_k^s)}(\delta) \leqslant -1$ then $d_{R^q(B_k^q)}(\delta) \leqslant 0$. In case $d_{R^q(B_k^q)}(\delta) = 0$ we get $\delta(\tau^{-(k(c+1)+1)}P_v) = 1$.*
(c) *We have $\delta = \delta_m$.*

Proof. (a) To obtain an upper bound for the defect we repeatedly add to it the subadditivity inequation for a $\preccurlyeq$-maximal indecomposable W such that $\delta(W)$ occurs in it with multiplicity greater than 0, thus making its multiplicity 0. We do not replace $\delta(\tau^{-1}P_v)$. We use the fact that $\delta(W)=0$ for any $W\notin \operatorname{supp}\delta_m$:

$$\begin{aligned} d_{R^q(B_0^q)}(\delta) &= -2\delta\big(\tau^{-1}P_{z_{n-3}}\big)+\delta\big(\tau^{-1}P_v\big)+\delta\big(\tau^{-1}P_d\big)+\delta\big(\tau^{-n-3}P_a\big)+\delta\big(\tau^{-n-3}P_b\big)\\ &\leqslant -2\delta\big(\tau^{-1}P_{z_{n-3}}\big)+\delta\big(\tau^{-1}P_v\big)+\delta(P_d)+\delta\big(\tau^{-n-3}P_a\big)\\ &\quad -\delta\big(\tau^{-n-4}P_b\big)+\delta\big(\tau^{-n-3}P_{z_1}\big)\\ &\leqslant \cdots\\ &\leqslant \delta\big(\tau^{-1}P_v\big)-\delta(P_{z_1})\leqslant 0. \end{aligned}$$

(b) As in (a) we prove the inequality

$$d_{R^q(B_k^q)}(\delta)-d_{\sigma_v R^s(B_k^s)}(\delta)\leqslant \delta\big(\tau^{-(k(c+1)+1)}P_v\big).$$

(c) We have $d_{R^q(B_k^q)}(\delta)\leqslant 0$ for all $k\in\{0,\dots,m-2\}$: use that the assumption $d_{\sigma_v R^s(B_{(k+1)}^s)}(\delta)=0$ leads to $\delta(\tau^{-(k+1)(c+1)}P_v)=\delta(\tau^{-(k(c+1)+1)}P_v)=1$ because of the strict additivity of δ provided by Lemma 4.1(b) as well as part (a) and (b) of this lemma. This contradicts $\delta\leqslant\delta_m$.

Furthermore, $\delta_m=\overline{\delta_m}$, hence $\bar\delta\leqslant\delta_m$. From $d_{R^q(B_k^q)}(\bar\delta)\leqslant 0$ for all $k\in\{0,\dots,m-2\}$ we obtain $d_{R^s(B_{(k+1)}^s)}(\delta)\geqslant 0$ for all $k\in\{0,\dots,m-2\}$. Thus the fact $d_{R^q(B_k^q)}(\delta)\leqslant 0$ can be sharpened to $d_{R^q(B_k^q)}(\delta)=0$. This shows $\delta(\tau^{-(k(c+1)+1)}P_v)=1$ for all $k\in\{0,\dots,m-2\}$. But from $d_{\sigma_v R^s(B_{(k+1)}^s)}(\bar\delta)\leqslant -1$ we obtain $d_{\sigma_v^{-1}R^q(B_k^q)}(\delta)\geqslant 1$ for all $k\in\{0,\dots,m-2\}$. Thus the subadditivity inequality is in each $\tau^{-(k(c+1)+1)}P_v$ proper subadditive while $\delta(\tau^{-(k(c+1)+1)}P_v)=1$. This forces δ to have codimension at least m. But any $\delta<\delta_m$ must have codimension strictly smaller than m. We conclude $\delta=\delta_m$. □

We sketch for the other quivers that consist of a tame quiver as a full subquiver plus an additional vertex w the construction of the families of minimal disjoint degenerations of arbitrary codimension: In the case of $\tilde{\tilde{E}}_6$ set $U:=P_z$, $V:=\tau^{-(8m-4)}P_z$, $B_i^q:=\tau^{-(8i+2)}P_z$, $B_i^s:=\tau^{-(8i+1)}P_z$ and construct δ_m in the same way as in the case $\tilde{\tilde{D}}_n$. For $\tilde{\tilde{E}}_7$ set $U:=P_z$, $V:=\tau^{-(15m-9)}P_z$, $B_i^q:=\tau^{-(15i+3)}P_z$, $B_i^s:=\tau^{-(15i+2)}P_z$. The quiver $\tilde{\tilde{E}}_8$ requires some, the quivers $\tilde{\tilde{A}}_{p,q}$ a lot of further technical adaption and can be found in [8].

The deformation shapes of a quiver with underlying unoriented graph S have a nicer property than those of $\tilde{\tilde{D}}_n$: We fix the orientation of S such that z is the only sink. To a pair $U:=P_z$, $V:=\tau^{-(2m+1)}P_z$ we can construct minimal deformation shapes of U and V to any codimension $n\in\{1,\dots,m\}$. Let $\delta_{m,n}$ be additive in any indecomposable except

- $\delta_{m,n}(U)=\delta_{m,n}(\tau V)=1$,
- $\delta_{m,n}(\tau^{-(4i+1)}P_d)=\delta_{m,n}(\tau^{-(4i+2)}P_d)=1$ for $i\in\{i\in\mathbb{N}_0\mid 2i+2\leqslant n\}$ and, if n is even, $\delta_{m,n}(\tau^{-i}P_d)=1$ for $i\in\{2n-1,\dots,2m-1\}$,
- $\delta_{m,n}(\tau^{-(4i+3)}P_e)=\delta_{m,n}(\tau^{-(4i+4)}P_e)=1$ for $i\in\{i\in\mathbb{N}_0\mid 2i+3\leqslant n\}$ and, if n is odd, $\delta_{m,n}(\tau^{-i}P_e)=1$ for $i\in\{2n-1,\dots,2m-1\}$,
- $\delta_{m,n}$ is zero for any other indecomposable in the τ-orbit of P_d or P_e.

Then $\delta_{m,n}$ is a deformation shape of U and V and has codimension m. We can prove the minimality by a lemma analogous to Lemma 4.2.

Acknowledgments

This paper answers a problem proposed by Klaus Bongartz and grew out of a Diploma thesis [8] written under his supervision. Markus Reineke has given me a lot of hints how to improve the readability.

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