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# Neutron Branching Processes

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## I. INTRODUCTION

A quantitative description of the neutron population in a nuclear reactor can be formulated mathematically in several ways. Probably the most familiar is in terms of the neutron flux obtained by solving a linear transport equation [1]. Another formulation makes use of the invariance principles of radiative transfer [2] to replace linear transport equations by nonlinear integro-differential equations with initial conditions [3–5].

Still another approach, which we take in this paper, is to study the neutron multiplication as a branching process [6, Chap. V, p. 15]. This is a probabilistic model that should give more detailed results than an expected-value model. In particular, the results presented in this paper for bare homogeneous reactors are expressed in terms of fundamental quantities that the transport theory seeks to determine, namely, the critical dimension and the steady-state flux. As a consequence, we obtain a new computational method for estimating these quantities by the iteration of a nonlinear operator.

The processes we study are relatively simple in comparison with any realistic fission process, in that we assume (1) all neutrons have the same energy, (2) the fissionable material is either a homogeneous sphere, an infinite homogeneous slab of finite thickness, or a homogeneous rod, and (3) the collision of a neutron with a nucleus results either in absorption or in the production of a random number of neutrons emitted with an isotropic angular distribution.[1] These assumptions permit a reduction to one dimension variable.

A source neutron is introduced into a body of fissionable material, and in moving with constant velocity it can either escape the body or upon

[1] See [1, Chap. I] for a discussion of the approximation of these assumptions to reality.

collision with a nucleus produce a random number of neutrons, each of which obeys the same laws as the original neutron independently of the others. The time variable is disguised by considering random variables $Z_n$ that denote the number of neutrons in the $n$th generation, i.e., those neutrons that can be traced to the trigger neutron through $n$ fissions.

The mathematical formulation, which is an extension of the familiar Galton-Watson process [7, 6], is given in Section II in terms of the probability-generating functions for the $Z_n$. These functions are obtained by the iteration of a nonlinear operator. A study is made of their dependence both on $n$ and on the dimension of the reactor.

The primary objective of this paper is to present Theorem 6, which in this generality is new. Harris obtains similar analytic results by other methods for the homogeneous rod [8] and conjectures these results for the sphere from numerical evidence [6, Chap. V], applying the general theory of branching processes [6, Chap. III] to neutron multiplication in a homogeneous bounded convex body in three space.

The other theorems in this paper are known, but, for the sake of completeness, proofs are sketched for most of them. Theorems 1 and 3 can be found in the work of Harris [6]. Theorem 2, which is certainly known for self-adjoint operators, is proved here on the assumption merely of the property of positivity. Theorem 4 and 5 are due to Sevast'yanov [9] for a different branching process, and are given by Harris [6] for the homogeneous rod.

## II. Mathematical Models

We shall study branching processes that result from the introduction of a neutron into a body of fissionable material. This "trigger neutron," moving with constant velocity, can either escape the body or upon collision with a nucleus produce a random number of neutrons. Each of these then obeys the same laws as the original neutron independently of the others.

As examples we take an infinite homogeneous slab of finite thickness and a homogeneous sphere. We assume that the neutrons resulting from a collision are emitted with an isotropic angular distribution, and that all neutrons have the same constant energy. The trigger neutron is to be thought of as a neutron produced spontaneously at some point in the body.

The collision process is described by a probability-generating function

$$h(s) = \sum_{n=0}^{\infty} p_n s^n,$$

where $s$ is a complex parameter and $p_n$ is the constant probability that $n$ neutrons result from a collision. Absorption, with probability $p_0$, and scattering, with probability $p_1$, are allowed. We assume that $h$ is analytic in $|s| \leqslant 1 + \alpha$, for some $\alpha > 0$, and that $h'(1) > 1$.

With the assumptions of homogeneity and isotropy, the functions will depend only on the radius for the sphere and on the depth for the slab. We then treat both as one-dimensional problems depending on a variable $x$, $0 \leqslant x \leqslant L$. A branching process is defined in which a neutron of *type* $x$, i.e., produced at the point $x$, with a certain probability produces a random number of neutrons between $y$ and $y + \Delta y$, and each of these in turn obeys the same probabilistic laws as the original neutron.

The probability that a neutron emitted at $x$ will have a first collision between $y$ and $y + \Delta y$ is given by $K(x, y)\mu(\Delta y)$, where $\mu$ is a finite nonnegative measure on $[0, L]$, $L < \infty$, absolutely continuous relative to ordinary Lebesgue measure. With isotropy of the physical process and with the unit of length a mean free path, for the homogeneous slab we have

$$K(x, y) = \tfrac{1}{2} \int_{|x-y|}^{\infty} \frac{e^{-t}}{t}\, dt, \quad \text{and } \mu = \text{ordinary Lebesgue measure}, \tag{2.1}$$

and for the homogeneous sphere,

$$K(x, y) = \frac{1}{2xy} \int_{|x-y|}^{x+y} \frac{e^{-t}}{t}\, dt, \qquad x, y > 0, \quad \text{and} \quad d\mu = y^2\, dy, \tag{2.2}$$

(see [1]). For the sphere, $x$ denotes radial distance from the center, and for the slab, $x$ denotes depth in the finite dimension.

The analysis in the following sections applies not only to (2.1) and (2.2) but also to any kernel $K(x, y)$ that satisfies the following conditions:

(i) $K(x, y) \geqslant 0$, and for each $x$, $K(x, y) = 0$ at most on a set of $y$'s of $\mu$ measure zero;

(ii) $$\int_0^L K(x, y)\, d\mu(y) \leqslant m(L) < 1, \qquad 0 \leqslant x \leqslant L < \infty;$$

(iii) $$\int_0^L K(x, y)\, d\mu(x) \leqslant M(L) < \infty, \qquad 0 \leqslant y \leqslant L < \infty;$$

(iv) there is a positive integer $n_0$ such that the $n_0$th iterate

$$K_{n_0}(x, y) = \int_0^L \cdots \int_0^L K(x, y_1)K(y_1, y_2) \ldots K(y_{n_0-1}, y)\, d\mu(y_1) \ldots d\mu(y_{n_0-1}) \tag{2.3}$$

is a continuous function of $x$ and $y$ satisfying

$$0 < k \leqslant K_{n_0}(x, y) \leqslant K < \infty, \qquad 0 \leqslant x, y \leqslant L;$$

(v) $\int_0^L K(x, y)f(y)\, d\mu(y)$ is a continuous function of $x$ if $f$ is continuous.

For the kernels (2.1) and (2.2) the only condition difficult to establish is (iv), and this follows from an application of the results on p. 29 of [10] to the three-dimensional kernel from which (2.1) and (2.2) are derived.

In particular, the kernel

$$K(x, y) = \tfrac{1}{2} e^{-|x-y|},$$

which caracterizes the homogeneous rod, satisfies the conditions of (2.3). This model is rather uninteresting as a physical process, but easier to treat mathematically than (2.1) or (2.2).

The branching process is described by probability-generating functions for random variables that specify the various generations of the process. The trigger neutron makes up the zeroth generation, the products of a collision of the trigger neutron constitute the first generation, etc. The location of origin of the trigger neutron is specified by $x$, and generating functions for the probabilities $p_{n,k}(x)$ of $k$ neutrons in the $n$th generation are defined recursively.[2]

The probability-generating function for the number of neutrons produced in one generation in $[0, L]$ by a source neutron produced at $x$ is

$$f_1(x, s) = 1 - \int_0^L K(x, y)\, d\mu(y) + \int_0^L K(x, y) h(s)\, d\mu(y). \tag{2.4}$$

To simplify the writing of such equations, we define the linear operator $T_L$ in the Banach space $L_\infty([0, L], \mu)$, of essentially bounded complex-valued functions relative to the measure $\mu$, as follows:

Definition 1. $T_L f(x) = \int_0^L K(x, y) f(y)\, d\mu(y).$

[2] The location of origin of the trigger neutron is specified by $x$, but that of neutrons in subsequent generations is specified only as being in the interval $[0, L]$. See Harris [6] for a precise treatment of general branching processes by means of moment-generating functionals, which allows for a continuum of types in each generation.

From condition (ii) of (2.3) we see that $T_L$ is a bounded operator with $||T_L|| \leqslant m(L) < 1$.

For the second generation we obtain the generating function

$$f_2(\cdot, s) = 1 - T_L(1 - h[f_1(\cdot, s)]) \tag{2.5}$$

by considering that the source neutron either escapes or starts a new set of neutron chains by collision at some point $y$.[3] In general,

$$f_{n+1}(\cdot, s) = 1 - T_L(1 - h[f_n(\cdot, s)]). \tag{2.6}$$

If $F_n(\cdot, s)$ is the generating function for the *total* number of neutrons produced through the first $n$ generations but not counting the initial neutron, then we have the recursive formula

$$F_1 = f_1,$$
$$F_{n+1}(\cdot, s) = 1 - T_L(1 - h[sF_n(\cdot, s)]). \tag{2.7}$$

In the following sections we study these sequences and their limits in the Banach spaces $L_\infty([0, L], \mu)$ as $L$ varies, $0 \leqslant L < \infty$.

## III. A Limit Theorem

We consider the open unit disc $D = \{s: |s| < 1\}$ in the complex plane. A mapping from $D$ into $L_\infty([0, L], \mu)$ is said to be analytic provided it has a power series in $s$ with coefficients in $L_\infty$ that converges in the norm of $L_\infty$ uniformly for $s$ in any compact subset of $D$.

Theorem 1. *The sequence $\{F_n\}$ of (2.7) converges uniformly in $x$ and $s$, for $s$ in any compact subset of $D$, to a function $F$ with the following properties:*

(i) *$F$ is analytic in $s$ and continuous in $x$, and $F \not\equiv 1$;*

(ii) *$F$ is the unique function analytic in a neighborhood of $s = 0$ satisfying*

$$F(\cdot, s) = 1 - T_L(1 - h[sF(\cdot, s)]); \tag{3.1}$$

(iii) *$\partial^n F/\partial s^n(\cdot, 0) \geqslant 0$ and is continuous in $x$ for $n = 0, 1, \ldots$ .*

Proof: From the positivity of the kernel of $T_L$, condition $(v)$ of (2.3), analyticity of $h$ on $D$, and $|h(s)| \leqslant 1$, it follows by induction that $F_n$ is analytic in $s$ and continuous in $x$, and $||F_n(s)||_\infty \leqslant 1$.

Since $h$ is an analytic function of $s$ with nonnegative coefficients, (2.7) gives

$$||F_{n+1}(s) - F_n(s)||_\infty \leqslant |s| h'(|s|)\, ||F_n(s) - F_{n-1}(s)||_\infty. \tag{3.2}$$

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[3] We shall use the notation $f(\cdot, s)$ to denote a function of $x$ for the parameter $s$. We shall sometimes write this simply as $f(s)$, or even as $f$ when it is clear from the context that $f$ is a function with values in a Banach space.

Therefore for each $s$ in the complex region

$$|s|h'(|s|) \leqslant k < 1$$

it follows that $\{F_n(s)\}$ is a Cauchy sequence with a limit function $F(s)$. By the theorem of Vitali [11], the sequence $\{F_n\}$ converges to an analytic function $F$ uniformly on any compact subset of $D$. Now $F$ is a continuous function of $x$ for each $s$, and it satisfies (3.1) since the limit can be taken in (2.7). Clearly $F$ is the only solution of (3.1) analytic in a neighborhood of $s = 0$ since any other solution must equal $F$ on the region $|s|h'(|s|) < 1$. Property (iii) can be obtained from (3.1) by induction. This completes the proof.

An analysis of the boundary values of $F$ for $|s| = 1$ as well as the behavior of the sequence $\{f_n\}$ of (2.6) is more delicate, and use must be made of the spectral properties of the linear operator $T_L$. In the next section we give those spectral properties that can be inferred from the fact that $T_L$ is a positive operator.

## IV. Spectral Properties

The Perron-Frobenius-Jentzsch theory of positive operators is a fundamental tool in the theory of branching processes [6]. Its general importance in the study of neutron fission has been emphasized by G. Birkhoff [12–16].

We list the following well-known results of this theory concerning the spectrum of $T_L$ defined by (2.3) and Definition 1:

(i) The spectrum consists of isolated points $\lambda$ with the possible exception of $\lambda = 0$.

(ii) There is a positive real eigenvalue $\lambda_L$ greater than the modulus of any other eigenvalue.

(iii) The eigenvalue $\lambda_L$ is simple, with continuous eigenfunctions $\phi_L$ of $T_L$ and $\phi_L^*$ of the adjoint $T_L^*$, which can be chosen real and uniformly positive on $[0, L]$. (4.1)

(iv) Two characterizations of $\lambda_L$ are

(a) $\lambda_L = \sup \{\lambda > 0\colon \text{ for some } f \geqslant 0, \ T_L f \geqslant \lambda f\}$,

(b) $\lambda_L = \inf \{\lambda > 0\colon \text{ for some } f \geqslant 0, \ T_L f \leqslant \lambda f\}$.

Proofs of these facts can be found in refs. [15, 6, 17], and are not difficult with the assumptions of (2.3), the compactness of the unit ball of $L_\infty$ in the $L_1$ topology, and the lattice structure of the subspace of $L_\infty$ of real-valued functions. The continuity requirements in (iv) and (v) of (2.3) are used only to guarantee continuity of the eigenfunctions.

*Note.* To avoid repetition, throughout the remainder of this paper we shall reserve the notation $\phi_L$ and $\phi_L^*$ for real positive continuous eigenfunctions of $T_L$ and $T_L^*$, respectively, with normalizations to be specified as desired. The linear functional obtained by multiplying by $\phi_L^*$ and integrating will be denoted by $\Phi_L^*$, and its value on a function $f$ will be written $\Phi_L^*(f)$. This is done to avoid confusion of $\phi_L^*(1)$ and

$$\Phi_L^*(1) = \int_0^L \phi_L^*(x)\, d\mu(x).$$

In subsequent sections we shall study the dependence of certain quantities on the dimension parameter $L$, and for this we shall need the following result:

THEOREM 2. *The maximal real eigenvalue $\lambda_L$ of $T_L$ is a continuous monotone increasing function of $L$ with $\lambda_L < 1$ for $L < \infty$.*

PROOF: For $0 \leqslant x \leqslant L$ and $t > 0$, we write

$$\lambda_{L+t}\, \phi_{L+t}(x) = T_L\, \phi_{L+t}(x) + \int_L^{L+t} K(x, y)\phi_{L+t}(y)\, d\mu(y), \tag{4.2}$$

and apply $\Phi_L^*$ to both sides, where $\phi_{L+t}$ is normalized by

$$\Phi_{L+t}^*(\phi_{L+t}) = 1.$$

Then

$$\lambda_{L+t} = \lambda_L + \int_0^L \phi_L^*(x) \int_L^{L+t} K(x, y)\phi_{L+t}(y)\, d\mu(y)\, d\mu(x), \tag{4.3}$$

and this shows that $\lambda_L$ is monotone increasing and continuous from the right.

Also for $0 \leqslant x \leqslant L - t,\ t > 0$, and $\min\limits_{0 \leqslant x \leqslant L} \phi_L(x) = 1$, we have

$$\lambda_L\, \phi_L = T_{L-\varepsilon}\, \phi_L + \int_{L-\varepsilon}^{L} K(\cdot, y)\phi_L(y)\, d\mu(y)$$

$$\leqslant T_{L-\varepsilon}\, \phi_L + \phi_L \max_{L-\varepsilon \leqslant x \leqslant L} \int_{L-\varepsilon}^{L} K(x, y)\phi_L(y)\, d\mu(y). \tag{4.4}$$

This gives

$$T_{L-\varepsilon}\, \phi_L \geqslant (\lambda_L - \alpha(\varepsilon))\phi_L, \tag{4.5}$$

where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. With (iv a) of (4.1), this shows that $\lambda_L$ is continuous also from the left.

That $\lambda_L < 1$ for $L < \infty$ follows from (ii) of (2.3) and (iv b) of (4.1), since the first of these is just $T_L(1) < 1$, $L < \infty$.

## V. Extinction Probability

We now investigate the boundary value of $F_L$ at $s = 1$. Since for each $x$, $F(x, s)$ is a monotone increasing function of real $s$ bounded by 1, a meaningful statement is made in the following definition:

Definition 2. $Q_L(x) \equiv \lim F_L(x, s)$ for real $s \uparrow 1$.

This is the familiar *extinction probability* of our branching process, i.e., the probability that the process is of finite duration.

As is well known [6, 8, 9], and demonstrated in the next theorem, the function $Q_L$ changes character as $L$ passes through the *critical dimension* $L_c$ determined by $\lambda_L$ and the function $h$. We have assumed $h'(1) > 1$ and demonstrated that $\lambda_L$ is monotone increasing and continuous in $L$. We also assume that $T_L$ is such that $L_c$ is determined, and hence uniquely, by the following equation:

Definition 3.[4] $\lambda_{L_c} h'(1) = 1$.

The concept of a critical dimension is explained by the behavior of the expected number $E_n$ of neutrons in the $n$th generation as $n \to \infty$. From (2.6) we obtain

$$E_n = (h'(1)T_L)^n 1.$$

Since the eigenvalue $\lambda_L$ of $T_L$ dominates the modulus of all other eigenvalues of $T_L$, $E_n$ increases exponentially like $(h'(1)\lambda_L)^n$, as $n \to \infty$, for $L > L_c$, but remains bounded for $L \leqslant L_c$.

Our next result is due to Harris [6, 8].

Theorem 3. *The extinction probability $Q_L$ satisfies the following conditions:*

(i) $Q_L \equiv 1$ *for* $L \leqslant L_c$,
$Q_L < 1$ for $L > L_c$,
*and $Q_L$ is the only function satisfying both these relations and*

$$Q_L = 1 - T_L(1 - h(Q_L)). \tag{5.1}$$

[4] This agrees with the definition in neutron transport theory [1], where this definition implies the existence of a steady-state neutron flux.

(ii) *The sequence* $\{f_n\}$ *of* (2.6) *converges uniformly in* $x$, $0 \leqslant x \leqslant L$, *and* $s$ *in any compact subset of* $|s| < 1$, *to* $Q_L$. *Hence* $Q_L$ *is continuous in* $x$.

PROOF: The following method of proof was suggested by R. N. Snow.

1. It follows readily from (3.1) that $Q_L$ satisfies (5.1), and this shows that either $Q_L \equiv 1$ or $1 - Q_L$ is uniformly positive. We also observe that whenever $Q_L \equiv 1$, Dini's theorem on monotone increasing sequences of continuous functions with continuous limits can be applied to show that $F(s)$ converges uniformly to 1.

From (3.1), absolute monotonicity of $h$, and uniform positivity of $\Phi_L^*$, it follows that

$$\lambda_L h'(1)\Phi_L^*(1-Q_L) \geqslant \Phi_L^*(1-Q_L), \tag{5.2}$$

$$\Phi_L^*(1-F(s)) \geqslant \lambda_L h'(\min_{0\leqslant x\leqslant L} F(x,s))\Phi_L^*(1-F(s)), \qquad 0 \leqslant s \leqslant 1. \quad (5.2)$$

If $L < L_c$, then $\lambda_L h'(1) < 1$ and the first inequality in (5.2) show that $Q_L = 1$ almost everywhere. If $L > L_c$, then $\lambda_L h'(1) > 1$, $F(s) \not\equiv 1$ for $s < 1$, and the second inequality in (5.2) show that

$$\min_{0\leqslant x\leqslant L} F(x,s) \leqslant \alpha < 1,$$

for $s < 1$ and $\alpha$ defined by $h'(\alpha)\lambda_L = 1$. In this case, $Q_L \equiv 1$ is impossible since $F(s)$ would then converge uniformly to it.

For $L = L_c$, (3.1) and properties of the function $h$ can be used to obtain

$$\Phi_{L_c}^*[h''(Q_{L_c})(1-Q_{L_c})^2] = 0 \tag{5.3}$$

and hence $Q_{L_c} \equiv 1$.

It follows from (ii) that any other solution $P_L$ of (5.1) satisfying the inequalities of (i) will also satisfy $P_L \geqslant Q_L$. For $L > L_c$ we have $P_L < 1$ by hypothesis, and therefore

$$\begin{aligned} 1 - P_L &> T_L[h'(P_L)(1-P_L)], \\ P_L - Q_L &\leqslant T_L[h'(P_L)(P_L - Q_L)]. \end{aligned} \tag{5.4}$$

Unless $P_L = Q_L$ we apply (iv) of (4.1) to the positive operator with kernel $K(x, y)h'(P_L(y))$ to obtain the contradictory results that the maximal real eigenvalue is both less than 1 and greater than or equal to 1.

2. Since $\{f_n\}$ is a sequence of functions holomorphic in $|s| < 1$, to prove convergence to $Q_L$ uniformly on any compact subset of $|s| < 1$ we need only prove it on an interval of the real $s$-axis.

From (ii) of (2.3), we have $0 < s_0 = \min_{0 \leqslant x \leqslant L}(1 - T_L(1))$, and it follows by induction that, on the real interval $0 \leqslant s \leqslant s_0$,

$$Q_L \geqslant f_n(s), \qquad f_{n+1}(s) \geqslant f_n(s), \qquad n = 1, 2, \ldots . \tag{5.5}$$

If for $s$ in this interval we define $P_L(s)$ by

$$P_L(x, s) = \lim_{n \to \infty} f_n(x, s), \tag{5.6}$$

then it follows that $P_L(s)$ satisfies (5.1) and $P_L(s) \leqslant Q_L(s)$. But then $P_L = Q_L \equiv 1$ for $L \leqslant L_c$, since for this case the arguments in (i) for $Q_L$ can be applied to $P_L$. If $L > L_c$ then also $P_L = Q_L$, since otherwise the inequalities

$$1 - Q_L > T_L[h'(Q_L)(1 - P_L)], \tag{5.7}$$

$$Q_L - P_L \leqslant T_L[h'(Q_L)(Q_L - P_L)] \tag{5.8}$$

give contradictory statements about the maximal eigenvalue of the operator with kernel $K(x, y)h'[Q_L(y)]$. This completes the proof.

## VI. Asymptotic Forms

The mapping $F_L$ is represented by a power series in $s$, $|s| < 1$, with nonnegative coefficients $p_n(L)$, and hence has a positive real singularity on its circle of convergence. It follows readily from Theorems 1 and 3 that $F_L$ can be extended to $|s| = 1$ in such a way as to agree at $s = 1$ with $Q_L$, and that $F_L(s)$ is a continuous function of $x$ for each $s$, $|s| = 1$. The theory of branch points of functional equations [18–20] provides a method for locating and describing the nature of the singularities of $F_L$.[5]

If the functional-equation analogue of the implicit-function theorem is applied to (3.1), then it follows that a necessary and sufficient condition for the solvability for $F_L$ as a holomorphic function of $s$ in some neighborhood of $s = 1$ is that 1 not be in the spectrum of the positive operator $\tilde{T}_L$ with kernel $K(x, y)h'[Q_L(y)]$ (see [18]). But for $L \leqslant L_c$ this is trivially so, and for $L > L_c$ it follows from

$$1 - Q_L > \tilde{T}_L(1 - Q_L) \tag{6.1}$$

by (iv, b) of (4.1). So for $L \neq L_c$, $F_L$ is holomorphic in some region $|s| < S(L)$ for $S(L) > 1$.

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[5] Apparently the first application of this theory to branching processes is made by Sevast'yanov in [9] to obtain results similar to the theorems of this section (see also Harris [8]).

For $L = L_c$, $s = 1$ is a singular point of $F_L(s)$. If we expand $h$ in (3.1), apply $\Phi^*_{L_c}$ to both sides and use $\lambda_{L_c} h'(1) = 1$, for $|s| < 1$ we obtain

$$(1 - s)\Phi^*_{L_c}(F_L(s)) = \frac{h''(1)}{2h'(1)} \Phi^*_{L_c}\{[1 - F_{L_c}(s) + (1 - s)F_{L_c}(s)]^2 g(s)\}, \quad (6.2)$$

where $g(s)$ is a positive function of $x$ and $s$, with $g(1) \equiv 1$. Since $h''(1) > 0$, this shows that $[1 - F_{L_c}(s)]/(1 - s)$ is unbounded as $s \to 1$, and suggests that $1 - F_{L_c}(s) = O(\sqrt{1 - s})$ as $s \to 1$, i.e., that $s = 1$ is a branch point of order two.

A complete description of $F_L(s)$ on $|s| = 1$ for $L = L_c$ is given in the following theorem (Sevast'yanov [9]).

THEOREM 4. *If $h$ is a function of $s^k$ then the kth roots of unity, $\omega_1 = 1, \ldots, \omega_k$, are branch points of order two for $F_{L_c}(s)$, all other $s$ on $|s| = 1$ being regular points. With appropriate cuts in the complex plane, $F_{L_c}(s)$ is analytic in $(1 - s/\omega_i)^{1/2}$ in some neighborhood of each $\omega_i$, with*

$$F_{L_c}(s) = 1 - a\phi_{L_c}\left(1 - \frac{s}{\omega_i}\right)^{1/2} + O\left(1 - \frac{s}{\omega_i}\right), \quad (6.3)$$

*where*

$$a^2 = \frac{2h'(1)}{h''(1)} \frac{\Phi^*_{L_c}(1)}{\Phi^*_{L_c}(\phi^2_{L_c})} . \quad (6.4)$$

PROOF: A point $s$, $|s| = 1$, is a regular (singular) point of the function $F_{L_c}(s)$ defined by (3.1) if 1 is not (is) in the spectrum of the operators $\hat{T}_s$ with kernel $K(x, y)sh'[sF_{L_c}(y, s)]$. We shall show that 1 is in the spectrum of $\hat{T}_s$ for a given $s$, $|s| = 1$, if and only if $h(s) = 1$.

We first consider any $s$ for which $|s| = 1$ and $h(s) \neq 1$ and suppose that, for some $x_0$, $|F_{L_c}(x_0, s)| = 1$. In (3.1) we take absolute values, cancel 1's, consider $K(x_0, y)\, d\mu(y)$ as a measure, and apply the Schwartz inequality to obtain

$$\begin{aligned}\int_0^{L_c} K(x_0, y)\, d\mu(y) &\leqslant \left| \int_0^{L_c} K(x_0, y) h(sF_{L_c}(y, s))\, d\mu(y) \right| \\ &\leqslant \left[ \int_0^{L_c} K(x_0, y)\, d\mu(y) \int_0^{L_c} K(x_0, y)\, |h(sF_{L_c}(y, s))|^2\, d\mu(y) \right]^{1/2} \\ &\leqslant \int_0^{L_c} K(x_0, y)\, d\mu(y). \end{aligned} \quad (6.5)$$

Therefore equality holds throughout, and this implies, in the Schwartz inequality statement between the second and third terms, that $h(sF_{L_c}(s))$ is proportional to the identity function, i.e.,

$$h(sF_{L_c}(s)) \equiv e^{i\theta}.$$

If this is put back in (3.1) if follows that

$$\theta = 0, \qquad F_{L_c}(s) \equiv 1,$$

and so $h(s) = 1$, a contradiction. Therefore, by continuity,

$$\|F(s)\|_\infty \leqslant \alpha(s) < 1$$

for each $s$, $|s| = 1$, for which $h(s) \neq 1$.

For $s$, $|s| = 1$, such that $h(s) \neq 1$, the spectral radius of $\hat{T}_s$ is less than 1 since [11, p. 125]

$$\|\hat{T}_s{}^n\|^{1/n} \leqslant h'(\alpha(s))\|T^n_{L_c}\|^{1/n} \to \frac{h'(\alpha(s))}{h'(1)} < 1 \text{ as } n \to \infty. \tag{6.6}$$

So each such $s$ is a regular point for $F_{L_c}(s)$.

If $h(s_0) = 1$ for any $s_0 \neq 1$, $|s_0| = 1$, it follows from the fact that $h$ can be represented as a power series with nonnegative coefficients that $h$ is a function of $s^k$ for some $k > 1$ and that $s_0$ is one of the $k$th roots of unity, $\omega_1 = 1, \ldots, \omega_k$. It follows readily from (3.1) that $F_{L_c}(s)$ is also a function of $s^k$ and $F_{L_c}(\omega_i) \equiv 1$. The operator $\hat{T}_{\omega_i}$ is just $h'(1)T_{L_c}$ with 1 in its spectrum. Each such $s$ is a singular point.

We merely refer to [18] for methods to prove the validity of (6.3). The value of $a$ in (6.4) is readily obtained by substituting (6.3) into a statement similar to (6.2) for each $\omega_i$, dividing by $1 - s/\omega_i$ and passing to the limit. This completes the proof.

The singularities described in the above theorem can be used to obtain asymptotic forms for $p_n(x, L_c)$ by the method used by Otter in [21]. For example, if $h(s) = 1$ on $|s| = 1$ only at $s = 1$, then $F_{L_c}(s)$ can be defined as an analytic function in a disc $|s| \leqslant a$, for some $a > 1$, with a cut along real $s \geqslant 1$. Then (6.3) and contour integration can be used to compute the $p_n(x, L)$ ([8] and [9]).

We state the following result without proof.

THEOREM 5. *If $h(s) = 1$ on $|s| = 1$ only for $s = 1$, and*

$$F_{L_c}(x, s) = \sum_{n=0}^{\infty} p_n(x, L_c)s^n,$$

*then*

$$p_n(x, L_c) = \frac{1}{\sqrt{2\pi}n^{3/2}} \left[\frac{h'(1)\Phi^*_{L_c}(1)}{h''(1)\Phi^*_{L_c}(\phi^2_{L_c})}\right]^{1/2} \phi_{L_c}(x) + O\left(\frac{1}{n^{5/2}}\right)$$

*as* $n \to \infty$.

For $L \neq L_c$ one can obtain for $s = s(L) > 1$ results similar to those at $s = 1$ for $L_c$, where $S(L)$ denotes the first singularity on the real axis. The asymptotic results for $L \neq L_c$ are that $p_n(x, L) = O([s(L)]^{-n})$ as $n \to \infty$; i.e., $p_n$ tends to zero exponentially with $n$.

## VII. Extinction Probability as a Function of Dimension

The number $Q_L(x)$ is the probability that the branching process is finite for a source neutron at $x$. We have seen that at the critical length $L_c$ the point $s = 1$ is a singular point of the integral equation (3.1). We shall now investigate $Q_L$ as a function of $L$ and show that $L_c$ is a singular point for this function. Throughout this section all quantities and Banach spaces are real.

We assume that the kernel $K(x, y)$ of (2.3) has in addition the properties

$$\text{(i)} \quad \lim_{x \downarrow L_c} \int_0^{L_c} |K(x, y) - K(L_c, y)| \, d\mu(y) = 0,$$

$$\text{(ii)} \quad \lim_{L \downarrow L_c} \int_0^{L_c} \left| K(x, L_c) - \frac{1}{\mu(L - L_c)} \int_{L_c}^{L} K(x, y) \, d\mu(y) \right| d\mu(x) = 0. \tag{7.1}$$

A brief calculation shows this to be true of the kernels in (2.1) and (2.2).

We now prove the following result.

Theorem 6. (i) *$Q_L(x)$ is a continuous function of $x$, and as a function of $L$ it is continuous from the right,* $0 \leqslant x \leqslant L < \infty$.[6]

(ii) *$Q_L(x)$ has a discontinuity in the derivative at $L_c$, with*

$$\lim_{L \downarrow L_c} \frac{1 - Q_L}{\mu(L_c - L)} \equiv 0$$

[6] We do not mean to imply that $Q_L$ has any discontinuities as a function of $L$, but we do not have a proof of continuity.

*and*

$$\lim_{L \downarrow L_c} \frac{Q_L(x) - 1}{\mu(L - L_c)} = - \frac{2h'(1)}{h''(1)\Phi^*_{L_c}(\phi^2_{L_c})} \phi_{L_c}(x),$$

*with the normalization* $\phi_{L_c}(L_c) = \phi^*_{L_c}(L_c) = 1$, *where* $\phi_{L_c}$ *and* $\phi^*_{L_c}$ *are positive eigenfunctions of* $T_{L_c}$ *and* $T^*_{L_c}$, *respectively.*

PROOF: Since $Q_L \equiv 1$ for $L \leqslant L_c$, we need consider only $L \geqslant L_c$. We define $U_L$ by $U_L = 1 - Q_L$. This is the *survival probability.*

(i) From the proof of Theorem 3 we see that $f_n(x, 0)$ increases to $Q_L(x)$ as $n \to \infty$ uniformly in $x$. Now from (2.6) we have, by induction,

$$f_n(x, 0, L) \geqslant f_n(x, 0, L + t), \qquad 0 \leqslant x \leqslant L, \quad t > 0, \tag{7.2}$$

whence

$$Q_L(x) \geqslant Q_{L+t}(x), \qquad 0 \leqslant x \leqslant L, \quad t > 0. \tag{7.3}$$

If $Q_{L^+}(x)$ denotes the limit as $t \downarrow 0$ of $Q_{L+t}(x)$ for $0 \leqslant x \leqslant L$, then by the Lebesgue dominated-convergence theorem we have, from (5.1),

$$1 - Q_{L^+} = T_L(1 - h(Q_{L^+})). \tag{7.4}$$

But by the uniqueness proved in Theorem 3, we have $Q_{L^+} = Q_L$. Since $Q_{L+t}$, which is continuous in $x$, increases monotonically for $0 \leqslant x \leqslant L$ to the continuous function $Q_L$, the convergence is uniform in $x$, $0 \leqslant x \leqslant L$.

(ii) The integral equation (5.1) for $Q_L$ gives by the absolute monotonicity of $h$,

$$h'(1)T_L(U_L) > U_L > T_L(h'(Q_L)U_L), \qquad L > L_c. \tag{7.5}$$

By part (i), for any $\varepsilon > 0$ we can choose $\delta > 0$ such that $U_L(x) < \varepsilon$ for $L - L_c < \delta$ and $0 \leqslant x \leqslant L_c$, since $U_{L_c} \equiv 0$. The left-hand side of (7.5) readily shows that $\delta$ can be chosen so that $U_L(x) < \varepsilon$ for $0 \leqslant x \leqslant L$. So $h'(Q_L) > h'(1 - \varepsilon)$ in (7.5) for $L - L_c < \delta$.

If we iterate (7.5) $n_0$ times and use condition (iv) of (2.3), we obtain

$$C_1 \int_0^L U_L(x)\, d\mu(x) \leqslant U_L(x) \leqslant C_2 \int_0^L U_L(x)\, d\mu(x), \qquad 0 < C_1 < C_2 < \infty, \tag{7.6}$$

where $C_1$ and $C_2$ can be chosen independent of $L$ for $L - L_c < \delta$.

We define the functions $f_L$ by

$$f_L(x) = \frac{U_L(x)}{\int_0^L U_L(x)\, d\mu(x)}, \qquad 0 \leqslant x \leqslant L_c, \qquad L > L_c. \tag{7.7}$$

From (7.6),

$$\int_0^{L_c} f_L(x)\, d\mu(x) \geqslant C_1 \mu([0, L_c]) > 0.$$

These are normalized projections of $U_L$, $L_c \leqslant L \leqslant L_c + \delta$, into $L_\infty([0, L_c], \mu)$.

If we let

$$q_L = \min_{0 \leqslant x \leqslant L_c} Q_L(x),$$

restrict $x$ to $[0, L_c]$, and divide by $\int_0^L U_L(x)\, d\mu(x)$ in (7.5), we obtain

$$T_{L_c}(f_L) < \frac{1}{h'(1-q_L)} f_L, \qquad L > L_c. \tag{7.8}$$

As $L \downarrow L_c$, $h'(1 - q_L)$ increases to $h'(1)$, and $1/h'(1)$ is the maximal eigenvalue of $T_{L_c}$. Since by (7.6) the functions $f_L$ are bounded in $L_\infty([0, L_c], \mu)$ by $C_2$, and any bounded sphere in $L_\infty$ is weak* sequentially compact in the $L_1$ topology ([11], p. 37), it follows that any sequence $\{f_{L_n}\}$, $L_n \downarrow L_c$, has a weak* convergent subsequence with a limit $f_{L_c}$.

It follows from (i) and (ii) of (2.3) and Tonelli's theorem ([22], p. 194) that $T^*_{L_c}$ takes the $L_1$ subspace of $L^*_\infty$ into itself. Therefore $T_{L_c}$ is continuous in the $L_1$ topology on $L_\infty$, and the limit of $f_{L_n}$ can be taken in (7.7) and (7.8) to give

$$T_{L_c}(f_{L_c}) \leqslant \frac{1}{h'(1)} f_{L_c}, \qquad \int_0^{L_c} f_{L_c}(y)\, d\mu(y) = 1. \tag{7.9}$$

An application of $\Phi^*_{L_c}$ to (7.9) shows that equality must hold. Therefore $f_{L_c}$ is uniquely specified as

$$f_{L_c} = \phi_{L_c}, \qquad \int_0^{L_c} \phi_{L_c}(x)\, d\mu(x) = 1. \tag{7.10}$$

The above argument can be repeated to show that for any $\{f_{L_n}\}$, $L_n \downarrow L_c$, every subsequence has a convergent subsequence with the unique limit $\phi_{L_c}$, normalized as in (7.10). Therefore $\{f_{L_n}\}$ converges weak* to $\phi_{L_c}$ for every sequence $\{L_n\}$ tending to $L_c$. This proves that $f_L$ tends weak* to $\phi_{L_c}$ as $L \downarrow L_c$.

Using this fact, we can show that $f_L$ converges strongly to $\phi_{L_c}$. To see this we write, using analyticity of $h$,

$$U_L = T_L\left[h'(1)U_L - \frac{h''(1)}{2}U_L^2 g(U_L)\right], \tag{7.11}$$

where $g$ is analytic near zero and $g(0) = 1$. If we restrict $x$ to $[0, L_c]$ and write (7.11) in terms of $f_L$ of (7.7), then

$$(I - h'(1)T_{L_c})f_L \tag{7.12}$$

$$= \frac{1}{\int_0^L U_L(x)\,d\mu(x)}\left[h'(1)\int_{L_c}^L K(x, y)U_L(y)\,d\mu(y) - \frac{h''(1)}{2}T_L(U_L^2 g(U_L))\right].$$

The null space of $I - h'(1)T_{L_c}$ is the one-dimensional eigenspace of the isolated eigenvalue $1/h'(1)$ of $T_{L_c}$. If we add the projection operator $P$ defined by

$$P(f) = \Phi^*_{L_c}(f)\phi_{L_c}, \tag{7.13}$$

where

$$\int_0^{L_c} \phi_{L_c}(x)\,d\mu(x) = 1 \qquad \text{and} \qquad \Phi^*_{L_c}(\phi_{L_c}) = 1,$$

then $I - h'(1)T_{L_c} + P$ has a bounded inverse $R$, and we find

$$\|f_L - P(f_L)\|_\infty \leqslant \|R\|\left[h'(1)C_2\int_{L_c}^L K(x, y)\,d\mu(y) + \frac{h''(1)}{2}C_2\|U_L\,g(U_L)\|_\infty\right]. \tag{7.14}$$

Therefore we have

$$\lim_{L\downarrow L_c}\|f_L - \phi_{L_c}\|_\infty \leqslant \lim_{L\downarrow L_c}[\|f_L - P(f_L)\|_\infty + \|P(f_L) - \phi_{L_c}\|_\infty] = 0. \tag{7.15}$$

We also need the fact that if $x(L)$ denotes a path, $L_c \leqslant x(L) \leqslant L$, then $U_L(x(L))/\int_0^L U_L(x)\,d\mu(x)$ tends to $\phi_{L_c}(L_c)$ as $L \downarrow L_c$. From (7.11) we get

$$\left| \frac{U_L(x(L))}{\int_0^L U_L(x)\, d\mu(x)} - \phi_{L_c}(L_c) \right| \leqslant h'(1) \int_0^{L_c} |K(x(L), y) f_L(y) - \tag{7.16}$$

$$K(L_c, y)\phi_{L_c}(y)| d\mu(y) + h'(1) C_2 \int_{L_c}^{L} K(x(L), y)\, d\mu(y) + \frac{h''(1)}{2} C_2 \|U_L\, g(U_L)\|_\infty.$$

Since $f_L$ approach $\phi_{L_c}$ uniformly on $[0, L_c]$, we obtain

$$\lim_{L \downarrow L_c} \left| \frac{U_L(x(L))}{\int_0^L U_L(x)\, d\mu(x)} - \phi_{L_c}(L_c) \right| \tag{7.17}$$

$$\leqslant h'(1) \phi_{L_c}(L_c) \lim_{L \downarrow L_c} \int_0^{L_c} |K(x(L), y) - K(L_c, y)|\, d\mu(y) = 0,$$

by (i) of (7.1).

Thus we have determined the behavior of $U_L / \int_0^L U_L(x)\, d\mu(x)$ as $L \downarrow L_c$; we want, however, to consider the difference quotient $U_L/\mu(L - L_c)$. To determine this limit we return to (7.12), apply $\Phi^*_{L_c}$ to both sides, and divide by $\mu(L - L_c)$ to get

$$\frac{h''(1)}{2h'(1)} \Phi^*_{L_c} \left( f_L \frac{U_L}{\mu(L - L_c)} g(U_L) \right) \tag{7.18}$$

$$= \frac{1}{\mu(L - L_c)} \int_{L_c}^{L} \frac{U_L}{\int_0^L U\, d\mu} \left[ h'(1) - \frac{h''(1)}{2} U_L\, g(U_L) \right] \int_0^{L_c} K(x, y) \phi^*_{L_c}(x)\, d\mu(x)\, d\mu(y).$$

We shall not write it out, but by using (7.17) and property (ii) of (7.1) we can show from (7.18) that

$$\lim_{L \downarrow L_c} \Phi^*_{L_c} \left( \frac{U}{\mu(L - L_c)} \phi_{L_c} \right) = \frac{2h'(1)}{h''(1)} \phi_{L_c}(L_c) \phi^*_{L_c}(L_c). \tag{7.19}$$

It is then essentially a repeat of the type of arguments used in (7.6) to (7.15) to show from (7.19) that $U_L/\mu(L - L_c)$ is bounded above and below and converges pointwise uniformly, $0 \leqslant x \leqslant L_c$, to $\alpha\phi_{L_c}$, where

$$\alpha = \frac{2h'(1)}{h''(1)} \frac{\phi_{L_c}(L_c)\phi^*_{L_c}(L_c)}{\Phi^*_{L_c}(\phi^2_{L_c})}. \tag{7.20}$$

Since $\phi_{L_c}$ and $\phi^*_{L_c}$ are continuous functions of $x$, uniformly bounded away from zero, we can normalize them by

$$\phi_{L_c}(L_c) = \phi^*_{L_c}(L_c) = 1. \tag{7.21}$$

This completes the proof.

The results in Theorems 4–6 are expressed in terms of real positive eigenfunctions of $T_L$ and $T_L{}^*$ at the critical dimension $L_c$. For the homogeneous rod the number $L_c$ as well as the eigenfunctions can easily be determined since the integral equation is related to a linear ordinary differential equation with boundary conditions. For the homogeneous sphere and the homogeneous slab, however, the determination of these quantities is one of the central problems of neutron transport theory ([18], p. 96 ff.).

One interpretation of the results expressed in Theorems 3 and 6 is that they provide a new computational method for estimating the critical dimension $L_c$ and the steady-state flux $\phi_{L_c}$ for self-adjoint operators $T_L$. This requires the solution, by iteration, of Eq. (5.1) for $Q_L'$ for a set of values of $L$ that surround $L_c$. The practicality of this method will be considered elsewhere.

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## References

1. Davison, B. "Neutron Transport Theory." Oxford Univ. Press., London and New York, 1957.
2. Chandrasekhar, S. "Radiative Transfer." Dover, New York, 1960.
3. Bellman, R., Kalaba, R., and Wing, G. M. Invariant imbedding and neutron transport theory, I. *J. Math. and Mech.* **7**, 149–162 (1958).
4. Bellman, R., Kalaba, R., and Wing, G. M. Invariant imbedding and neutron transport theory, II. *J. Math. and Mech.* **7**, 741–756 (1958).
5. Bellman, R., Kalaba, R., and Wing, G. M. Invariant imbedding and mathematical physics, I. Particle processes. *J. Math. and Phys.* **1**, 280–308 (1960).
6. Harris, T. E. "The Theory of Branching Processes." Springer, Berlin, to appear.
7. Feller, W. "Probability Theory and Its Applications." Wiley, New York, 1950.

8. Harris, T. E. On one-dimensional neutron multiplication. The RAND Corporation, Research Memorandum RM-2317 (ASTIA No. AD **233143**), January 18, 1960.
9. Sevast'yanov, B. A. Branching random processes for particles diffusing in a bounded region with absorbing boundaries. *Teoriya Veroyatnosteĭ i ee Primeneniya* **3**, 121–136 (1958).
10. Petrovskii, I. G. "Integral Equations." Graylock Press, Rochester, 1957 (translation).
11. Hille, E. and Phillips, R. S. "Functional Analysis and Semi-groups," (revised edition). Am. Math. Soc. Colloquium Publications. Vol. **31**, 1957.
12. Birkhoff, G. Criticality and positivity. *Proc. Symposia in Applied Math.* Vol. XI, Am. Math. Soc., **1961**.
13. Birkhoff, G. Some mathematical problems of nuclear reactor theory. *Proc. Symposium on Frontiers of Numerical Mathematics*, 1959. Univ. of Wisconsin Press, 1960.
14. Birkhoff, G. Reactor criticality in transport theory. *Proc. Natl. Acad. Sc. US.* **45**, 567–569 (1959).
15. Birkhoff, G. Extensions of Jentzsch's theorem. *Trans. Am. Math. Soc.* **85**, 219–227 (1957).
16. Birkhoff, G. and Varga, R. S. Reactor criticality and non-negative matrices. *J. Soc. Ind. and Appl. Math.* **6**, 354–377 (1958).
17. Krein, M. G. and Rutman, M. A. Linear operators leaving invariant a cone in a Banach space. Am. Math. Soc. Translation No. **26**, 1950.
18. Cronin, Jane. Branch points of solutions of equations in a Banach space. *Trans. Am. Math. Soc.* **69**, 208–231 (1950).
19. Lefschetz, S. "Differential Equations: Geometric Theory." Interscience, New York, London, 1957.
20. Schmidt, E. Zur Theorie der Linear und Nichtlinearen Integralgleichungen. *Math. Ann.* **65**, 370–399 (1908).
21. Otter, R. The multiplicative process. *Ann. Math. Statist.* **20**, 206–224 (1949).
22. Dunford, N. and Schwartz, J. "Linear Operators. Part I, General Theory." Interscience, New York, 1959.