The interpolation method of Sprague-Karup

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ABSTRACT

The usual interpolation method is that of Lagrange. The disadvantage of the method is that in the given points the derivatives of the interpolating polynomials are not equal one to the other. In the method of Hermite, polynomials of a higher degree are used, whose derivatives in the given points are supposed to be equal to the derivatives of the function at the given points. This means that those derivatives must be known.
If those derivatives are not known, then in the given points the derivatives may be replaced by approximative values, e. g. based on the interpolating polynomials of Lagrange. Such a method has been described by T. B. Sprague (1880) and in a simplified form by J. Karup (1898). In this paper the formulae are derived. Both methods are illustrated with an example. Some properties and theorems are stated. Tables to simplify the computational work are given. Subroutines for these interpolation methods will be published in a next article.

1. INTRODUCTION

Let be given $n+1$ points of a known or unknown function $f(x)$ of the argument x (cfr. fig.) :

x_0	y_0
x_1	y_1
....	....
x_i	y_i
....	....
x_n	y_n

with $x_0 < x_1 < \ldots < x_i < \ldots < x_n$.

It is asked to determine by interpolation an approximating value $y(x)$ of $f(x)$ for :

$$x_i \leqslant x \leqslant x_{i+1}.$$

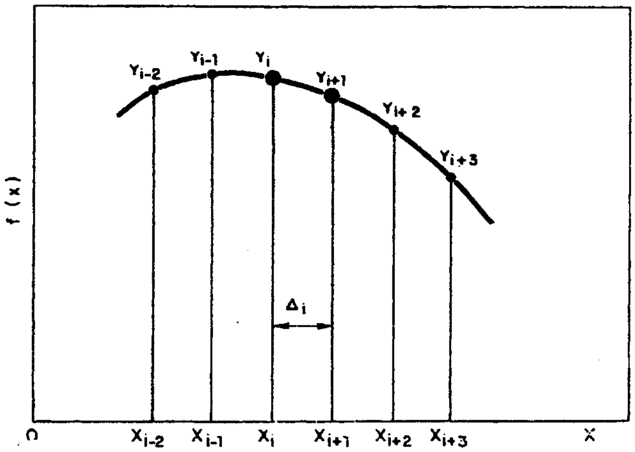

Fig. 1.

The most employed method is that of Lagrange, according to which the function $f(x)$ in the interval considered is simulated with the help of a polynomial $y(x)$ through the given points in the neighbourhood of this interval. The disadvantage of this method is that in the given points the derivatives of the interpolating polynomials $y(x)$ are not equal one to the other. According to the method of Hermite, polynomials $y(x)$ of a higher degree are used, going through the given points, but of which the derivatives in the given points are supposed to be equal to the derivatives of the function $f(x)$. The mentioned discontinuity is thus removed but the derivatives of the function $f(x)$ have to be known in the given points.
If these derivatives are not known, then in the method of Hermite the derivatives in the given points can be replaced by approximative values, e. g. based on the interpolating polynomials $y(x)$ of Lagrange, in order to avoid the mentioned discontinuity. Such methods have been described by T. B. Sprague [1] and in a simplified form by J. Karup [2]. Among other things they are applied to actuarial

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mathematics [3-7] and to colour computations [8-11].
These methods are of special importance when an interpolation has to be carried out over experimental points, wherein certain demands are made as to the continuity of the derivatives. So far these methods have not been treated in the actual literature on numerical analysis. It is the intention of this article to fill this gap. In this connection it should be emphasized explicitly that there is a clear difference with the use of spline functions.

Remark
For simplicity's sake we assume that the intervals between the abscissae $\Delta_i = x_{i+1} - x_i$ equal each other so that by linear interpolation Δ_i can be put equal to 1.

2. THE METHOD OF KARUP

2.1. Principle

A polynomial of third degree is constructed through the end points of the interval considered, wherein moreover the derivatives of first order in these end points are put equal to the approximative value obtained by the three-point-rule of Lagrange. By taking herein for any point the foregoing and the following point the first order derivatives of the interpolating polynomials become continuous in the given points.

2.2. Formulae

We use thus four points :

$$\begin{array}{ll} x_{i-1} & y_{i-1} \\ x_i & y_i \\ x_{i+1} & y_{i+1} \\ x_{i+2} & y_{i+2} \end{array}$$

We put : $y(h) = a + bh + ch^2 + dh^3$
with $h = x - x_i$ $(0 \leqslant h \leqslant 1)$.
The derivative reads :

$$y'(h) = b + 2ch + 3dh^2.$$

The interpolating polynomial y(h) has to go through the points x_i, y_i $(h = 0)$ and x_{i+1}, y_{i+1} $(h = 1)$.
The derivatives y'(h) in these points should be equal to :

$$\frac{1}{2}(-y_{i-1} + y_{i+1}) \text{ and } \frac{1}{2}(-y_i + y_{i+2}).$$

So we obtain the following set of four equations :

$$\begin{aligned} y(0) &= a &&= y_i \\ y(1) &= a + b + c + d &&= y_{i+1} \\ y'(0) &= b &&= \frac{1}{2}(-y_{i-1} + y_{i+1}) \\ y'(1) &= b + 2c + 3d &&= \frac{1}{2}(-y_i + y_{i+2}) \end{aligned}$$

The solution of this set of four equations gives :

$$\begin{aligned} a &= y_i \\ b &= \frac{1}{2}(-y_{i-1} + y_{i+1}) \\ c &= \frac{1}{2}(2y_{i-1} - 5y_i + 4y_{i+1} - y_{i+2}) \\ d &= \frac{1}{2}(-y_{i-1} + 3y_i - 3y_{i+1} + y_{i+2}) \end{aligned}$$

The interpolating polynomial finally becomes :

$$\begin{aligned} y(h) = y_i & + \frac{1}{2}(-y_{i-1} + y_{i+1})h \\ & + \frac{1}{2}(2y_{i-1} - 5y_i + 4y_{i+1} - y_{i+2})h^2 \\ & + \frac{1}{2}(-y_{i-1} + 3y_i - 3y_{i+1} + y_{i+2})h^3 \end{aligned}$$

or, after separation of the y-values used :

$$\begin{aligned} y(h) = & -\frac{1}{2}h(h-1)^2 y_{i-1} \\ & + \frac{1}{2}(h-1)(3h^2 - 2h - 2)y_i \\ & - \frac{1}{2}h(3h^2 - 4h - 1)y_{i+1} \\ & + \frac{1}{2}h^2(h-1)y_{i+2} \end{aligned} \qquad \text{(K)}$$

Finally the formula can be expressed as a function of the differences :

$$\begin{aligned} y_i &= y_{i-1} + \Delta y_{i-1} \\ y_{i+1} &= y_{i-1} + 2\Delta y_{i-1} + \Delta^2 y_{i-1} \\ y_{i+2} &= y_{i-1} + 3\Delta y_{i-1} + 3\Delta^2 y_{i-1} + \Delta^3 y_{i-1} \end{aligned}$$

Substitution of these expressions in the foregoing formulae yields :

$$\begin{aligned} y(h) &= y_i + h(\Delta y_{i-1} + \frac{1}{2}\Delta^2 y_{i-1}) \\ & \quad + \frac{1}{2}h^2(\Delta^2 y_{i-1} - \Delta^3 y_{i-1}) \\ & \quad + \frac{1}{2}h^3(\Delta^3 y_{i-1}) \\ &= y_i + \Delta y_{i-1}(h) \\ & \quad + \frac{1}{2}\Delta^2 y_{i-1}(h)(h+1) \\ & \quad + \frac{1}{2}\Delta^3 y_{i-1}(h^2)(h-1) \end{aligned}$$

As a function of h this form of the formula is simpler. The disadvantage however is, that the y-values cannot be used directly : the differences have to be calculated.

2.3. Example

The accompanying table 1 represents the results that have been obtained for the function :

$$Si(x) = \int_0^x \frac{\sin(x)}{x} dx, \text{ with } x = 0.0(0.2)10.0,$$

with the help of the third degree polynomial according to Lagrange and with the help of the described method of Karup respectively.

Herein we started from the accurate values of this function for the arguments $x = -1.0(1.0)11.0$, accurate to the tenth decimal place :

x	Si(x)
x = − 1.0	Si(x) = − 0.9460830704
0.0	0.0000000000
1.0	0.9460830704
2.0	1.6054129768
3.0	1.8486525280
4.0	1.7582031389
5.0	1.5499312449
6.0	1.4246875513
7.0	1.4545966142
8.0	1.5741868217
9.0	1.6650400758
10.0	1.6583475942
11.0	1.5783068069

Table 1 also contains the accurate values of the function for the calculated interpolations, as well as the errors of both methods. If we put :

E_1 = the mean error,

E_2 = the mean of the absolute value of the error,

E_3 = the quadratic mean error,

E_4 = the absolute value of the maximal error,

then we obtain for Lagrange and Karup respectively:

$E_1 = 0.0005765047$ and 0.0005765047

$E_2 = 0.0016253029$ and 0.0020352119

$E_3 = 0.0021137131$ and 0.0026632572

$E_4 = 0.0048115117$ and 0.0057931350

The accuracy is for both methods of the same order of magnitude. It should be noted however, that the mean error for the two methods is precisely the same.

Remark

The accurate values of the function Si(x) have been taken from the tables of Glaisher [12] and of Abramowitz-Stegun [13].

2.4. Theorems

Theorem 1 : If we integrate the interpolation formula of Karup K over the interval $h = [0,1]$, the corrected trapezoid rule is obtained.

If this integral is worked out, we obtain indeed :

$$\int_0^1 y(h)\,dh = \frac{1}{24}(-y_{i-1} + 13y_i + 13y_{i+1} - y_{i+2})$$

Theorem 2 : If the interpolation formula of Karup K is integrated over the interval $h = [-1, 2]$, the four-point-rule of Newton-Cotes (closed form) is obtained.

If this integral is worked out, we obtain indeed :

$$\int_{-1}^2 y(h)\,dh = \frac{3}{8}(y_{i-1} + 3y_i + 3y_{i+1} + y_{i+2})$$

Although the third degree polynomials used in the methods of Lagrange and Karup are totally different, the same numerical integration formulae are obtained for the two methods. The first theorem explains why the mean error is the same for both methods, as has been established in the treated example.

3. THE METHOD OF SPRAGUE

3.1. Principle

A polynomial of fifth degree is constructed through the end points of the interval considered, wherein moreover the derivatives of the first and the second order in these end points are equalled to the approximative value that is obtained with the five-point-rule of Lagrange. By taking here, the two foregoing and the two following points together with the point considered, the derivatives of the first and the second order of the interpolating polynomials in the given points become continuous.

As the curvature and the radius of curvature are only a function of the derivatives of the first and second order :

$$R = \frac{(1 + y'^2)^{3/2}}{|y''|}$$

those become also continuous for the interpolating polynomials in the given points.

As the method of Karup has been published after the method of Sprague, the former can be considered as a simplification of the latter.

3.2. Formulae

We use thus six points :

x_{i-2}	y_{i-2}
x_{i-1}	y_{i-1}
x_i	y_i
x_{i+1}	y_{i+1}
x_{i+2}	y_{i+2}
x_{i+3}	y_{i+3}

We put : $y(h) = a + bh + ch^2 + dh^3 + eh^4 + fh^5$,
with $h = x - x_i$ $(0 \leqslant h \leqslant 1)$.

TABLE 1

x	Si(x)	Lagrange	Error	Karup	Error
0.0	0.0000000000				
0.2	0.1995560885	0.1983927153	0.0011633732	0.1938046647	0.0057514238
0.4	0.3964614648	0.3944914053	0.0019700595	0.3921973800	0.0042640848
0.6	0.5881288096	0.5860020447	0.0021267649	0.5882960700	-0.0001672604
0.8	0.7720957855	0.7706306082	0.0014651773	0.7752186588	-0.0031228733
1.0	0.9460830704				
1.2	1.1080471990	1.1050280949	0.0030191041	1.1029586999	0.0050884991
1.4	1.2562267328	1.2514682953	0.0047584375	1.2504335978	0.0057931350
1.6	1.3891804859	1.3843689742	0.0048115117	1.3854036717	0.0037768142
1.8	1.5058167803	1.5026954338	0.0031213465	1.5047648289	0.0010519514
2.0	1.6054129768				
2.2	1.6876248272	1.6847112702	0.0029135570	1.6860296928	0.0015951344
2.4	1.7524855008	1.7480251607	0.0044603401	1.7486843720	0.0038011288
2.6	1.8003944505	1.7960138596	0.0043805909	1.7953546483	0.0050398022
2.8	1.8320965891	1.8293365783	0.0027600108	1.8280181556	0.0040784335
3.0	1.8486525280				
3.2	1.8514008970	1.8503500395	0.0010508575	1.8538039024	-0.0024030054
3.4	1.8419139833	1.8404269248	0.0014870585	1.8421538563	-0.0002398730
3.6	1.8219481156	1.8206101155	0.0013380001	1.8188831840	0.0030649316
3.8	1.7933903548	1.7926265430	0.0007638118	1.7891726801	0.0042176747
4.0	1.7582031389				
4.2	1.7183685637	1.7195473379	-0.0011787742	1.7227609492	-0.0043923855
4.4	1.6758339594	1.6777854424	-0.0019514830	1.6793922480	-0.0035582886
4.6	1.6324603525	1.6345242579	-0.0020639054	1.6329174523	-0.0004570998
4.8	1.5899752782	1.5913705902	-0.0013953120	1.5881569790	0.0018182992
5.0	1.5499312449				
5.2	1.5136709468	1.5159322644	-0.0022613176	1.5170862573	-0.0034153105
5.4	1.4823000826	1.4858314083	-0.0035313257	1.4864084047	-0.0041083221
5.6	1.4566683847	1.4602056731	-0.0035372884	1.4596286767	-0.0029602920
5.8	1.4373591823	1.4396320553	-0.0022728730	1.4384780624	-0.0011188801
6.0	1.4246875513				
6.2	1.4187068241	1.4203522349	-0.0016454108	1.4193046892	-0.0005978651
6.4	1.4192229740	1.4216992559	-0.0024762819	1.4211754831	-0.0019525091
6.6	1.4258161486	1.4282048414	-0.0023886928	1.4287286143	-0.0029124657
6.8	1.4378684161	1.4393452195	-0.0014768024	1.4403927643	-0.0025243482
7.0	1.4545966142				
7.2	1.4750890554	1.4751295433	-0.0000404879	1.4732348537	0.0018542017
7.4	1.4983447533	1.4983023733	0.0000423800	1.4973550286	0.0009897247
7.6	1.5233137914	1.5231677596	0.0001460318	1.5241151044	-0.0008013130
7.8	1.5489374581	1.5487783573	0.0001591008	1.5506730469	-0.0017355888
8.0	1.5741868217				
8.2	1.5980985106	1.5968583098	0.0012402008	1.5957573693	0.0023411413
8.4	1.6198065968	1.6178298496	0.0019767472	1.6172793793	0.0025272175
8.6	1.6385696454	1.6365509706	0.0020186748	1.6371014409	0.0014682045
8.8	1.6537921861	1.6524712028	0.0013209833	1.6535721433	0.0002200428
9.0	1.6650400758				
9.2	1.6720494480	1.6707309206	0.0013185274	1.6711180795	0.0009313685
9.4	1.6747291725	1.6727135154	0.0020156571	1.6729070948	0.0018220777
9.6	1.6731569801	1.6711814396	0.0019755405	1.6709878602	0.0021691199
9.8	1.6675696169	1.6663282727	0.0012413442	1.6659411139	0.0016285030
10.0	1.6583475942				

The derivatives of the first and of the second order are :

$$y'(h) = b+2ch+3dh^2+4eh^3+5fh^4,$$

$$y''(h) = 2c+6dh+12eh^2+20fh^3.$$

The interpolating polynomial y(h) has to go through the points x_i, y_i (h = 0) and x_{i+1}, y_{i+1} (h = 1). The derivatives y'(h) in these points should be equal to :

$$\frac{1}{12}(y_{i-2} - 8y_{i-1} + 8_{i+1} - y_{i+2}) \text{ and}$$

$$\frac{1}{12}(y_{i-1} - 8y_i + 8y_{i+2} - y_{i+3}).$$

The derivatives y''(h) in these points should be equal to :

$$\frac{1}{12}(-y_{i-2}+16y_{i-1}-30y_i+16y_{i+1}-y_{i+2}) \text{ and}$$

$$\frac{1}{12}(-y_{i-1}+16y_i-30y_{i+1}+16y_{i+2}-y_{i+3}).$$

So we obtain the following set of six equations :

$$y(0) = a = y_i$$

$$y(1) = a+b+c+d+e+f = y_{i+1}$$

$$y'(0) = b = \frac{1}{12}(y_{i-2}-8y_{i-1} + 8y_{i+1} - y_{i+2})$$

$$y'(1) = b+2c+3d+4e+5f = \frac{1}{12}(y_{i-1}-8y_i+8y_{i+2} - y_{i+3})$$

$$y''(0) = 2c = \frac{1}{12}(-y_{i-2}+16y_{i-1} - 30y_i+16y_{i+1}-y_{i+2})$$

$$y''(1) = 2c+6d+12e+20f = \frac{1}{12}(-y_{i-1}+16y_i - 30y_{i+1}+16y_{i+2}-y_{i+3})$$

The solution of this set of six equations gives :

$$a = y_i$$

$$b = \frac{1}{12}(y_{i-2}-8y_{i-1}+8y_{i+1}-y_{i+2})$$

$$c = \frac{1}{24}(y_{i-2}+16y_{i-1}-30y_i+16y_{i+1}-y_{i+2})$$

$$d = \frac{1}{24}(-9y_{i-2}+39y_{i-1}-70y_i+66y_{i+1}-33y_{i+2} + 7y_{i+3})$$

$$e = \frac{1}{24}(13y_{i-2}-64y_{i-1}+126y_i-124y_{i+1}+61y_{i+2} -12y_{i+3})$$

$$f = \frac{1}{24}(-5y_{i-2}+25y_{i-1}-50y_i+50y_{i+1}-25y_{i+2}+5y_{i+3})$$

The interpolating polynomial finally becomes :

$$\begin{aligned} y(h) = y_i & + \frac{1}{12}(y_{i-2}-8y_{i-1}+8y_{i+1}-y_{i+2})h \\ & + \frac{1}{24}(-y_{i-2}+16y_{i-1}-30y_i+16y_{i+1}-y_{i+2})h^2 \\ & + \frac{1}{24}(-9y_{i-2}+39y_{i-1}-70y_i+66y_{i+1}-33y_{i+2} + 7y_{i+3})h^3 \\ & + \frac{1}{24}(13y_{i-2}-64y_{i-1}+126y_i-124y_{i+1} + 61y_{i+2}-12y_{i+3})h^4 \\ & + \frac{1}{24}(-5y_{i-2}+25y_{i-1}-50y_i+50y_{i+1}-25y_{i+2} + 5y_{i+3})h^5 \end{aligned}$$

or, after separation of the y-values used :

$$\begin{aligned} y(h) = & -\frac{1}{24}h(h-1)^3(5h+2)y_{i-2} \\ & + \frac{1}{24}h(h-1)(25h^3-39h^2+16)y_{i-1} \\ & - \frac{1}{12}(h-1)(25h^4-38h^3-3h^2+12h+12)y_i \\ & + \frac{1}{12}h(25h^4-62h^3+33h^2+8h+8)y_{i+1} \\ & - \frac{1}{24}h(h-1)(25h^3-36h^2-3h-2)y_{i+2} \\ & + \frac{1}{24}h^3(h-1)(5h-7)y_{i+3} \end{aligned} \qquad (S)$$

Finally the formula can be expressed as a function of the differences :

$$y_{i-1} = y_{i-2} + \Delta y_{i-2}$$

$$y_i = y_{i-2} + 2\Delta y_{i-2} + \Delta^2 y_{i-2}$$

$$y_{i+1} = y_{i-2} + 3\Delta y_{i-2} + 3\Delta^2 y_{i-2} + \Delta^3 y_{i-2}$$

$$y_{i+2} = y_{i-2} + 4\Delta y_{i-2} + 6\Delta^2 y_{i-2} + 4\Delta^3 y_{i-2} + \Delta^4 y_{i-2}$$

$$y_{i+3} = y_{i-2} + 5\Delta y_{i-2} + 10\Delta^2 y_{i-2} + 10\Delta^3 y_{i-2} + 5\Delta^4 y_{i-2} + \Delta^5 y_{i-2}$$

Substitution of these expressions in the foregoing formulae yields :

$$y(h) = y_i + \frac{1}{12}h(12\Delta y_{i-2} + 18\Delta^2 y_{i-2} + 4\Delta^3 y_{i-2} - \Delta^4 y_{i-2})$$
$$+\frac{1}{24}h^2(12\Delta^2 y_{i-2} + 12\Delta^3 y_{i-2} - \Delta^4 y_{i-2})$$
$$+\frac{1}{24}h^3(4\Delta^3 y_{i-2} + 2\Delta^4 y_{i-2} + 7\Delta^5 y_{i-2})$$
$$+\frac{1}{24}h^4(\Delta^4 y_{i-2} - 12\Delta^5 y_{i-2})$$
$$+\frac{5}{24}h^5(\Delta^5 y_{i-2})$$
$$= y_i + \Delta y_{i-2}(h)$$
$$+\frac{1}{2}\Delta^2 y_{i-2}(h)(h+3)$$
$$+\frac{1}{6}\Delta^3 y_{i-2}(h)(h+1)(h+2)$$
$$+\frac{1}{24}\Delta^4 y_{i-2}(h)(h-1)(h+1)(h+2)$$
$$+\frac{1}{24}\Delta^5 y_{i-2}(h^3)(5h-7)(h-1)$$

Originally Sprague has published his interpolation formula as a function of these differences. As a function of h these formulae are simpler. The disadvantage however is, that the y-values cannot be used directly : the differences have to be calculated.

3.3. Example

The accompanying table 2 represents the results that have been obtained for the function :

$$Si(x) = \int_0^x \frac{\sin(x)}{x}\,dx, \text{ with } x = 0.0(0.2)10.0,$$

respectively with the method of the fifth degree polynomial of Lagrange and with the described method of Sprague.

Herein we started from the accurate values of this function for the arguments $x = -2.0(1.0)12.0$, accurate to the tenth decimal place :

x	Si(x)
x = − 2.0	Si(x) = − 1.6054129768
− 1.0	− 0.9460830704
0.0	0.0000000000
1.0	0.9460830704
2.0	1.6054129768
3.0	1.8486525280
4.0	1.7582031389
5.0	1.5499312449
6.0	1.4246875513
7.0	1.4545966142
8.0	1.5741868217
9.0	1.6650400758
10.0	1.6583475942
11.0	1.5783068069
12.0	1.5049712415

Table 2 also contains the accurate values of the function for the calculated interpolations, as well as the errors of both methods. If we put :

E_1 = the mean error,

E_2 = the mean of the absolute value of the error,

E_3 = the quadratic mean error,

E_4 = the absolute value of the maximal error,

then we obtain for Lagrange and Sprague respectively:

E_1 = 0.0000776753 and 0.0000776753

E_2 = 0.0002407607 and 0.0003472044

E_3 = 0.0003047908 and 0.0004429708

E_4 = 0.0006526439 and 0.0009115076

The accuracy is for both methods of the same order of magnitude. It should be noted however, that the mean error for the two methods is precisely the same.

Remark

The accurate values of the function Si(x) have been taken from the tables of Glaisher [12] and of Abramowitz-Stegun [13].

3,4. Theorems

Theorem 1 : If we integrate the interpolation formula of Sprague S over the interval h = [0,1], the twice corrected trapezoid rule is obtained.

If this integral is worked out, we obtain indeed :

$$\int_0^1 y(h)\,dh = \frac{1}{1440}(11y_{i-2} - 93y_{i-1} + 802y_i + 802y_{i+1} - 93y_{i+2} + 11y_{i+3})$$

Theorem 2 : If the interpolation formula of Sprague S is integrated over the interval h = [-2,3], the six-point-rule of Newton-Cotes (closed form) is obtained.

If this integral is worked out, we obtain indeed :

$$\int_{-2}^{3} y(h)dh = \frac{5}{288}(19y_{i-2} + 75y_{i-1} + 50y_i + 50y_{i+1} + 75y_{i+2} + 19y_{i+3})$$

Although the fifth degree polynomials used in the methods of Lagrange and Sprague are totally different, the same numerical integration formulae are obtained again for the two methods. The first theorem explains why, as has been stated in the treated example, the mean error is the same for both methods. May be the same integration formulae are also obtained when the method of Sprague-Karup would be extended to interpolation polynomials of the higher degree.

4. TABLES K AND S

In order to simplify the computational work in the

TABLE 2

x	Si(x)	Lagrange	Error	Sprague	Error
0.0	0.0000000000				
0.2	0.1995560885	0.1993901029	0.0001659856	0.1986445809	0.0009115076
0.4	0.3964614648	0.3961839419	0.0002775229	0.3957507331	0.0007107317
0.6	0.5881288096	0.5878356260	0.0002931836	0.5882688347	-0.0001400251
0.8	0.7720957855	0.7719000106	0.0001957749	0.7726455326	-0.0005497471
1.0	0.9460830704				
1.2	1.1080471990	1.1076390731	0.0004081259	1.1073818011	0.0006653979
1.4	1.2562267328	1.2555784901	0.0006482427	1.2554289942	0.0007977386
1.6	1.3891804859	1.3885278420	0.0006526439	1.3886773379	0.0005031480
1.8	1.5058167803	1.5054002815	0.0004164988	1.5056575535	0.0001592268
2.0	1.6054129768				
2.2	1.6876248272	1.6872643647	0.0003604625	1.6876350684	-0.0000102412
2.4	1.7524855008	1.7519265079	0.0005589929	1.7521419168	0.0003435840
2.6	1.8003944505	1.7998450736	0.0005493769	1.7996296647	0.0007647858
2.8	1.8320965891	1.8317544160	0.0003421731	1.8313837123	0.0007128768
3.0	1.8486525280				
3.2	1.8514008970	1.8513311617	0.0000697353	1.8520343666	-0.0006334696
3.4	1.8419139833	1.8418200762	0.0000939071	1.8422286953	-0.0003147120
3.6	1.8219481156	1.8218702282	0.0000778874	1.8214616092	0.0004865064
3.8	1.7933903548	1.7933510906	0.0000392642	1.7926478857	0.0007424691
4.0	1.7582031389				
4.2	1.7183685637	1.7186106422	-0.0002420785	1.7191491747	-0.0007806110
4.4	1.6758339594	1.6762264756	-0.0003925162	1.6765394067	-0.0007054473
4.6	1.6324603525	1.6328634066	-0.0004030541	1.6325504756	-0.0000901231
4.8	1.5899752782	1.5902374029	-0.0002621247	1.5896988704	0.0002764078
5.0	1.5499312449				
5.2	1.5136709468	1.5140224074	-0.0003514606	1.5140644158	-0.0003934690
5.4	1.4823000826	1.4828525721	-0.0005524895	1.4828769824	-0.0005768998
5.6	1.4566683847	1.4572188894	-0.0005505047	1.4571944791	-0.0005260944
5.8	1.4373591823	1.4377068709	-0.0003476886	1.4376648625	-0.0003056802
6.0	1.4246875513				
6.2	1.4187068241	1.4189071905	-0.0002003664	1.4185062896	0.0002005345
6.4	1.4192229740	1.4195272552	-0.0003042812	1.4192942992	-0.0000713252
6.6	1.4258161486	1.4261086868	-0.0002925382	1.4263416427	-0.0005254941
6.8	1.4378684161	1.4380464487	-0.0001780326	1.4384473496	-0.0005789335
7.0	1.4545966142				
7.2	1.4750890554	1.4750169074	0.0000721480	1.4745312032	0.0005578522
7.4	1.4983447533	1.4982190520	0.0001257013	1.4979368185	0.0004079348
7.6	1.5233137914	1.5231763283	0.0001374631	1.5234585619	-0.0001447705
7.8	1.5489374581	1.5488429379	0.0000945202	1.5493286422	-0.0003911841
8.0	1.5741868217				
8.2	1.5980985106	1.5978476467	0.0002508639	1.5976421190	0.0004563916
8.4	1.6198065968	1.6194077017	0.0003988951	1.6192882734	0.0005183234
8.6	1.6385696454	1.6381677064	0.0004019390	1.6382871346	0.0002825108
8.8	1.6537921861	1.6535355295	0.0002566566	1.6537410572	0.0000511289
9.0	1.6650400758				
9.2	1.6720494480	1.6718348269	0.0002146211	1.6720107698	0.0000386782
9.4	1.6747291725	1.6743974165	0.0003317560	1.6744996536	0.0002295189
9.6	1.6731569801	1.6728320542	0.0003249259	1.6727298171	0.0004271630
9.8	1.6675696169	1.6673679837	0.0002016332	1.6671920407	0.0003775762
10.0	1.6583475942				

interpolation method of Lagrange, tables have been elaborated with the coefficients as a function of the argument. Such tables have been published among others in the work of Abramowitz and Stegun [14]. Similar tables can be obtained for the method of Sprague-Karup.

4.1. Table K

If in the interpolation formula of Karup (cfr. 2.2) we put :

$$k_{i-1}(h) = -\frac{1}{2}h(h-1)^2$$

$$k_i(h) = \frac{1}{2}(h-1)(3h^2-2h-2)$$

$$k_{i+1}(h) = -\frac{1}{2}h(3h^2-4h-1)$$

$$k_{i+2}(h) = \frac{1}{2}h^2(h-1)$$

we obtain :

$$y(h) = k_{i-1}(h)y_{i-1} + k_i(h)y_i + k_{i+1}(h)y_{i+1} + k_{i+2}(h)y_{i+2}$$

If the k-values are known for a given value of h, the interpolation is obtained by evaluating the sum of four products. The following identities should be considered :

$$k_{i-1}(h) = k_{i+2}(1-h)$$

$$k_i(h) = k_{i+1}(1-h)$$

$$k_{i+1}(h) = k_i(1-h)$$

$$k_{i+2}(h) = k_{i-1}(1-h)$$

In table K these k-values are tabulated for $h = 0.00(0.01)1.00$. Herein use has been made of the above identities, which explains the double entry.

4.2. Table S

If in the interpolation formula of Sprague S (cfr. 3.2) we put :

$$s_{i-2}(h) = -\frac{1}{24}h(h-1)^3(5h+2)$$

$$s_{i-1}(h) = \frac{1}{24}h(h-1)(25h^3-39h^2+16)$$

$$s_i(h) = -\frac{1}{12}(h-1)(25h^4-38h^3-3h^2+12h+12)$$

$$s_{i+1}(h) = -\frac{1}{12}h(25h^4-62h^3+33h^2+8h+8)$$

$$s_{i+2}(h) = \frac{1}{24}h(h-1)(25h^3-36h^2-3h-2)$$

$$s_{i+3}(h) = \frac{1}{24}h^3(h-1)(5h-7)$$

we obtain :

$$y(h) = s_{i-2}(h)y_{i-2} + s_{i-1}(h)y_{i-1} + s_i(h)y_i + s_{i+1}(h)y_{i+1} + s_{i+2}(h)y_{i+2} + s_{i+3}(h)y_{i+3}.$$

If the s-values are known for a given value of h, the interpolation is obtained by evaluating the sum of six products. The following identities should be considered :

$$s_{i-2}(h) = s_{i+3}(1-h)$$

$$s_{i-1}(h) = s_{i+2}(1-h)$$

$$s_i(h) = s_{i+1}(1-h)$$

$$s_{i+1}(h) = s_i(1-h)$$

$$s_{i+2}(h) = s_{i-1}(1-h)$$

$$s_{i+3}(h) = s_{i-2}(1-h)$$

In table S these s-values are tabulated for $h = 0.00(0.01)1.00$, rounded off to the tenth decimal place. Herein use has been made of the above identities, which explains the double entry.

5. SUBROUTINES

FORTRAN-subroutines have been programmed for the interpolation methods described : KARUP and SPRAG. These routines will be submitted for publication to the editors of the Journal of Computational and Applied Mathematics.

REFERENCES

1. T. B. Sprague, "Explanation ov a new formula for interpolation", Journal of the Institute of Actuaries, Vol. 22, 1880, pp. 270-285.
2. J. Karup, "On a new mechanical method of graduation", Transactions of the Second International Actuarial Congress, London, 1899.
3. G. King, "On the construction of mortality tables from census returns and records of deaths", Journal of the Institute of Actuaries, Vol. 42, 1908, pp. 238-246.
4. J. Buchanan, "Osculatory interpolation by central differences, with an application to life table construction", Journal of the Institute of Actuaries, Vol. 42, 1908, pp. 369-394.
5. G. J. Lidstone, "Alternative demonstration of the formula for osculatory interpolation", Journal of the Institute of Actuaries, Vol. 42, 1908, pp. 394-397.
6. G. King, "On a new method of construction and of graduating mortality and other tables", Journal of the Institute of Actuaries, Vol. 43, 1909, pp. 109-184.
7. J. W. Glover, "Derivation of the United States mortality table by osculatory interpolation", Quarterly Publications of the American Statistical Association, Vol. 12, 1910, pp. 87-93.
8. D. B. Judd, "Extension of the standard visibility function to intervals of 1 millimicron by third-difference osculatory interpolation", Bureau of Standards Journal of Research, Vol. 6, No. 3, March 1931, pp. 465-471.

TABLE K

H	→ K(I-1)	K(I)	K(I+1)	K(I+2)	
0.00	0.0	1.0000000	0.0	0.0	1.00
0.01	-0.0049005	0.9997514	0.0051985	-0.0000495	0.99
0.02	-0.0096040	0.9990120	0.0107880	-0.0001960	0.98
0.03	-0.0141135	0.9977905	0.0167595	-0.0004365	0.97
0.04	-0.0184320	0.9960960	0.0231040	-0.0007680	0.96
0.05	-0.0225625	0.9939375	0.0298125	-0.0011875	0.95
0.06	-0.0265080	0.9913239	0.0368760	-0.0016920	0.94
0.07	-0.0302715	0.9882644	0.0442855	-0.0022785	0.93
0.08	-0.0338560	0.9847680	0.0520320	-0.0029440	0.92
0.09	-0.0372645	0.9808435	0.0601065	-0.0036855	0.91
0.10	-0.0405000	0.9765000	0.0685000	-0.0045000	0.90
0.11	-0.0435655	0.9717464	0.0772035	-0.0053845	0.89
0.12	-0.0464640	0.9665920	0.0862080	-0.0063360	0.88
0.13	-0.0491985	0.9610454	0.0955045	-0.0073515	0.87
0.14	-0.0517720	0.9551160	0.1050839	-0.0084280	0.86
0.15	-0.0541875	0.9488125	0.1149375	-0.0095625	0.85
0.16	-0.0564480	0.9421440	0.1250560	-0.0107520	0.84
0.17	-0.0585565	0.9351195	0.1354305	-0.0119935	0.83
0.18	-0.0605160	0.9277480	0.1460519	-0.0132840	0.82
0.19	-0.0623295	0.9200385	0.1569115	-0.0146205	0.81
0.20	-0.0640000	0.9119999	0.1680000	-0.0160000	0.80
0.21	-0.0655305	0.9036415	0.1793085	-0.0174195	0.79
0.22	-0.0669240	0.8949720	0.1908280	-0.0188760	0.78
0.23	-0.0681835	0.8860005	0.2025495	-0.0203665	0.77
0.24	-0.0693120	0.8767360	0.2144639	-0.0218880	0.76
0.25	-0.0703125	0.8671875	0.2265625	-0.0234375	0.75
0.26	-0.0711880	0.8573640	0.2388360	-0.0250120	0.74
0.27	-0.0719415	0.8472745	0.2512755	-0.0266085	0.73
0.28	-0.0725760	0.8369280	0.2638720	-0.0282240	0.72
0.29	-0.0730945	0.8263335	0.2766165	-0.0298555	0.71
0.30	-0.0735000	0.8155000	0.2895000	-0.0315000	0.70
0.31	-0.0737955	0.8044364	0.3025135	-0.0331545	0.69
0.32	-0.0739840	0.7931520	0.3156480	-0.0348160	0.68
0.33	-0.0740685	0.7816555	0.3288945	-0.0364815	0.67
0.34	-0.0740520	0.7699560	0.3422440	-0.0381480	0.66
0.35	-0.0739375	0.7580625	0.3556875	-0.0398125	0.65
0.36	-0.0737280	0.7459840	0.3692160	-0.0414720	0.64
0.37	-0.0734265	0.7337295	0.3828205	-0.0431235	0.63
0.38	-0.0730360	0.7213080	0.3964919	-0.0447640	0.62
0.39	-0.0725595	0.7087285	0.4102215	-0.0463905	0.61
0.40	-0.0720000	0.6960000	0.4240000	-0.0480000	0.60
0.41	-0.0713605	0.6831315	0.4378185	-0.0495895	0.59
0.42	-0.0706440	0.6701320	0.4516680	-0.0511560	0.58
0.43	-0.0698535	0.6570105	0.4655395	-0.0526965	0.57
0.44	-0.0689920	0.6437760	0.4794240	-0.0542080	0.56
0.45	-0.0680625	0.6304375	0.4933125	-0.0556875	0.55
0.46	-0.0670680	0.6170040	0.5071959	-0.0571320	0.54
0.47	-0.0660115	0.6034845	0.5210655	-0.0585385	0.53
0.48	-0.0648960	0.5898880	0.5349120	-0.0599040	0.52
0.49	-0.0637245	0.5762235	0.5487265	-0.0612255	0.51
0.50	-0.0625000	0.5625000	0.5625000	-0.0625000	0.50
	K(I+2)	K(I+1)	K(I)	K(I-1) ←	H

TABLE S

→	S(I-2)	S(I-1)	S(I)	S(I+1)	S(I+2)	S(I+3)
00	0.0000000000	0.0000000000	1.0000000000	0.0000000000	0.0000000000	0.0000000000
01	0.0008287971	-0.0065984016	0.9998721356	0.0067360319	-0.0008388497	0.0000002867
02	0.0016470860	-0.0130540900	0.9994775000	0.0136211800	-0.0016939300	0.0000022540
03	0.0024528087	-0.0193582597	0.9988004519	0.0206701156	-0.0025725916	0.0000074751
04	0.0032440320	-0.0255027200	0.9978265600	0.0278963200	-0.0034816000	0.0000174080
05	0.0040189453	-0.0314798828	0.9965425781	0.0353121094	-0.0044271484	0.0000333984
06	0.0047758580	-0.0372827500	0.9949364200	0.0429286600	-0.0054148700	0.0000566820
07	0.0055131969	-0.0429049009	0.9929971344	0.0507560331	-0.0064498503	0.0000883868
08	0.0062295040	-0.0483404800	0.9907148800	0.0588032000	-0.0075366400	0.0001295360
09	0.0069234336	-0.0535841841	0.9880809006	0.0670780669	-0.0086792672	0.0001810502
10	0.0075937500	-0.0586312500	0.9850875000	0.0755875000	-0.0098812500	0.0002437500
11	0.0082393252	-0.0634774422	0.9817280169	0.0843373506	-0.0111456091	0.0003183586
12	0.0088591360	-0.0681190400	0.9779968000	0.0933324800	-0.0124748800	0.0004055040
13	0.0094522618	-0.0725528253	0.9738891831	0.1025767844	-0.0138711259	0.0005057219
.14	0.0100178820	-0.0767760700	0.9694014600	0.1120732200	-0.0153359500	0.0006194580
.15	0.0105552734	-0.0807865234	0.9645308594	0.1218238281	-0.0168705078	0.0007470703
.16	0.0110638080	-0.0845824000	0.9592755200	0.1318297600	-0.0184755200	0.0008888320
.17	0.0115429501	-0.0881623666	0.9536344656	0.1420913019	-0.0201512847	0.0010449337
.18	0.0119922540	-0.0915255300	0.9476075800	0.1526079000	-0.0218976900	0.0012154860
.19	0.0124113617	-0.0946714247	0.9411955819	0.1633781856	-0.0237142266	0.0014005221
.20	0.0128000000	-0.0976000000	0.9344000000	0.1744000000	-0.0256000000	0.0016000000
.21	0.0131579783	-0.1003116078	0.9272231481	0.1856704194	-0.0275537434	0.0018138054
.22	0.0134851860	-0.1028069900	0.9196681000	0.1971857800	-0.0295738300	0.0020417540
.23	0.0137815899	-0.1050872659	0.9117386644	0.2089417031	-0.0316582853	0.0022835938
.24	0.0140472320	-0.1071539200	0.9034393600	0.2209331200	-0.0338048000	0.0025390080
.25	0.0142822266	-0.1090087891	0.8947753906	0.2331542969	-0.0360107422	0.0028076172
.26	0.0144867580	-0.1106540500	0.8857526200	0.2455988600	-0.0382731700	0.0030889820
.27	0.0146610782	-0.1120922072	0.8763775469	0.2582598206	-0.0405888441	0.0033826056
.28	0.0148055040	-0.1133260800	0.8666572800	0.2711296000	-0.0429542400	0.0036879360
.29	0.0149204148	-0.1143587903	0.8565995131	0.2842000544	-0.0453655609	0.0040043689
.30	0.0150062500	-0.1151937500	0.8462125000	0.2974625000	-0.0478187500	0.0043312500
.31	0.0150635064	-0.1158346484	0.8355050294	0.3109077381	-0.0503095028	0.0046678773
.32	0.0150927360	-0.1162854400	0.8244864000	0.3245260800	-0.0528332800	0.0050135040
.33	0.0150945431	-0.1165503316	0.8131663956	0.3383073719	-0.0553853197	0.0053673407
.34	0.0150695820	-0.1166337700	0.8015552600	0.3522410200	-0.0579606500	0.0057285580
.35	0.0150185547	-0.1165404297	0.7896636719	0.3663160156	-0.0605541016	0.0060962891
.36	0.0149422080	-0.1162752000	0.7775027200	0.3805209600	-0.0631603200	0.0064696320
.37	0.0148413313	-0.1158431728	0.7650838781	0.3948440894	-0.0657737784	0.0068476524
.38	0.0147167540	-0.1152496300	0.7524189800	0.4092733000	-0.0683887900	0.0072293860
.39	0.0145693429	-0.1145000309	0.7395201944	0.4237961731	-0.0709995203	0.0076138408
.40	0.0144000000	-0.1136000000	0.7264000000	0.4384000000	-0.0736000000	0.0080000000
0.41	0.0142096596	-0.1125553141	0.7130711606	0.4530718069	-0.0761841372	0.0083868242
0.42	0.0139992860	-0.1113718900	0.6995467000	0.4677983800	-0.0787457300	0.0087732540
0.43	0.0137698712	-0.1100557722	0.6858398769	0.4825662906	-0.0812784791	0.0091582126
0.44	0.0135224320	-0.1086131200	0.6719641600	0.4973619200	-0.0837760000	0.0095406080
0.45	0.0132580078	-0.1070501953	0.6579332031	0.5121714844	-0.0862318359	0.0099193359
0.46	0.0129776580	-0.1053733500	0.6437608200	0.5269810600	-0.0886394700	0.0102932820
0.47	0.0126824594	-0.1035890134	0.6294609594	0.5417766081	-0.0909923378	0.0106613243
0.48	0.0123735040	-0.1017036800	0.6150476800	0.5565440000	-0.0932838400	0.0110223360
0.49	0.0120518961	-0.0997238966	0.6005351256	0.5712690419	-0.0955073547	0.0113751877
0.50	0.0117187500	-0.0976562500	0.5859375000	0.5859375000	-0.0976562500	0.0117187500
	S(I+3)	S(I+2)	S(I+1)	S(I)	S(I-1)	S(I-2) ←

9. D. B. Judd, "Interpolation of the O.S.A. 'excitation' data by the fifth-difference osculatory formula", Bureau of Standards Journal of Research, Vol. 7, No. 1, July 1931, pp. 85-91.

10. A. C. Hardy, "Handbook of Colorimetry", The Technology Press, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1936.

11. J. L. F. De Kerf, "De mathematische nauwkeurigheid van de benaderingsmethoden ter berekening van de trichromatische coëfficiënten en de relatieve helderheid van kleurmonsters", Master's Degree Dissertation, Central Jury of Belgium, Brussels, 1956.

12. J. W. L. Glaisher, "Tables of the numerical values of the sine-integral, cosine-integral, and exponential-integral", Philosophical Transactions of the Royal Society of London, Vol. 160, Part II, 1870, pp. 367-388.

13. M. Abramowitz and I. A. Stegun (eds.), "Handbook of mathematical functions (with formulas, graphs, and mathematical tables)", Chapter 5: Exponential integral and related functions (W. Gautschi and W. F. Cahill), Dover Publications, New York, 1965, pp. 238-243.

14. M. Abramowitz and I. A. Stegun (eds.), "Handbook of mathematical functions (with formulas, graphs and mathematical tables)", Chapter 25 : Numerical interpolation, differentiation, and integration (P. J. Davis and I. Polonsky); Dover Publications, New York, 1965, pp. 900-913.