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MULTICOREFLECTIVE SUBCATEGORIES AND COPRIME OBJECTS

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Multicoreflective subcategories of a given category can be viewed as "categories of connected objects" for suitable connectivity notions. If one tries to define "connected objects" in an absolute way, one is led to the notion of coprime object. Both notions are treated using the same tool, which is parallel to the description of coreflective subcategories by projectivity. These methods can be applied to characterize multicoreflective subcategories of Comp. They also lead to some special results about reflective and multireflective subcategories of the category of unital rings. In particular, the category of powers of $\mathbb{Z}$ is reflective if and only if there exists no uncountable measurable cardinal.

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multicoreflective subcategory	measurable cardinal	connected colimit
compact Hausdorff space	multiorthogonality	coprime object
ultraconnected ring	connected cocone	unital ring

Introduction

Kaput's [14] notion of "locally adjunctable functor" led to the study of various modifications [7, 10, 24] using different terminologies. Important examples are furnished by component categories [2, 12, 19, 21, 22, 25] in topological categories: They are multicoreflective, i.e. their inclusion functors are multileft-adjoint [18]. This result can be generalized to other categories—like Hausdorff spaces—if the notion of "component category" is defined in a suitable way [2, 4]. In the present paper, we consider multicoreflective subcategories of arbitrary categories. Our results can be used to prove well-known properties of component categories. Thus we will mention only examples not covered by [2, 19], particularly in the category Comp of compact Hausdorff spaces and in the dual of the category Rng of unital rings. So it turns out that a full replete subcategory of Comp is multicoreflective if and only if it is closed under connected colimits.

Another approach to describe connectivity are Hoffmann's [13] "Z-objects", which we will call "coprime objects".

While a component category is a generalization of "the category of all objects with a given connectedness property", a coprime object can be viewed as a "connected object" in a category with coproducts.

Now the notions of multicoreflectivity and coprimality can be handled by a common tool, namely the multiorthogonality relation between objects and sinks. The treatment is analogous to the description of coreflective subcategories by orthogonality, i.e. projectivity [16, 18, 24]. Combining these results we obtain some results on reflective and multireflective subcategories of Rng. In particular, the category of all powers of $\mathbb{Z}$ (or $\mathbb{Q}$, or $\mathbb{R}$) is multireflective if and only if there are no uncountable measurable cardinals.

According to the referee's suggestions, the investigation of multicoreflective subcategories of Comp was refined; in particular Example 2.4 was added in the revised version. Some results are taken from the author's thesis [4].

1. Multicoreflective subcategories

A category $\mathscr{D}$ is called connected, if $\mathscr{D} \neq \emptyset$ and all functors from $\mathscr{D}$ to discrete categories are constant. The latter condition means that for all objects $D, D' \in |\mathscr{D}|$ there are finitely many $D = D_1, D_2, \ldots, D_n = D'$ with $\mathscr{D}(D_i, D_{i+1}) \cup \mathscr{D}(D_{i+1}, D_i) \neq \emptyset$ for all $i < n$. A diagram in a category $\mathscr{X}$ is a functor $H: \mathscr{D} \to \mathscr{X}$, and we call it connected, if $\mathscr{D}$ is connected. A cocone $\varphi: H \to \Delta A$ is called connected, if the diagram H is connected; here ΔA is the constant functor with value $A \in |\mathscr{X}|$.

For any class $\mathscr{M} \subset \operatorname{Mor} \mathscr{X}$, $\mathscr{M}^{\perp}$ denotes the conglomerate of all cocones $\varepsilon: H \to \Delta Z$, such that for any $m: X \to Y$ in $\mathscr{M}$ and any cocone $\varphi: H \to \Delta X$ and any $g: Z \to Y$ with $\Delta g \cdot \varepsilon = \Delta m \cdot \varphi$ there is a unique $l: Z \to X$ with $ml = g$ and $\Delta l \cdot \varepsilon = \varphi$. Here $\Delta l: \Delta Z \to \Delta X$ is the constant natural transformation with value l. $\hat{\mathscr{M}}$ denotes the collection of all sinks $(X, (m_i: A_i \to X)_{i \in I})$ with $X \in |\mathscr{X}|$ and $|A_i| \in \mathscr{X}$, $m_i \in \mathscr{M}$ for all $i \in I$, where I is a class. We sometimes write the above sink shortly as $(m_i)_{i \in I}$.

1.1. Definition. (i) Let $A \in |\mathscr{X}|$ and let $(X, (u_j: B_j \to X)_{j \in J})$ be a sink in $\mathscr{X}$. Then A is called *multiorthogonal* to $(X, (u_j)_{j \in J})$ (written $A \perp (u_j)_j$), if for every $f: A \to X$ there is a unique pair (j, h) with $j \in J$, $h: A \to B_j$ and $u_j h = f$.

(ii) If $\mathscr{A} \subset \mathscr{X}$ is a full subcategory, then $\mathscr{A}_{\perp}$ denotes the conglomerate of all sinks $(X, (u_j)_{j \in J})$ with $A \perp (X, (u_j)_{j \in J})$ for all $A \in |\mathscr{A}|$.

(iii) If $\mathscr{S}$ is a conglomerate of sinks, then $\mathscr{S}^{\perp}$ denotes the full subcategory of all objects A, such that $A \perp (u_j)_j$ for all $(X, (u_j)_{j \in J}) \in \mathscr{S}$.

(iv) If $\mathscr{M} \subset \operatorname{Mor} \mathscr{X}$, a full replete subcategory $\mathscr{A} \subset \mathscr{X}$ is called *$\mathscr{M}$-multicoreflective*, if for every $X \in |\mathscr{X}|$ there is a sink $(X, (m_i: A_i \to X)_{i \in I}) \in \mathscr{A}_{\perp} \cap \hat{\mathscr{M}}$ with $A_i \in |\mathscr{A}|$ for all $i \in I$. If $\mathscr{M} = \operatorname{Mor} \mathscr{X}$, we simply say that $\mathscr{A}$ is multicoreflective, and $(m_i)_i$ is called a multicoreflection sink.

Every coreflective subcategory is multicoreflective in the multicoreflection sink consisting of the coreflection morphism only. The empty category is multicoreflective in any category with all multicoreflection sinks being empty. Any component category in a topological category is multicoreflective [2, 19], e.g. the category of connected topological spaces is multicoreflective in Top. An analogous result holds for Hausdorff spaces. An example for the dual situation is the category of fields as a multireflective subcategory of the category of commutative unital rings.

1.2. Theorem. *Let $\mathscr{A} \subset \mathscr{X}$ be a full replete subcategory and let $\mathscr{M} \subset \mathrm{Mor}\,\mathscr{X}$. Then each of the statements* (i)–(iii) *below implies the next one. Moreover, if $\mathscr{X}$ has $(\mathscr{M}^{\perp}, \mathscr{M})$-factorizations for connected cocones, i.e. if for every connected cocone $\varphi : H \to \Delta Z$ there are $\varepsilon \in \mathscr{M}^{\perp}$, $m \in \mathscr{M}$ with $\varphi = \Delta m \cdot \varepsilon$, then all three statements are equivalent*:

(i) *$\mathscr{A}$ is $\mathscr{M}$-multicoreflective,*

(ii) *$\mathscr{A} = (\mathscr{A}_{\perp} \cap \hat{\mathscr{M}})^{\perp}$,*

(iii) *If $\varepsilon : H \to \Delta Z$, $H : \mathscr{D} \to \mathscr{X}$, is a connected cocone with $H[\mathscr{D}] \subset \mathscr{A}$, $\varepsilon \in \mathscr{M}^{\perp}$, then $Z \in |\mathscr{A}|$.*

Proof. (i)$\Rightarrow$(ii): "$\subset$" is trivial. So assume $X \in (\mathscr{A}_{\perp} \cap \hat{\mathscr{M}})^{\perp}$. By (i), there is a sink

$$(m_i : A_i \to X)_{i \in I} \in \mathscr{A}_{\perp} \cap \hat{\mathscr{M}},$$

with $A_i \in |\mathscr{A}|$ for all $i \in I$, hence $X \perp (m_i)_i$. Thus there are $i_0 \in I$, $t : X \to A_{i_0}$ with $m_{i_0} t = 1_X$. Now $m_{i_0} t m_{i_0} = m_{i_0}$, the uniqueness condition in Definition 1.1(i), gives $t m_{i_0} = 1_{A_{i_0}}$, hence $t \in \mathrm{Iso}\,\mathscr{X}$ and therefore $X \in |\mathscr{A}|$.

(ii)$\Rightarrow$(iii): By (ii) it suffices to prove $Z \perp (u_j)_j$ for every $(u_j : B_j \to X)_{j \in J} \in \mathscr{A}_{\perp} \cap \hat{\mathscr{M}}$. For $f : Z \to X$, $D \in |\mathscr{D}|$, we have $HD \in |\mathscr{A}|$, and thus there is a unique pair $(j_D, \varphi D)$ with $u_{j_D} \cdot \varphi D = f \cdot \varepsilon D$. If $d \in \mathscr{D}(D, D')$, we have $f \cdot \varepsilon D = f \cdot \varepsilon D' \cdot Hd = u_{j_{D'}} \cdot \varphi D' \cdot Hd$, hence $j_{D'} = j_D$ and $\varphi D' \cdot Hd = \varphi D$. As $\mathscr{D}$ is connected, there is a unique $j_0 \in J$ with $j_D = j_0$ for all $D \in |\mathscr{D}|$, and $\varphi : H \to \Delta B_{j_0}$ is a cocone with $\Delta u_{j_0} \cdot \varphi = \Delta f \cdot \varepsilon$. As $u_{j_0} \in \mathscr{M}$ and $\varepsilon \in \mathscr{M}^{\perp}$ there is a unique $l : Z \to B_{j_0}$ with $u_{j_0} l = f$ and $\Delta l \cdot \varepsilon = \varphi$. The uniqueness follows immediately.

(iii)$\Rightarrow$(i): Here we assume the existence of $(\mathscr{M}^{\perp}, \mathscr{M})$-factorizations for arbitrary (not necessarily small) connected cocones. For $X \in |\mathscr{X}|$ we consider the comma category $\mathscr{A}/X$, whose objects are pairs (Z, f) with $Z \in |\mathscr{A}|$, $f \in \mathscr{X}(Z, X)$, the morphisms from (Z, f) to (Y, g) being given by $\mathscr{X}$-morphisms $h : Z \to Y$ with $f = gh$. Let $\mathscr{D}_i$ denote the distinct connected components—i.e. maximal connected subcategories—of $\mathscr{A}/X$, where i ranges over a suitable indexing class I. For $i \in I$ fixed, consider the connected cocone $\varphi : H \to \Delta X$, where $H(Z, f) := Z$, $\varphi(Z, f) := f$ for all $(Z, f) \in |\mathscr{D}_i|$. By hypothesis we have $\varphi = \Delta m_i \cdot \varepsilon$ for some $m_i : A_i \to X$ in $\mathscr{M}$ and $\varepsilon : H \to \Delta A_i$ in $\mathscr{M}^{\perp}$, in particular $A_i \in |\mathscr{A}|$ by (iii). This leads to $f = \varphi(Z, f) = m_i \cdot \varepsilon(Z, f)$ for all $(Z, f) \in |\mathscr{D}_i|$, proving the existence condition of $(m_i)_{i \in I} \in \mathscr{A}_{\perp}$ and also $m_i = m_i \cdot \varepsilon(A_i, m_i)$.

In order to show uniqueness, assume $f = m_j h$ with $j \in I$, $h \in \mathscr{X}(A_j, X)$. Then $\varepsilon(Z, f) : Z \to A_i$, $h : Z \to A_j$ give rise to $\mathscr{A}/X$-morphisms $(Z, f) \to (A_i, m_i)$, $(z, f) \to (A_j, m_j)$. Therefore (A_i, m_i), (Z, f), (A_j, m_j) lie in the same connected component of $\mathscr{A}/X$, i.e. in $\mathscr{D}_i$, hence $i = j$, and only the uniqueness of h remains to be shown.

The cocone property of ε gives $\varepsilon(Z, f) = \varepsilon(A_i, m_i) \cdot h = \varepsilon(A_i, m_i) \cdot \varepsilon(Z, f)$, proving $\Delta\varepsilon(A_i, m_i) \cdot \varepsilon = \varepsilon$. Since $m_i \in \mathcal{M}$ and $m_i = m_i \cdot \varepsilon(A_i, m_i)$, the uniqueness property in $\varepsilon \in \mathcal{M}^{\perp}$ leads to $\varepsilon(A_i, m_i) = 1_{A_i}$, hence $h = \varepsilon(A_i, m_i) \cdot h = \varepsilon(Z, f)$, and the proof is complete. □

1.3. Corollary. (ii) ([6, Theorem 20], cf. [10]) *Every multicoreflective category of an arbitrary category is closed under the formation of connected colimits.*

Proof. Apply 1.2(i)⇒(iii) to $\mathcal{M} := \operatorname{Mor} \mathcal{X}$. □

Note that Theorem 1.2(iii)⇒(i) is a special case of [22, Corollary 4.5]. We gave direct proof here for the reader not familiar with the abstract machinery used there. Our proof obviously becomes easier if we additionally assume $\mathcal{M} \subset \operatorname{Mono} \mathcal{X}$, which would follow from the existence of $(\mathcal{M}^{\perp}, \mathcal{M})$-factorizations for arbitrary cones [8]. However, the following counterexample shows that this is no longer valid in the situation of Theorem 1.2.

1.4. Example. Let $\mathcal{X}$ consist of two objects A, B and two nonidentity morphisms $s : A \to A$, $e : A \to B$ with $s^2 = 1_A$, $es = e$. Then $\mathcal{X}$ has $(\mathcal{M}^{\perp}, \mathcal{M})$-factorizations for all connected cocones, where $\mathcal{M} := \operatorname{Mor} \mathcal{X} \not\subset \operatorname{Mono} \mathcal{X}$.

Proof. We firstly show that every—possibly large—connected diagram $H : \mathcal{D} \to \mathcal{X}$ has a colimit. Indeed, if H admits a cocone $\varepsilon : H \to \Delta A$, then $\varepsilon D \in \{s, 1_A\} \subset \operatorname{Iso} \mathcal{X}$ for all $D \in |\mathcal{D}|$, and by connectedness of $\mathcal{D}$ it easily follows that ε is a colimit; otherwise the colimit is the unique cocone $H \to \Delta B$. For any connected cocone $\varphi : H \to \Delta X$, $H : \mathcal{D} \to \mathcal{X}$ take the colimit $\varepsilon : H \to \Delta Z$ and the unique factorization $\varphi : \Delta m \cdot \varepsilon$. Then $m \in \mathcal{M}$, and the colimit property gives $\varepsilon \in \mathcal{M}^{\perp}$. On the other hand, from $es = e$, $s \neq 1_A$ we see $e \notin \operatorname{Mono} \mathcal{X}$, hence $\operatorname{Mor} \mathcal{X} \not\subset \operatorname{Mono} \mathcal{X}$. □

2. Multicoreflective subcategories of Comp

In Comp we can easily characterize multicoreflective subcategories. In this section $U : \mathrm{Comp} \to \mathrm{Set}$ always denotes the forgetful functor.

2.1. Theorem. *Let $\mathcal{A} \subset \mathrm{Comp}$ be a nonempty full replete subcategory with $\emptyset \notin |\mathcal{A}|$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is mono-multicoreflective.*
(ii) *$\mathcal{A}$ is multicoreflective.*
(iii) *$\mathcal{A}$ is closed under the formation of small connected colimits.*
(iv) *There are a functor $P : \mathrm{Comp} \to \mathrm{Set}$ and a pointwise surjective natural transformation $\zeta : U \to P$ with the following properties:*

(a) *For all* $X \in |\mathrm{Comp}|$, $k \in PX$, $(\zeta X)^{-1}[\{k\}] \subset X$ *is closed and belongs to* $\mathscr{A}$ *as a compact subspace of* $\mathscr{X}$.
(b) $\# PZ = 1$ *for all* $Z \in |\mathscr{A}|$.

Proof. (i)$\Rightarrow$(ii) is trivial, and (ii)$\Rightarrow$(iii) follows immediately from Corollary 1.3.

(iii)$\Rightarrow$(iv): By assumption, $\mathscr{A}$ contains a nonempty space A. Then the unique map from A to a singleton is the coequalizer of 1_A and a constant map, therefore $\mathscr{A}$ contains all singletons.

Now we prove that for any surjective continuous map $e : A \to B$ in Comp, $A \in |\mathscr{A}|$ implies $B \in |\mathscr{A}|$. W.l.o.g. assume $A \cap B = \emptyset$ and define a small connected category $\mathscr{D}$ in the following way:

$$|\mathscr{D}| := B \cup \{c\} \quad \text{for some } c \notin A \cup B,$$

and the only nonidentity morphisms are $x : e(x) \to c$ for $x \in A$. Now we define a functor $H : \mathscr{D} \to \mathrm{Comp}$ and a cocone $\lambda : H \to \Delta B$ by $Hc := A$, $\lambda c := e$ and

$$Hy := \{0\}, \qquad Hx(0) := x, \qquad \lambda y(0) := y, \quad \text{for all } x \in A,\ y \in B.$$

Then $H[\mathscr{D}] \subset \mathscr{A}$, and λ is a colimit cocone, thus $B \in |\mathscr{A}|$ by (iii).

Now we want to construct PX, ζX for an arbitrary $X \in |\mathrm{Comp}|$. Let $\mathscr{E}$ be the category of all closed subspaces $D \subset X$, which belong to $|\mathscr{A}|$, with set-theoretic inclusions as morphisms. We claim that every connected component $\mathscr{D}$ of $\mathscr{E}$ has a terminal object.

Indeed, consider the canonical functor $H : \mathscr{D} \to \mathrm{Comp}$ with $HD := D$, where $Hd : D \hookrightarrow D'$ is the inclusion map for $d \in \mathscr{D}(D, D')$. Then there exists a colimit $\lambda : H \to \Delta Z$ in Comp, and by (iii) we have $Z \in |\mathscr{A}|$. For the cocone $\varphi : H \to \Delta X$ (with $\varphi D : D \hookrightarrow X$) we get a unique continuous map $p : Z \to X$ with $\varphi = \Delta p \cdot \lambda$. Now $K := p[Z]$ is compact as a continuous image of $Z \in |\mathscr{A}|$ and Hausdorff as a subspace of X, therefore $K \in |\mathscr{A}|$ and thus $K \in |\mathscr{D}|$.

Moreover, we have

$$D = \varphi D[D] = (p \circ \lambda D)[D] \subset p[Z] = K \quad \text{for all } D \in |\mathscr{D}|,$$

thus K is a terminal object of $\mathscr{D}$.

Now let PX be the set of all $K \subset X$ such that K is terminal in some component of $\mathscr{D}$. Then for any $x \in X$ there is a unique $K \in PX$ with $x \in K$, namely the terminal object K of the component of $\{x\}$. Thus there is a unique map $\zeta X : X \to PX$ with $x \in \zeta X(x)$ for all $x \in X$.

For every $X, Y \in |\mathrm{Comp}|$, every continuous map $f : X \to Y$ and every $u, v \in X$ with $\zeta X(u) = \zeta X(v) =: K$ we have $K \in |\mathscr{A}|$, hence $f[K] \in |\mathscr{A}|$. For the unique $L \subset PY$ with $f[K] \subset L$ we obtain $f(u), f(v) \in f[K] \subset L$, hence $(\zeta Y \circ f)(u) = L = (\zeta Y \circ f)(v)$. Thus P is a functor in a unique way such that $\zeta : U \to P$ is a natural transformation.

For every $X \in |\mathrm{Comp}|$, $K \in PX$, we have $(\zeta X)^{-1}[\{K\}] = K \in |\mathscr{A}|$. Thus (a) is satisfied, and (b) is obvious from the construction.

(iv)$\Rightarrow$(i): For $X \in |\mathrm{Comp}|$ and $k \in PX$, define $A_k := (\zeta X)^{-1}[\{k\}] \subset X$. Then (a) yields $A_k \in |\mathscr{A}|$, and we consider the inclusion $m_k : A_k \hookrightarrow X$. It remains to prove $Z \perp (X, (m_k)_{k \in K})$ for any $Z \in |\mathscr{A}|$. For any continuous $f : Z \to X$ we obtain from (b):

$$\#(\zeta X \circ f)[Z] = \#(Pf \circ \zeta Z)[Z] = \# Pf[PZ] = 1,$$

i.e.

$$f[Z] \subset (\zeta X)^{-1}[\{k\}] = A_k \quad \text{for a unique } k \in PX,$$

hence $f = m_k \circ h$ for some continuous $h : Z \to A_k$. The uniqueness of k and h is obvious. $\square$

We will now compare different notions of connectivity properties in Comp. In the sequel, consider an $\mathscr{A} \subset \mathrm{Comp}$ with $\emptyset \notin |\mathscr{A}| \neq \emptyset$. We call $\mathscr{A}$ left-constant, if there exists a full subcategory $\mathscr{B} \subset \mathrm{Comp}$ such that $\mathscr{A}$ is the category of those nonempty compact Hausdorff spaces, for which all continuous maps $A \to B$ with $B \in |\mathscr{B}|$ are constant.

A sink $(u_j : B_j \to X)_{j \in J}$ is called chained [24], if each $f \in \mathrm{Comp}(X, Y)$ ($Y \in |\mathrm{Comp}|$) is constant whenever all fu_j are constant. $\mathscr{A}$ is called a component category in the sense of Tiller [25], if for every chained sink $(u_j : B_j \to X)_{j \in J}$ with $X \neq \emptyset$ and all B_j in $\mathscr{A}$ it follows that $X \in |\mathscr{A}|$. (We modified Tiller's definition by excluding $\emptyset$, since $\emptyset \in |\mathscr{A}|$ would destroy multicoreflectivity.) $\mathscr{A}$ is called a component category in the sense of the author [2], if there exist a functor $P : \mathrm{Comp} \to \mathrm{Set}$ and a pointwise surjective natural transformation $\zeta : U \to P$ with

$$|\mathscr{A}| = \{A \in |\mathrm{Comp}| \mid \# PA = 1\}.$$

2.2. Corollary. *For $\mathscr{A} \subset \mathrm{Comp}$ with $\emptyset \notin |\mathscr{A}| \neq \emptyset$, each of the statements below implies the following one*:

(i) *$\mathscr{A}$ is left-constant.*
(ii) *$\mathscr{A}$ is a component category in the sense of Tiller.*
(iii) *$\mathscr{A}$ is multicoreflective.*
(iv) *$\mathscr{A}$ is a component category in the sense of the author.*

Proof. (i)$\Rightarrow$(ii) is trivial. (ii)$\Rightarrow$(iii) follows immediately from Theorem 2.1 and the easy observation that $(\lambda D : HD \to X)_{D \in |\mathscr{D}|}$ is chained for each connected colimit $\lambda : H \to \Delta X$, $H : \mathscr{D} \to \mathrm{Comp}$.

(iii)$\Rightarrow$(iv): Choose P, ζ as in Theorem 2.1(iv). Then by (b) we have $\# PA = 1$ for all $A \in |\mathscr{A}|$, and the converse follows from (a). $\square$

In the rest of this section we will show that none of the implications in Corollary 2.2 can be reversed. In all examples below $I := [0, 1]$ denotes the unit interval.

The first example is based on ideas of Salicrup [20], who used the same space to prove a slightly different result:

2.3. Example. Consider $X := I \times I$ with the lexicographic order (i.e. $(r, s) < (r', s')$, if either $r < r'$ or both $r = r'$ and $s < s'$), equipped with the interval topology (for which the open intervals $]x, z[$, $x < z$ form a basis) and define $p : X \to I$ by $p(r, s) := r$. Let $\mathscr{A} \subset \mathrm{Comp}$ be the full subcategory of all $A \neq 0$ with $p \circ f : A \to I$ constant for all $f \in \mathrm{Comp}(A, X)$. Then $\mathscr{A}$ is a component category in the sense of Tiller, but not left-constant.

Proof. Firstly, X is a compact connected Hausdorff space by essentially the same proof as for I, and this easily implies that $\mathscr{A}$ is a component category in the sense of Tiller. Moreover, $p : X \to I$ is continuous, and for each $r \in I$, $u_r : I \to X$ with $u_r(s) := (r, s)$ for all s is continuous. Since p is not constant, we have $X \notin |\mathscr{A}|$.

Next we prove $I \in |\mathscr{A}|$. If this were false, we would have $f(t_\nu) = (r_\nu, s_\nu)$ ($\nu \in \{0, 1\}$) with $r_0 < r_1$ for some $t_\nu \in I$, $f \in \mathrm{Comp}(I, X)$. Since $f[I]$ is connected, it contains the closed interval $[(r_0, s_0), (r_1, s_1)]$ and therefore the open interval $G_r :=](r, 0), (r, 1)[$ for each $r \in]r_0, r_1[$. Now $\{f^{-1}[G_r] \mid r \in]r_0, r_1[\}$ is an uncountable collection of pairwise disjoint nonempty open subsets of I. But this is impossible, because each such set contains a rational number and $\mathbb{Q}$ is countable $\square$

2.4. Example. $\{I\}_\perp{}^\perp$ is multicoreflective in Comp, but not a component category in the sense of Tiller.

Proof. $\{I\}_\perp{}^\perp$ is closed under connected colimits by Theorem 1.2(ii)$\Rightarrow$(iii) and therefore multicoreflective by Theorem 2.1. Now consider

$$A_{1/n} := \left\{\left(\frac{1}{n}\cos t, \frac{1}{n}\sin t\right) \middle| 0 \leq t \leq \tfrac{3}{2}\pi\right\} \cup \left\{\left(r, -\frac{1}{n}\right) \middle| r \in I\right\} \subset \mathbb{R}^2$$

for all $n \in \mathbb{N}$. Then all $A_{1/n}$ are homeomorphic to I, because $((1/n)\cos\frac{3}{2}\pi, (1/n)\sin\frac{3}{2}\pi) = (0, -1/n)$.

Now $Z = \bigcup \{A_{1/n} \mid n \in \mathbb{N}\}$ is bounded in $\mathbb{R}^2$. Let $X := \beta Z$ be its Stone–Čech compactification.

We claim that $(u_{1/n} : A_{1/n} \hookrightarrow X)_{n \in \mathbb{N}}$ is chained. Consider $Y \in |\mathrm{Comp}|$, $f \in \mathrm{Comp}(X, Y)$ with all $f \circ u_{1/n}$ constant. For all $n, m \in \mathbb{N}$ we have

$$\left(\frac{1}{n}, 0\right) = \left(\frac{1}{n}\cos 0, \frac{1}{n}\sin 0\right) \in A_{1/n}, \quad \left(\frac{1}{m}, 0\right) \in A_{1/m}, \left(\frac{1}{n}, -\frac{1}{m}\right) \in A_{1/m}.$$

Since $(1/n, -1/m)$ goes to $(1/n, 0)$ for $m \to \infty$, $f(1/m, 0) = f(1/n, -1/m)$ must tend to $f(1/n, 0)$ for $m \to \infty$. But this implies $f(1/n, 0) = f(1/m, 0)$ for all $n, m \in \mathbb{N}$. Since $f \circ u_{1/n}$ is constant for all $n \in \mathbb{N}$, f must be constant on Z and hence on X, proving that $(u_{1/n})_{n \in \mathbb{N}}$ is chained.

Thus every component category in the sense of Tiller containing I—and hence all $A_{1/n}$—will also contain X. Since $I \in |\{I\}_{\perp}{}^{\perp}|$, it suffices to prove $X \notin |\{I\}_{\perp}{}^{\perp}|$. We will do this by giving a sink $(u_i)_{i \in I} \in |\{I\}_{\perp}|$ that does not satisfy $X \perp (u_i)_{i \in I}$.

The inclusion $Z \hookrightarrow Y := \mathrm{cl}_{\mathbb{R}^2} X = X \cup (I \times \{0\})$ can be extended to a continuous surjection $p : X \to Y$. If $a \in X/Z$, then there exists an ultrafilter $\mathscr{F}$ on Z converging to a, and its image ultrafilter $\mathscr{F}'$ under p converges to $p(a) \in Y$. If we had $p(a) = p(b) \in Z$ for some $b \in Z$, then $\mathscr{F}$ would converge to $b \in Z \subset X$, since the composite $Z \hookrightarrow \beta Z = Y \xrightarrow{p} X$ is an embedding. As limit points of ultrafilters in the Hausdorff space Z are unique, this would lead to $a = b \in Z$, contradicting $a \in X/Z$. Therefore we have $p(a) \in Y/Z$, i.e. $p(a) = (j, 0)$ for some $j \in J := I \backslash \{1/n | n \in \mathbb{N}\}$. For each $j \in J$ we define $A_j := p^{-1}[\{(j, 0)\}] \subset X$. Then the above argument shows that $(A_i)_{i \in I}$ is a partition of X. Now for all $i \in I$, let $u_i : A_i \hookrightarrow X$ be the inclusion. If $j \in J$, then A_j is closed by continuity of p and therefore compact. For $n \in \mathbb{N}$, $A_{1/n}$ is also compact. Hence $(u_i)_{i \in I}$ is a sink in Comp. By considering the identity map $1_X : X \to X$ we see that $X \perp (u_i)_{i \in I}$ cannot hold.

It remains to be shown that $(u_i)_{i \in I} \in |\{I\}_{\perp}|$, i.e. that $I \perp (u_i)_{i \in I}$. Consider $f \in \mathrm{Comp}(I, X)$. Then we have to show $f[I] \subset A_i$ for some $i \in I$. So assume this to be false. Then one easily sees that there must be a subinterval $I' \subset I$ with $p \circ f[I'] \subset X/Z$ and $\# I' \geq 2$ (hence $I' \cong I$) such that the restriction $p \circ f|_{I'} : I' \to Y$ is not constant. So w.l.o.g. we can assume $f[I] \subset X/Z$, and

$$p \circ f(c) = (r, 0), \quad p \circ f(d) = (t, 0), \quad r < t \quad \text{for some } c, d, r, t \in I.$$

Since $p \circ f[I] \subset Y \backslash Z$ must be connected, we have

$$p \circ f[I] \subset \left]\frac{1}{n+1}, \frac{1}{n}\right[\times \{0\} \quad \text{for some } n \in \mathbb{N}.$$

Now let $s \in]r, t[\subset p \circ f[I] \subset]1/(n+1), 1/n[$ be arbitrary and define a continuous map $\bar{g}_s : Z \to [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ by

$$\bar{g}_s(x, y) := \begin{cases} \arctan(x - s)/|y| & \text{if } y \neq 0, \\ \frac{1}{2}\pi & \text{if } y = 0,\ x > s, \\ -\frac{1}{2}\pi & \text{if } y = 0,\ x < s. \end{cases}$$

(Note that $(s, 0) \notin Z$.) By the universal property of $X = \beta Z$, $\bar{g}_s$ can be extended to a continuous map $g_s : X \to [-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Now $g_s \circ f(c) = -\frac{1}{2}\pi$, $g_s \circ f(d) = \frac{1}{2}\pi$, and by connectedness of I the open set

$$C_s := \{k \in I | -\tfrac{1}{2}\pi < g_s \circ f(k) < \tfrac{1}{2}\pi\}$$

must be nonempty. Now let $k \in C_s$ be arbitrary. Then

$$p \circ f(k) \in p \circ f[I] \subset \left(\left]\frac{1}{n+1}, \frac{1}{n}\right[\right) \times \{0\},$$

hence $p \circ f(k) = (x, 0)$ for some $x \in]1/(n+1), 1/n[\subset I$. But one easily sees that $x < s$ would lead to the contradiction $g_s \circ f(k) = -\frac{1}{2}\pi$, while $x > s$ would imply $g_s \circ f(k) = \frac{1}{2}\pi$, contradicting $k \in C_s$. Therefore we must have $x = s$, proving $C_s \subset p^{-1}[\{(s, 0)\}]$.

In particular, we have $C_s \cap C_{s'} = \emptyset$ for $r < s < s' < t$. But then $(C_s)_{s \in]r,t[}$ is an uncountable family of pairwise disjoint nonempty open subsets of I, which is impossible. So $p \circ f$ must be constant, proving our claim. □

2.5. Example. The category of pathwise connected (nonempty!) compact Hausdorff spaces is a component category in the sense of the author, but not multicoreflective.

Proof. For $X \in |\mathrm{Comp}|$, let PX be the set of its path-components ($P\emptyset = \emptyset$) and let $\zeta X : UX \to PX$ be the canonical projection. Then P can be extended to a functor with ζ natural in a unique way, and it follows that the category $\mathscr{A}$ of pathwise connected objects of Comp is a component category in the sense of the author.

On the other hand, $\mathscr{A}$ is not multicoreflective by the following argument due to Preuß [17]:

$$Z := \left\{ \left(x, \sin \frac{1}{x} \right) \,\middle|\, x \in \mathbb{R},\, 0 < x \leq 1 \right\}$$

is bounded in $\mathbb{R}^2$, hence its closure $X := \mathrm{cl}\, Z = Z \cup (\{0\} \times [-1, 1])$ is compact.

If $(m_j : A_j \to X)_{j \in J}$ were a multicoreflection sink, the images $m_j[A_j]$ would have to be the path-components of X. But this is impossible, because the path-component $Z \subset X$ is not compact. □

3. Coprime objects

When we look for coproduct decompositions of an object of a category with coproducts, we like to have some "uniqueness up to canonical isomorphism". It may not be clear, what "canonical" should mean in this context. But consider the dual situation in Set. A set Z is product-indecomposable (i.e. $\#Z \neq 1$ and $Z \cong X \times Y$ implies $\#X = 1$ or $\#Y = 1$) if and only if Z is finite and $\#Z$ is prime.

A finite set is isomorphic to a finite product of product-indecomposable sets, but the isomorphism is by no means canonical, although for any two such decompositions there is a bijection between the index sets such that corresponding factors are isomorphic. A denumerable set is not a product of product-indecomposable sets at all, because each such product must have infinitely many factors and therefore be uncountable. In Top no uniqueness of product decompositions can be expected; if I_k denotes an indiscrete space with k points, we have

$$(I_1 \amalg I_2) \sqcap (I_1 \amalg I_4 \amalg I_{16}) \cong (I_1 \amalg I_8) \sqcap (I_1 \amalg I_2 \amalg I_4),$$

and all factors are indecomposable. This example was found in a conversation with A. Möbus and G. Richter; an earlier version is due to Möbus. These remarks should make clear that the notion of a coproduct-indecomposable object is not of much help in an arbitrary category.

The notion of coprime object (called "Z-object" by Hoffman [13]) in a category with all small coproducts seems to be much more adequate in many situations.

Simultaneously, we will define and investigate finitely coprime objects. Definition 3.1 and all results in this section—together with their proofs—are to be read either with all words put in parentheses or without all of them. It is also possible to consider coproducts, whose indexing sets are smaller than a given infinite regular cardinal, but by [3] this is not much more general.

3.1. Definition. Let $\mathscr{X}$ be a category with (finite) coproducts.

Then we define:

(i) An $A \in |\mathscr{X}|$ is called *(finitely) coprime*, if A is multiorthogonal to all (finite) coproduct sinks.

(ii) (Finite) coproducts are *universal* in $\mathscr{X}$, if for any (finite) family $(A_i)_{i \in I}$ of objects and any $\mathscr{X}$-morphism $f: Z \to \coprod_{i \in I} A_i$ the pullback

$$\begin{tikzcd} A_{i_0} \arrow[r, "u_{i_0}"] & \coprod_i A_i \\ C_{i_0} \arrow[u, "g_{i_0}"] \arrow[r, "m_{i_0}"'] & Z \arrow[u, "f"'] \end{tikzcd} \qquad (*)$$

exists for all $i_0 \in I$ (u_{i_0} being the injection) and $(Z, (m_i)_{i \in I})$ is a coproduct sink.

(iii) (Finite) coproducts are *totally disjoint* in $\mathscr{X}$, if for all (finite) families $(g_i: C_i \to A_i)_{i \in I}$ of $\mathscr{X}$-morphisms, the diagram $(*)$ is always a pullback for $i_0 \in I$, $Z := \coprod_{i \in I} C_i$ with injections $m_i: C_i \to Z$, and for $f := \coprod_{i \in I} g_i$.

(iv) $Z \in |\mathscr{X}|$ is called *pre-initial*, if $\# \mathscr{X}(Z, X) \leq 1$ for all $X \in |\mathscr{X}|$.

The notion of universality is well known, while the notion of "total disjointness" is stronger than the usual notion of "disjoint coproducts" and is studied in [5].

In categories like Top, Haus, Graph, Cat etc., the coprime objects are the connected objects in the familiar sense. This follows from Theorem 3.3 (or Theorem 3.4) below. If $\mathscr{X}$ has small hom-sets, then an $A \in |\mathscr{X}|$ is (finitely) coprime, if and only if the partial hom-functor $\mathscr{X}(A, -)$ preserves (finite) coproducts. If uncountable measurable cardinals do not exist, then in any category with arbitrary coproducts every object orthogonal to all countable coproduct sinks is already coprime. This follows from [3, Theorem 1.2].

Our first observation about coproducts is an immediate consequence of Theorem 1.2(ii)$\Rightarrow$(iii).

3.2. Proposition (Hoffmann [13]). *In any category the subcategory of coprime objects and the category of finitely coprime objects are closed under the formation of connected colimits.*

If an object is a coproduct $\coprod_{i \in I} Z_i$ of coprime objects, for any other coproduct decomposition $\coprod_{j \in J} A_j$ the multiorthogonality condition gives rise to a unique map

$\sigma: I \to J$ and unique morphisms $u_i: Z_i \to A_{\sigma(i)}$ commuting with the coproduct injections. If the A_j are also coprime, the symmetry of the argument yields that σ is a bijection and the u_i are isomorphisms. This is a very strong uniqueness property (even too strong for additive categories), and as suggested by Hoffmann [13], one often should look at coproduct decompositions into coprime objects. Even if such decompositions do not exist, the subcategory of coprime objects is often multicoreflective and the multicoreflection sink has many properties in common with coproduct sinks. In order to describe coprime objects, we can often restrict the multiorthogonality condition either to coproduct decompositions of the object itself (in which case "coprime" is equivalent to "coproduct indecomposable") or to coproducts of terminal objects.

3.3. Theorem. *Let $\mathscr{X}$ have pullbacks and (finite) coproducts, which are universal. Then for any $Z \in |\mathscr{X}|$ the following statements are equivalent:*

(i) *Z is (finitely) coprime.*

(ii) *Z is not pre-initial, and for every representation of Z as a binary coproduct one of the coproduct injections is an isomorphism.*

Proof. (i)$\Rightarrow$(ii): If Z were pre-initial, then $(1_Z)_{i\in\{0,1\}}$ would be a coproduct sink (and $Z \cong Z \amalg Z$), and the identity morphisms would factor over both injections, contradicting $Z \perp (1_Z)_{i\in\{0,1\}}$. Now let $(m_i: C_i \to Z)_{i\in\{0,1\}}$ be a binary coproduct sink. Since $Z \perp (m_i)_{i\in\{0,1\}}$ there is a unique pair (i, h) with $i \in \{0, 1\}$, $h: Z \to C_i$, $1_Z = m_i h$. By coproduct property, there is a $t: Z \to C_i$ with $tm_i = 1_{C_i}$, $tm_{1-i} = hm_{1-i}$, proving $m_i \in \operatorname{Iso} \mathscr{X}$.

(ii)$\Rightarrow$(i): For any (finite) coproduct sink $(u_i: A_i \to X)_{i\in I}$ we have to prove $Z \perp (u_i)_i$. For $f: Z \to X$ let C_i, m_i, g_i make the diagram (*) a pullback. As Z is not pre-initial, there are $Y \in |\mathscr{X}|$, $x, y: Z \to Y$ with $x \neq y$. Since coproducts are universal in $\mathscr{X}$, $(m_i)_i$ is a coproduct sink and therefore $xm_{i_0} \neq ym_{i_0}$ for some $i_0 \in I$. For $B := \coprod_{i\in I\setminus\{i_0\}} C_i$ and the canonical morphism $n: B \to Z$, n and m_{i_0} form a copruduct sink, and by (i) n or m_{i_0} is an isomorphism. By the coproduct property, there is a $z: Z \to Y$ with $zm_{i_0} = xm_{i_0}$ and $zn = yn$. Now $n \in \operatorname{Iso} \mathscr{X}$ (even $n \in \operatorname{Epi} \mathscr{X}$) would lead to the contradiction $z = y$, $xm_{i_0} = zm_{i_0} = ym_{i_0}$. Thus we have $m_{i_0} \in \operatorname{Iso} \mathscr{X} \subset \operatorname{Epi} \mathscr{X}$ and thus $f = fm_{i_0}m_{i_0}^{-1} = u_{i_0}g_{i_0}m_{i_0}^{-1}$, $z = x$, and $xn = zn = yn$.

It remains to be shown that $f = u_{i_1}h$, $i_1 \in I$, $h: Z \to A_{i_1}$ implies $i_1 = i_0$ and $h = g_{i_0}m_{i_0}^{-1}$. By the pullback property, there is a unique $k: Z \to C_{i_1}$ with $m_{i_1}k = 1_Z$ and $g_{i_1}k = h$. Now $i_1 \neq i_0$ would imply $m_{i_1} = nr$ for the coproduct injection $r: C_{i_1} \to B$ and hence lead to the contradiction $x = xm_{i_1}k = xnrk = ynrk = y$. Thus we have $i_0 = i_1$ and therefore $k = m_{i_0}^{-1}$, hence $h = g_{i_0}k = g_{i_0}m_{i_0}^{-1}$. $\square$

3.4. Theorem. *Let $\mathscr{X}$ have arbitrary (respectively finite) coproducts and let them be totally disjoint. Moreover let $\mathscr{X}$ have a terminal object T. Then a $Z \in \mathscr{X}$ is (finitely) coprime if and only if $Z \perp (S, (s_i)_{i\in I})$ for all (finite) sets I and $S := \coprod_{i\in I} T$ with injections $s_i: T \to S$.*

Proof. We only need to prove that the condition is sufficient. For a (finite) coproduct sink $(u_i : A_i \to X)_i$ and an $f \in \mathscr{X}(Z, X)$ consider the diagrams

$$\begin{tikzcd} A_i \arrow[r, "u_i"] \arrow[d, "e_i"'] & X \arrow[d, "q"] \\ T \arrow[r, "s_i"] & S \end{tikzcd} \tag{**}$$

for $i \in I$, where $e_i : A_i \to T$ is the unique morphism, and $q : X \to S$ is the unique arrow making $(**)$ commutative for all $i \in I$. As $Z \perp (s_i)_i$, there is a unique $i_0 \in I$ with $qf = s_{i_0} t$ for the unique morphism $t : Z \to T$. Since coproducts are totally disjoint in $\mathscr{X}$, $(**)$ is a pullback, and thus there is a unique $h : Z \to A_{i_0}$ with $u_{i_0} h = f$ and $e_{i_0} h = t$. If $f = u_{i_1} h'$ for some $i_1 \in I$, $h' : Z \to A_{i_1}$, then $qf = qu_{i_1} h' = s_{i_1} e_{i_1} h' = s_{i_1} t$, hence $i_1 = i_0$, $u_{i_0} h' = u_{i_1} h' = f$, $e_{i_0} h' = t$, thus $h = h'$. □

3.5. Proposition. *The following statements hold whenever $\mathscr{X}$ has arbitrary coproducts.*

(i) *If coproducts are universal, then all finitely coprime objects are coprime.*

(ii) *If $Z \in |\mathscr{X}|$ is finitely coprime with $GZ \neq \emptyset$ for some coproduct preserving functor $G : \mathscr{X} \to \mathrm{Set}$, then Z is coprime.*

(iii) *For $Z' \to Z$ in $\mathscr{X}$ with Z' coprime and Z finitely coprime, Z is coprime.*

Proof. (i) Follows immediately from Theorem 3.3, because Theorem 3.3(ii) is the same in both versions (with finite or with arbitrary coproducts).

(ii) First choose an $a \in GZ$. Let $X := \coprod_{i \in I} A_i$ be a coproduct with injections $u_i : A_i \to X$. Since it is preserved, GX is the disjoint union of all $Gu_i[GA_i]$.

For any morphism $f : Z \to X$ it follows that $Gf(a) \in Gu_{i_0}[GA_{i_0}]$ for a unique $i_0 \in I$. Now $X \cong A_{i_0} \amalg Y$ where $Y := \coprod_{i \in I \setminus \{i_0\}} A_i$ with injections u_{i_0} and a suitable $v : Y \to X$. Since Z is finitely coprime, there is either an $h : A_{i_0} \to X$ with $f = u_{i_0} h$ or a $k : Y \to X$ with $f = vk$. But the latter would lead to $Gf(a) = Gv(Gk(a)) \in Gv[GY]$ contradicting $Gf(a) \in Gu_{i_0}[GA_{i_0}]$. Therefore $f = u_{i_0} h$ for some h, and the uniqueness of i_0 and h is obvious.

(iii) If $\mathscr{X}$ has small hom-sets, apply (ii) to $G := \mathscr{X}(Z', -)$; otherwise change universes or simulate the proof. □

Note that (ii) and (iii) are special cases of [3, Proposition 1.3].

3.6. Proposition. *The finitely coprime objects in* Comp *are exactly the connected spaces.* Comp *has no coprime objects.*

Proof. The characterization of finitely coprime objects follows immediately from Theorem 3.3. If a coprime object would exist the "singleton space" $*$ would be coprime by Proposition 3.5(iii). But this is false, because $U : \mathrm{Comp} \to \mathrm{Set}$ is represented by $*$ and does not preserve coproducts. □

Now we look at the relationship between coprime objects and multicoreflective subcategories:

3.7. Theorem. *Let $\mathscr{X}$ have all small coproducts and let $\mathscr{A} \subset \mathscr{X}$ be a full subcategory consisting of coprime objects such that for every $X \in |\mathscr{X}|$ the conglomerate of connected components of $\mathscr{A}/X$ is small. Let $\mathscr{C} \subset \mathscr{X}$ be the category of all coproducts of $\mathscr{A}$-objects. Then $\mathscr{C}$ is coreflective if and only if $\mathscr{A}$ is multicoreflective.*

Proof. Firstly, let $\mathscr{C}$ be coreflective. For $X \in |\mathscr{X}|$ consider the coreflection morphism $r: C \to X$. Since $C \in |\mathscr{C}|$, we have $C \cong \coprod_{i \in I} A_i$ for suitable $A_i \in |\mathscr{A}|$ with injections, $u_i : A_i \to C$. Now $(ru_i : A_i \to X)_{i \in I}$ can easily be seen to be the desired multicoreflection sink, therefore $\mathscr{A} \subset \mathscr{X}$ is multicoreflective. Conversely, if $\mathscr{A}$ is multicoreflective in $\mathscr{X}$, consider a multicoreflection sink $(m_i : A_i \to X)_{i \in I}$ for a given $X \in |\mathscr{X}|$. I is in bijective correspondence with the connected components of $\mathscr{A}/X$ and therefore small.

Let C be the coproduct $\coprod_{i \in I} A_i$ with injections $u_i : A_i \to C$. Then the coproduct property renders a unique $r: C \to X$ with $m_i = ru_i$ for all $i \in I$. For $Z \in |\mathscr{C}|$ we shall see that each morphism $f: Z \to X$ factors uniquely over r. We can restrict our attention to the case $Z \in |\mathscr{A}|$. Now $Z \perp (m_i)_i$ yields $f = m_{i_0}h = ru_{i_0}h$ for some $i_0 \in I$, $h: Z \to A_{i_0}$. For the uniqueness, assume $f = rg$, where $g: Z \to C$. Since Z is coprime, we have $Z \perp (u_i)_{i \in I}$, hence $g = u_{i_1}h'$ for some $i_1 \in I$, $h': Z \to A_{i_1}$. Now $f = rg = ru_{i_1}h' = m_{i_1}h'$, and $Z \perp (m_i)_i$ leads to $i_1 = i_0$, $h' = h$, $g = u_{i_1}h' = u_{i_0}h$. □

Theorem 3.7 can also be proved using formal coproduct completions (cf. [10, 16]).

It looks a bit strange that both directions of the above proof require the hypothesis that $\mathscr{A}$ consists of coprime objects only. But indeed, if $\mathscr{X} = \text{Comp}$ and $\mathscr{A}$ consists of all singletons, then $\mathscr{A}$ is multicoreflective and even consists only of finitely coprime objects. On the other hand, $\mathscr{C}$ is the category of all Stone–Čech-compactifications of discrete spaces, which is not closed under coequalizers and hence not coreflective. Conversely, consider $\mathscr{X} := \text{Set}$ and let $\mathscr{A}$ be the category of finite sets, then $\mathscr{C} = \text{Set}$ is clearly coreflective in itself. But $\mathscr{A}$ is not multi-coreflective by Corollary 1.3, because it is not closed under filtered colimits (which are connected ones).

4. Reflective and multireflective subcategories of Rng

In this section, we consider the dual of the category Rng of rings. (We assume all rings and ring homomorphisms to be unital, but rings need not be commutative.) It follows immediately from Theorem 3.4 that a ring R is finitely prime in Rng (i.e. finitely coprime in Rng^{op}), if and only if there are exactly two ring homomorphisms from $\mathbb{Z}^2$ to R. Moreover, this property means that 0 and 1 are the only idempotents in R. We call such rings connected (this is a familiar notion in the commutative case). Moreover, if no uncountable measurable cardinal exists, it follows from

Theorem 3.4 and [3, Theorem 1.2] that the prime objects in Rng are the rings R admitting only trivial ring homomorphisms from $\mathbb{Z}^{\mathbb{N}}$ to R. In [6] these rings are called ultraconnected, the above assertions are proved directly, and some special rings are proved to be ultraconnected or not.

4.1. Proposition. *The following subcategories are multireflective in* Rng:
(i) *the category of connected rings,*
(ii) *the category of ultraconnected rings,*
(iii) *any full replete subcategory generated by a subset of* $\{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$,
(iv) *the category of skew fields* (*=division rings*).

Proof. (i) and (ii) follow immediately from Theorem 1.2(ii)$\Rightarrow$(i), because in Rng product projections are regular epimorphisms and Rng has (regular-epi, mono)-factorization even for arbitrary cones.

(iii) If $\mathscr{A} \subset \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, $R \in |\mathrm{Rng}|$, let I be the set of all ring homomorphisms $f: R \to S$ with $S \in |\mathscr{A}|$, which admit no factorization over a $T \in \mathscr{A}$ with $T \subsetneq S$. As the inclusions are the only morphisms between objects of $\mathscr{A}$, it follows easily that $(R, (f)_{f \in I})$ is a multireflection source.

(iv) For $R \in |\mathrm{Rng}|$ consider all ring homomorphisms $p: R \to S$, where S is a skew field generated (as a skew field) by $p[R]$. Let $(p_i: R \to S_i)_{i \in I}$ be a representative system under isomorphy of R-algebras. Then all p_i are Rng-epimorphisms, because for all Rng-morphisms f, g with $g \circ p_i = h \circ p_i$ the equalizer set $\{x \in S_i | g(x) = h(x)\}$ is a sub-skew field containing $p_i[R]$.

Now let K be a skew field and consider an arbitrary $f \in \mathrm{Rng}(R, K)$. Then the sub-skew field K' of K generated by $f[R]$ is isomorphic to a unique S_i, and the inclusion map $K' \hookrightarrow K$ gives rise to a Rng-morphism $g: S_i \to K$ with $f = p_i \circ g$. Now assume $f = p_j \circ h$ for some $j \in J$, $h: S_j \to K$. Consider the pullback of g and h, i.e. the ring $T := \{(x, y) \in S_i \times S_j | g(x) = h(y)\}$. As S_i is a skew field, g is injective, and for every $(x, y) \in T$ with $x \neq 0$ we have $h(y) = g(x) \neq 0$, hence $y \neq 0$, and (x^{-1}, y^{-1}) is inverse to (x, y). Analogously, $y \neq 0$ implies the invertibility of (x, y), proving that T is a skew field. Thus the projection $q: T \to S_i$, $(x, y) \mapsto x$ is a skew field homomorphism and hence injective. For all $z \in R$ we have $p_i(z) = q(p_i(z), p_j(z)) \in q[T]$; and since $p_i[R]$ generates S_i as a skew field, q is an isomorphism. Thus S_i is isomorphic to T as an R-algebra, and the same holds for S_j, therefore we have $i = j$. As p_i is epic, we obtain $g = h$. □

An explicit construction follows from Cohn's [8] localization of a ring with respect to a prime matrix ideal.

Using Theorem 3.7 we immediately obtain the following:

4.2. Corollary. *Assume that there exists no uncountable measurable cardinal. Then the following full replete subcategories of* Rng *are reflective*:
(i) *the category of all products of ultraconnected rings,*

(ii) *the category of all products of $\mathscr{A}$-objects, where $\mathscr{A} \subset$ Rng is any subcategory with $|\mathscr{A}| \subset \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.*

The hypothesis about measurable cardinals is necessary; indeed, if there are uncountable measurable cardinals, then (i) fails, and (ii) remains true only in the case $\mathscr{A} = \emptyset$ by the following:

4.3. Proposition. *Let α be a measurable cardinal and let $\mathscr{C} \subset$ Rng be a full reflective subcategory containing at least one ring R with $2 \leqslant \# R < \alpha$. Then $\mathscr{C}$ contains a ring, which is not a product of connected rings.*

Proof. Firstly, $R^\alpha \in |\mathscr{C}|$, since the reflective subcategory $\mathscr{C}$ is closed under products.

Let $\mathscr{F}_1$, $\mathscr{F}_2$ be different nonprincipal α-complete ultrafilters on α. For $(x_\xi)_{\xi<\alpha} \in R^\alpha$ let $f_\nu((x_\xi)_{\xi<\alpha})$ be the unique element $a \in R$ with $\{\xi < \alpha | x_\xi = a\} \in \mathscr{F}_\nu$, $\nu \in \{1, 2\}$. Then the $f_\nu : R^\alpha \to R$ are ring homomorphisms (cf. [6]).

As $\mathscr{C}$ is reflective and therefore closed under equalizers, $S := \{x \in R^\alpha | f_1(x) = f_2(x)\} \subset R^\alpha$ belongs to $\mathscr{C}$. We claim that S is not a product of connected rings. Indeed, in a product of connected rings all idempotents are central, and they form a complete Boolean algebra. On the other hand the Boolean algebra of central idempotents in S is not complete because for $A \in \mathscr{F}_1 \backslash \mathscr{F}_2$ the set $\{(\delta_{\gamma\xi})_{\xi<\alpha} | \gamma \in A\}$ has no supremum, where δ denotes the Kronecker symbol with values in suitable rings. $\square$

For $\alpha = \aleph_0$, Proposition 4.3 means that any reflective subcategory containing a finite ring $\neq \{0\}$ does not consist of products of connected rings. In particular, the category of all products of connected rings is not reflective. In the commutative case, connected rings are the rings with connected Zariski-spectrum, while fields are the reduced rings with one-point spectrum. Obviously the reduced connected rings form a multireflective subcategory, because the reflection into the category of reduced rings preserves connectedness. On the other hand, all rings with one-point spectrum (i.e. all rings, in which every element is invertible or nilpotent) do not form a multireflective subcategory.

Indeed, for every prime p and every natural number $n \geqslant 1$, $\mathbb{Z}/p^n\mathbb{Z}$ belongs to this category. The ring of p-adic integers is a connected limit of these rings, and its spectrum is not a singleton. Thus it follows from Corollary 1.3(ii) that the category is not multireflective.

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