SOLVING CAUCHY SINGULAR INTEGRAL EQUATIONS
BY USING GENERAL QUADRATURE-COLLOCATION NODES

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Abstract—We show that the specific nodes are not necessary for solving Cauchy singular integral equations by using quadrature-collocation methods. The solvability of the discrete system is proved for arbitrary selection of quadrature and collocation nodes. We also propose several special choices of these nodes. Especially, a weighted minimum norm-least square method is discussed.

1. INTRODUCTION

The classical theory for solving Cauchy Singular Integral Equations (CSIE) is based on the properties of sectionally holomorphic functions, which enable us to reduce the singular equation to a Fredholm equation of the second kind. (See N. I. Muskhelishvili [1] and F. D. Gakhov [2].) In many physical problems, when numerical solution is necessary, direct methods are often preferable. A method is called direct if the singular integral is replaced by a numerical approximation without resorting to regularization. Such methods initiated by F. Erdogan [3] and F. Erdogan and G. D. Gupta [4] have been developed subsequently by F. Erdogan, G. D. Gupta and T. S. Cook [5], P. S. Theocaris and N. I. Loakimidis [6].

Suppose the CSIE has the form

\[ a(x)g(x) + \frac{b(x)}{\pi} \int_{-1}^{1} \frac{g(t)dt}{t - x} + \int_{-1}^{1} k(t,x)g(t)dt = f(x), \quad -1 < x < 1. \quad (1.1) \]

All these numerical methods use Gauss-type formulae after expressing the unknown singular function as the product of a weight function \((1 - z)^{a-1}(1 + z)^{-b-1}\) and a smooth function to be computed. \(\alpha\) and \(\beta\) are determined by Noether's index theorems.

Let \(P_n^{(\alpha,\beta)}(x)\) be the Jacobi polynomial of degree \(n\) orthogonal with respect to the weight function \((1 - x)^{\alpha-1}(1 + x)^{-\beta-1}\). The zeros of \(P_n^{(\alpha,\beta)}(x)\) are used as quadrature nodes in the Gauss-Jacobi integral formula and the zeros of another related Jacobi polynomial are used as the corresponding collocation points. For example, when \(\alpha = \beta = \frac{1}{2}\), the Jacobi polynomial \(P_n^{(\alpha,\beta)}(x)\) is \(T_n(x)\), the Chebyshev polynomial of the first kind, and the related polynomial is \(U_{n-1}(x)\), the Chebyshev polynomial of the second kind.

There are three factors, which influence the size of the error in the computed solution: (a) accuracy of the quadrature formula, (b) choice of the collocation nodes, (c) "condition" of the linear algebraic system. The rate of convergence depends on the "Lebesgue constants" of the collocation and quadrature nodes, and the smoothness of the functions. Certain sets of quadrature-collocation nodes are inadequate to represent the intrinsic features of the problems, especially of the problems arising from the oscillatory behavior of \(f(x)\) or the kernel \(k(t,x)\), or the problems due to large derivatives of these functions. For example, for the function

\[ f(x) = \frac{e^x}{(x - c)^2 + \alpha^2}, \]
for very small values of $a$, a collocation point in the immediate vicinity of $c$ is essential. As discussed by Gerasoulis and Srivastav in [7], the methods based on orthogonal polynomials will require the quadrature rules of an excessively high degree, leading to extremely large systems of linear algebraic equations.

The paper by Tsamysphyros and Theocaris [11] appears to be the first to ask the rhetoric question "Are special collocation nodes necessary for the numerical solution of singular integral equations?". The examples are given showing that it is not so for the Gauss-Chebyshev quadrature and collocation. Our objective in this paper is to analyse the linear algebraic systems for solvability. The approximation characteristics of the computed solution will be discussed in a subsequent paper.

In order to make the paper self-sufficient, some well known results are included here. We organize the sections as follows:

Section 2 is used to construct the theory of orthogonal polynomials. The similar results can be found in S. Welstead’s Ph.D. thesis [12].

In Section 3 we prove the solvability of the system of equations derived from general quadrature-collocation nodes.

Section 4 is to discuss several optional selections of these nodes.

The paper is concluded with a numerical example in section 5.

2. TWO RELATED SEQUENCES OF ORTHOGONAL POLYNOMIALS

Consider the dominant part of the Cauchy singular integral operator $U$:

$$Ug(x) := ag(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{g(t)}{t-x} dt,$$

where $a$ and $b$ are assumed constants and $|a| \neq |b|$. For the general case of variable $a(x)$ and $b(z)$, Welstead has discussed in [12]. Here, we only rehearse some related results.

The operator $U$ is defined on the space of Hölder continuous functions. By using the sectionally holomorphic functions, we introduce

$$\phi(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{g(t)}{t-z} dt,$$

where $z = x + iy$, $x$ and $y$ are real numbers. Let

$$\phi^+(x) = \lim_{y \to 0^+} \phi(x + iy), \quad \phi^-(x) = \lim_{y \to 0^-} \phi(x + iy).$$

We have

$$g(x) = \phi^+(x) - \phi^-(x),$$

$$\frac{1}{\pi i} \int_{-1}^{1} \frac{g(t)}{t-x} dt = \phi^+(x) - \phi^-(x)$$

and

$$Ug(x) = (a + ib)\phi^+(x) - (a - ib)\phi^-(x).$$

Denote

$$G = \frac{a - ib}{a + ib} = \frac{a^2 - b^2 - i2ab}{a^2 + b^2} = e^{-i\theta},$$

we have

$$|G| = 1, \quad \tan \theta = \frac{b}{a}, \quad 0 \leq |\theta| \leq \frac{\pi}{2}, \quad |\theta| \neq \frac{\pi}{4}.$$

Let

$$\Gamma(x) = -\frac{\theta}{\pi} \log \frac{1-x}{1+x},$$
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\[ Z(x) = (1 - x)^{\lambda_1} (1 + x)^{\lambda_{-1}} e^{\Gamma(x)} = (1 - x)^{\lambda_1 - \frac{a}{2}} (1 + x)^{\lambda_{-1} + \frac{b}{2}}. \]

If \( a \geq 0 \) and \( b \geq 0 \), then \( 0 \leq \frac{a}{2} \leq \frac{1}{2} \). \( \lambda_1 \) and \( \lambda_{-1} \) can be chosen as following so that both \( Z(x) \) and \( Z^{-1}(x) \) are integrable on \((-1, 1)\).

1. \( \lambda_1 = 0 \) and \( \lambda_{-1} = -1; \\
2. \( \lambda_1 = 0 \) and \( \lambda_{-1} = 0 \) or \( \lambda_1 = 1 \) and \( \lambda_{-1} = -1; \\
3. \( \lambda_1 = 1 \) and \( \lambda_{-1} = 0. \)

The index of the operator \( U \) is defined as \( \chi = -(\lambda_1 + \lambda_{-1}). \)

In the case of 1-index,

\[ Z(x) = (1 - x)^{\frac{a}{2}} (1 + x)^{-\frac{b}{2}} = \frac{1}{\sqrt{1 - x^2}} \left( \frac{1 - x}{1 + x} \right)^{\frac{a}{2} - \frac{b}{2}}. \quad (2.2) \]

Let \( \{p_n(x)\}_{n=0}^{\infty} \) be a sequence of monic orthogonal polynomials with respect to the weight function \( Z(x) \) on the interval \((-1, 1)\), i.e.,

\[ \int_{-1}^{1} Z(x)p_n(x)p_m(x)dx = c_n \delta_{nm}, \quad (2.3) \]

where \( c_n \) is a constant, \( \delta_{nm} \) is the Kronecker notation. According to the general properties of orthogonal polynomials, there is a recursion formula between every three consecutive polynomials:

\[ p_0(x) = 1, \]
\[ p_1(x) = (x - \alpha_0)p_0(x), \]
\[ p_2(x) = (x - \alpha_1)p_1(x) - \beta_1 p_0(x), \]
\[ \ldots \ldots, \]
\[ p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \]

where \( \{\alpha_k\}_{n=0}^{\infty} \) and \( \{\beta_k\}_{n=0}^{\infty} \) are two sequences of numbers. Now, we can construct a new sequence of monic polynomials \( \{q_n(x)\}_{n=0}^{\infty} \) by using the \( \{\alpha_n\}_{n=0}^{\infty} \) and \( \{\beta_n\}_{n=0}^{\infty} \):

\[ q_0(x) = 1, \]
\[ q_1(x) = (x - \alpha_1)q_0(x), \]
\[ q_2(x) = (x - \alpha_2)q_1(x) - \beta_2 q_0(x), \]
\[ \ldots \ldots, \]
\[ q_{n+1}(x) = (x - \alpha_{n+1})q_n(x) - \beta_{n+1} q_{n-1}(x). \]

Our main arguments are

**Proposition 2.1.**

\[ U(Z(x)p_0(x)) = 0, \]
\[ U(Z(x)p_k(x)) = \mu q_{k-1}(x), \quad \text{for } k = 1, 2, \ldots \]

where

\[ \mu = \sqrt{a^2 + b^2}. \quad (2.4) \]

**Proposition 2.2.** \( \{q_n(x)\}_{n=0}^{\infty} \) is a sequence of orthogonal polynomials with respect to the weight function \( Z^{-1}(x) \) on \((-1, 1)\).

Consider the adjoint operator \( V \) of \( U \):

\[ Vg(x) := ag(x) - \frac{b}{\pi} \int_{-1}^{1} \frac{g(t)dt}{t - x}. \quad (2.5) \]

We have a similar result of \( V \) as the Proposition 2.1.

**Proposition 2.3.** For \( k = 1, 2, \ldots \)

\[ V \left( \frac{q_{k-1}(x)}{Z(x)} \right) = \mu p_k(x). \]
3. THE DISCRETE SCHEME OF GENERAL QUADRATURE NODES AND COLLOCATION POINTS

For the problem with 1-index, we shall find the solution \( g(x) \) satisfying

\[
\begin{aligned}
ag(x) + \frac{1}{\pi} \int_{-1}^{1} g(t) dt = f(x), \\
\frac{1}{\pi} \int_{-1}^{1} g(x) dx = c.
\end{aligned}
\]  

(3.1)

Let \( g(x) = Z(x) \phi(x) \). We choose \( x_1 < x_2 < \cdots < x_n \) as quadrature nodes and \( y_1, y_2, \cdots, y_{n-1} \) as collocation points. The \( \{x_j\}_1^n \) and the \( \{y_k\}_1^{n-1} \) are independent of the weight function \( Z(x) \).

Denote

\[
X(x) = \prod_{j=1}^{n} (x - x_j),
\]

(3.2)

\[
Y(y) = \prod_{k=1}^{n-1} (y - y_k).
\]

(3.3)

The quadrature coefficients are defined as

\[
W_j = \int_{-1}^{1} \frac{Z(x)X(x) dx}{(x - x_j)X'(x_j)},
\]

(3.4)

\[
V_k = \int_{-1}^{1} \frac{Y(y) dy}{Z(y)(y - y_k)Y'(y_k)}.
\]

(3.5)

We can construct the system of equations as following:

\[
A \vec{\phi} = \vec{f},
\]

(3.6)

where

\[
\vec{\phi} = (\phi(x_1), \phi(x_2), \cdots, \phi(x_n))^T,
\]

\[
\vec{f} = (f(y_1), f(y_2), \cdots, f(y_{n-1}), c)^T
\]

and

\[
A = (a_{kj})_{n \times n},
\]

(3.7)

\[
a_{kj} = \frac{1}{\pi} W_j - \frac{1}{X'(x_j)} U(Z(x)X(x)|_{x=y_k}), \text{ for } 1 \leq k \leq n - 1, \quad 1 \leq j \leq n,
\]

\[
an_{kj} = \frac{1}{\pi} W_j, \text{ for } 1 \leq j \leq n.
\]

We can also construct the corresponding matrix \( B \),

\[
B = (b_{kj})_{n \times n},
\]

(3.8)

\[
b_{kj} = \frac{1}{\pi} V_j + \frac{1}{Y'(y_j)} V(Y(y)|_{y=y_k}), \text{ for } 1 \leq k \leq n, \quad 1 \leq j \leq n - 1,
\]

\[
b_{kn} = \mu b, \text{ for } 1 \leq k \leq n.
\]

In order to prove the solvability of (3.6) and to find the inverse of (3.7), we introduce two general lemmas first.

**Lemma 3.1.** If \( A = (a_{kj}) \) is an \( n \times n \) matrix satisfying

\[
\sum_{j=1}^{n} a_{kj} \phi(x_j) = U(Z(x)\phi(x))|_{x=y_k}, \quad k = 1, 2, \cdots, n - 1,
\]

\[
\sum_{j=1}^{n} a_{nj} \phi(x_j) = \frac{1}{\pi} \int_{-1}^{1} Z(x) \phi(x) dx
\]

and

The proof is omitted due to space constraints. However, this lemma is crucial for proving the solvability of the system (3.6) and finding its inverse.
for any polynomial \( \phi(x) \) of degree \( \leq n - 1 \), then \( A \) is invertible.

**Lemma 3.2.** If \( B = (b_{kj}) \) is an \( n \times n \) matrix satisfying

\[
\sum_{j=1}^{n-1} b_{kj} \phi(y_j) = V \left( \frac{\phi(y)}{Z(y)} \right) |_{y=y_k}, \quad k = 1, 2, \ldots, n
\]

\[
b_{kn} = \mu b,
\]

for any polynomial \( \phi(y) \) of degree \( \leq n - 2 \) and \( A \) is an \( n \times n \) matrix satisfying the conditions of Lemma 3.1, then

\[AB = \mu^2 I.\]

That is

\[A^{-1} = \frac{1}{\mu^2} B.\]

Next, we need to check that the matrix of (3.7) satisfies the conditions of Lemma 3.1 and the matrix of (3.8) satisfies the conditions of Lemma 3.2.

**Proposition 3.3.** The matrix \( A \) in (3.7) satisfies the conditions of Lemma 3.1. \( A \) is invertible.

**Proposition 3.4.** The matrix \( B \) in (3.8) satisfies the conditions of Lemma 3.2. The inverse of the matrix \( A \) in (3.7) has a closed form as \( \frac{1}{\mu^2} B \).

**4. Several Special Choices of the Quadrature and Collocation Nodes**

Sometime we need a simpler form of the system of equations, or we require higher accurate quadrature rule and or we want the two groups of nodes to be the same. We can make various options about the quadrature and the collocation nodes. In this section we provide four different choices of these nodes.

1. **Gauss-Jacobi Scheme.**

Let \( \{x_1, x_2, \ldots, x_n\} = \{\xi_1, \xi_2, \ldots, \xi_n\} \) and \( \{y_1, y_2, \ldots, y_{n-1}\} = \{\eta_1, \eta_2, \ldots, \eta_{n-1}\} \), where \( \{\xi_k\}_{k=1}^{n} \) and \( \{\eta_k\}_{k=1}^{n-1} \) are the zeros of the \( p_n(x) \) and \( q_{n-1}(y) \) respectively. We have \( X(x) = p_n(x) \) and \( Y(y) = q_{n-1}(y) \).

Denote

\[W_j^* = \int_{-1}^{1} Z(z)p_n(z)dz, \]

\[V_j^* = \int_{-1}^{1} \frac{q_{n-1}(y)dy}{Z(y)(y-\eta_j)q_{n-1}'(\eta_j)}.\]

The matrix \( A \) in (3.7) will become the form

\[A = (a_{kj})_{n \times n},\]

where

\[a_{kj} = \frac{1}{\xi_j - \eta_k} \left( \frac{b}{\pi} W_j^* - \frac{1}{p_n(\xi_j)} U(Z(z)p_n(z))|_{z=\eta_k} \right)\]

\[= \frac{1}{\xi_j - \eta_k} \left( \frac{b}{\pi} W_j^* - \frac{1}{p_n'(\xi_j)} q_{n-1}(\eta_k) \right) = \frac{b}{\pi} \frac{W_j^*}{\xi_j - \eta_k}, \text{ for } 1 \leq k \leq n - 1, \quad 1 \leq j \leq n.\]

The matrix \( B \) in (3.8) has also a simpler form,

\[B = (b_{kj})_{n \times n},\]
where

\[ b_{kj}^* = -\frac{1}{\eta_j - \xi_k} \left( \frac{b}{\pi} V_j^* + \frac{1}{q_{j-1}(\eta_j)} V \left( \frac{q_{n-1}(y)}{Z(y)} \right) \right) \bigg|_{y=\xi_k} \]

\[ = \frac{1}{\xi_k - \eta_j} \left( \frac{b}{\pi} V_j^* + \frac{1}{q_{j-1}(\eta_j)} p_n(\xi_k) \right) = \frac{b}{\pi} \frac{V_j^*}{\xi_k - \eta_j}, \text{ for } 1 \leq k \leq n, \quad 1 \leq j \leq n - 1. \]

Similar to Proposition 3.3, we have

**Proposition 4.1.** For any polynomial \( \phi(t) \) of degree \( \leq 2n \), the matrix \( A \) in (4.3) satisfies

\[ \sum_{j=1}^{n} a_{kj}^* \phi(\xi_j) = U(Z(x)\phi(x)) \bigg|_{x=\eta_k}, \text{ for } 1 \leq k \leq n - 1, \]

\[ \sum_{j=1}^{n} a_{kj}^* \phi(\xi_j) = \frac{1}{\pi} \int_{-1}^{1} Z(x)\phi(x)dx. \]

Similar to Proposition 3.4 we have

**Proposition 4.2.** For any polynomial \( \phi(t) \) of degree \( \leq 2n - 2 \), the matrix \( B \) in (4.4) satisfies

\[ \sum_{j=1}^{n-1} b_{kj}^* \phi(\eta_j) = V \left( \frac{\phi(x)}{Z(x)} \right) \bigg|_{x=\xi_k}, \text{ for } k = 1, 2, \ldots, n. \]

Since the Gauss quadrature formulae have higher algebraic accuracy. This scheme may have higher approximation rate of convergence.

2. Lobatto-Jacobi Scheme.

Let \( \{-1, -\eta_{n-1}, -\eta_{n-2}, \ldots, -\eta_1, 1\} \) be quadrature nodes and let \( \{-\xi_n, -\xi_{n-1}, \ldots, -\xi_1\} \) be collocation points. We have

\[ X(x) = (x + 1)(x - 1) \prod_{j=1}^{n} (x + \eta_j) \]

\[ = (-1)^n(1 - x)(1 + x) \prod_{j=1}^{n-1} (-x - \eta_j) \]

\[ = (-1)^n(1 - x)(1 + x)q_{n-1}(-x), \]

\[ Y(y) = \prod_{j=1}^{n} (y + \xi_j) = (-1)^n p_n(-y). \]

Considering our assumption of \( a \geq 0, b > 0 \) and \( \theta \geq 0 \) in the definition of \( Z(x) = (1 - x)^{-\frac{\xi}{2}} (1 + x)^{-1+\frac{\xi}{2}} \), we can represent the quadrature coefficients \( W_j \) and \( V_j \) by using the \( W_j^* \) and \( V_j^* \).

\[ W_0 := \int_{-1}^{1} \frac{Z(x)X(x)dx}{(x + 1)X'(-1)} \]

\[ = \frac{1}{2q_{n-1}(1)} \int_{-1}^{1} (1 - x)^{-\frac{\xi}{2}} (1 + x)^{-1+\frac{\xi}{2}} (1 - x)(1 + x)q_{n-1}(-x)dx \]

\[ = \frac{1}{2q_{n-1}(1)} \int_{-1}^{1} (1 - x)^{-\frac{\xi}{2}} (1 + x)^{-1+\frac{\xi}{2}} q_{n-1}(x)dx \]
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\[ \int_{-1}^{1} \frac{g_{n-1}(z)dz}{(z - x)Z(x)} \]
\[ = -\frac{1}{2q_{n-1}(1)} \int_{-1}^{1} \frac{g_{n-1}(z)dz}{(z - 1)Z(x)} \]
\[ = -\frac{1}{2q_{n-1}(1)} \frac{\pi}{b} \left( \frac{a_{q_{n-1}(1)}}{Z(1)} - \mu p_n(1) \right) \]
\[ = \frac{\pi \mu p_n(1)}{2b_{q_{n-1}(1)}}, \]

where we use the fact that \( \frac{1}{Z(1)} = 0 \) because of \( \theta \geq 0 \). For \( j = 1, 2, \ldots, n - 1 \), we have

\[ W_j := \int_{-1}^{1} \frac{Z(x)X(x)dx}{(x + \eta_{n-j})X'(\eta_{n-j})} \]
\[ = -\int_{-1}^{1} \frac{(1 - z)^{-\frac{\theta}{2}}(1 + z)^{-1+\frac{\theta}{2}}(1 - x)(1 + z)g_{n-1}(x)dx}{(x + \eta_{n-j})q_{n-1}(\eta_{n-j})(1 - \eta_{n-j}^2)} \]
\[ = \frac{1}{1 - \eta_{n-j}^2} \int_{-1}^{1} \frac{(1 + z)^{1-\frac{\theta}{2}}(1 - z)^{\frac{\theta}{2}}g_{n-1}(x)dx}{(x - \eta_{n-j})q_{n-1}'(\eta_{n-j})} \]
\[ = \frac{V_{n-j}^*}{1 - \eta_{n-j}^2}. \]

and

\[ W_n := \int_{-1}^{1} \frac{Z(x)X(x)dx}{(x - 1)X'(1)} \]
\[ = -\frac{1}{2q_{n-1}(-1)} \int_{-1}^{1} \frac{g_{n-1}(x)dx}{z - 1} \]
\[ = \frac{1}{2q_{n-1}(-1)} \int_{-1}^{1} \frac{g_{n-1}(z)dz}{Z(z)(z + 1)} \]
\[ = \frac{1}{2q_{n-1}(-1)} \frac{\pi}{b} \left( \frac{a_{q_{n-1}(-1)}}{Z(-1)} - \mu p_n(-1) \right) \]
\[ = -\frac{\pi \mu p_n(-1)}{2b_{q_{n-1}(-1)}}, \]

where we also use \( \frac{1}{Z(-1)} = 0 \) because of \( \theta \geq 0 \).

For \( j = 1, 2, \ldots, n \), we have

\[ V_j := \int_{-1}^{1} \frac{Y(y)dy}{(y + \xi_{n-j+1})Z(y)Y'(-\xi_{n-j+1})} \]
\[ = -\int_{-1}^{1} \frac{p_n(-y)dy}{(y + \xi_{n-j+1})Z(y)p_n'(\xi_{n-j+1})} \]
\[ = \int_{-1}^{1} \frac{(1 + y)^{\frac{\theta}{2}}(1 - y)^{1-\frac{\theta}{2}}p_n(y)dy}{(y - \xi_{n-j+1})p_n(\xi_{n-j+1})} \]
\[ = \int_{-1}^{1} \frac{Z(y)(1 - y^2)p_n(y)dy}{(y - \xi_{n-j+1})p_n(\xi_{n-j+1})} \]
\[ = (1 - \xi_{n-j+1}^2)W_{n-j+1}^* - \int_{-1}^{1} \frac{Z(y)p_n(y)(y + \xi_{n-j+1})dy}{p_n'(\xi_{n-j+1})} \]
\[ = (1 - \xi_{n-j+1}^2)W_{n-j+1}^*. \]
We can construct the system of equations as following:

Let

$$\tilde{\phi} = (\tilde{\phi}(-1), \tilde{\phi}(-\eta_{n-1}), \ldots, \tilde{\phi}(-\eta_1), \tilde{\phi}(1))^T,$$

$$\tilde{f} = (c, f(\xi_n), f(-\xi_{n-1}), \ldots, f(-\xi_1))^T.$$

We have

$$A\tilde{\phi} = \tilde{f},$$

where

$$A = (a_{kj})_{(n+1)\times(n+1)}, \quad 0 \leq k \leq n, \quad 0 \leq j \leq n.$$

$$a_{kj} = \begin{cases}
\frac{b}{\pi} \frac{V_{n-j}^*}{1 - \eta_{n-j}^2} \frac{1}{\xi_{n-k+1} - \eta_{n-j}}, & \text{for } 1 \leq k \leq n, \quad 1 \leq j \leq n - 1,

-\mu \frac{p_n(1)}{q_{n-1}(1)} \frac{1}{\xi_{n-k+1}}, & \text{for } 1 \leq k \leq n,

-\mu \frac{p_n(-1)}{q_{n-1}(-1)} \frac{1}{1 + \xi_{n-k+1}}, & \text{for } 1 \leq k \leq n,

\frac{V_{n-j}^*}{\pi} \frac{1}{1 - \eta_{n-j}^2}, & \text{for } 1 \leq j \leq n - 1,

\frac{p_n(1)}{2b q_{n-1}(1)}, & \text{for } a_{0j},

\frac{p_n(-1)}{2b q_{n-1}(-1)} & \text{for } a_{0n}.
\end{cases}$$

$$a_{k0} = a_{nk}.$$

Similar to Proposition 4.1, we have

**PROPOSITION 4.3.** For any polynomial $\phi(t)$ of degree $\leq 2n - 3$, the matrix $A$ in (4.12) satisfies

$$a_{k0}\phi(-1) + \sum_{j=1}^{n-1} a_{kj} \phi(-\eta_{n-j}) + a_{kn} \phi(1) = U(Z(x)\phi(x))|_{x=-\xi_{n-k+1}}, \text{ for } 1 \leq k \leq n,$$

$$a_{00} \phi(-1) + \sum_{j=1}^{n-1} a_{0j} \phi(-\eta_{n-j}) + a_{0n} \phi(1) = \frac{1}{\pi} \int_{-1}^{1} Z(x)\phi(x)dx.$$

Similar to Proposition 4.2, we introduce a matrix

$$B = (b_{kj})_{(n+1)\times(n+1)}, \quad 0 \leq k \leq n, \quad 0 \leq j \leq n,$$

where

$$b_{kj} = \begin{cases}
\frac{b}{\pi} \frac{(1 - \xi_{n-j+1}^2 W_{n-j+1}^*)}{\xi_{n-j+1} - \eta_{n-k}}, & \text{for } 1 \leq k \leq n, \quad 1 \leq j \leq n,

\frac{b}{\pi} \frac{(1 - \xi_{n-j+1}^2 W_{n-j+1}^*)}{\xi_{n-j+1} + 1}, & \text{for } 1 \leq j \leq n,

b_{nj} = \frac{b}{\pi} \frac{(1 - \xi_{n-j+1}^2 W_{n-j+1}^*)}{\xi_{n-j+1} - 1}, & \text{for } 1 \leq j \leq n,

b_{k0} = b\mu, & \text{for } 0 \leq k \leq n.
\end{cases}$$

**PROPOSITION 4.4.** For any polynomial $\phi(t)$ of degree $\leq 2n - 1$, the matrix $B$ in (4.13) satisfying

$$\sum_{j=1}^{n} b_{kj} \phi(-\xi_{n-j+1}) = \begin{cases}
V(\frac{\phi(y)}{2\eta(y)})|_{y=-1}, & \text{for } k = 0,

V(\frac{\phi(y)}{2\eta(y)})|_{y=-\eta_{n-j}}, & \text{for } 1 \leq k \leq n - 1,

V(\frac{\phi(y)}{2\eta(y)})|_{y=1}, & \text{for } k = n.
\end{cases}$$
By using the same way as that in proof of Lemma 3.1 and Lemma 3.2, we can show that the coefficient matrix $A$ of Gauss-Jacobi scheme in (4.3) and the coefficient matrix $A$ of Lobatto-Jacobi scheme in (4.12) are invertible and that their inverses are of the closed form $\frac{1}{B}B$, $B$ is given in (4.4) and (4.13) respectively.

These two schemes can approximate the singular integral operator in higher algebraic accuracy. Are there other groups of quadrature nodes or collocation nodes which have similar properties? For example, can we construct Radau-Jacobi scheme? In the case of $a = 0$ and $b = 1$, we found four different choices of the nodes (see [9]).

3. Coincidence of Quadrature Nodes and Collocation Points.

If we choose the $n - 1$ collocation points the same as the first $n - 1$ quadrature nodes, we also can construct an invertible linear system of equations.

Let $y_j = x_j$, for $1 \leq j \leq n - 1$, we have

$$X(x) = \prod_{j=1}^{n}(x - x_j),$$
$$Y(y) = \prod_{j=1}^{n-1}(y - x_j).$$

The discrete system of equations has the form

$$A\tilde{\phi} = \tilde{f},$$

where

$$\tilde{\phi} = (\tilde{\phi}(x_1), \tilde{\phi}(x_2), \ldots, \tilde{\phi}(x_n))^T,$$
$$\tilde{f} = (f(x_1), f(x_2), \ldots, f(x_{n-1}), c)^T,$$

$$A = (a_{kj})_{n \times n}, \quad 1 \leq k \leq n, \quad 1 \leq j \leq n,$$

$$a_{kj} = \frac{b}{\pi X'(x_j)} W_j X'(x_j) - W_k X'(x_k), \quad 1 \leq k \leq n - 1, \quad 1 \leq j \leq n, \quad k \neq j,$$
$$a_{kk} = a Z(x_k) + \frac{b}{\pi X'(x_k)} \int_{-1}^{1} \frac{Z(t)X(t)dt}{(t - x_k)^2}, \quad 1 \leq k \leq n - 1,$$
$$a_{nj} = \frac{1}{\pi} W_j, \quad 1 \leq j \leq n.$$

PROPOSITION 4.5. For any polynomial $\phi(t)$ of degree $\leq n - 1$, the matrix $A$ in (4.14) satisfies

$$\sum_{j=1}^{n} a_{kj} \phi(x_j) = U(Z(x)\phi(x))|_{x=x_k}, \quad 1 \leq k \leq n - 1,$$
$$\sum_{j=1}^{n} a_{nj} \phi(x_j) = \frac{1}{\pi} \int_{-1}^{1} Z(x)\phi(x)dx.$$

By Lemma 3.1, the matrix $A$ in (4.14) is nonsingular. Next, we introduce a matrix $B$,

$$B = (b_{kj})_{n \times n}, \quad 1 \leq k \leq n, \quad 1 \leq j \leq n,$$

$$b_{kj} = \frac{b}{\pi Y'(x_j)} V_j Y'(x_j) - V_k Y'(x_k), \quad 1 \leq k \leq n, \quad 1 \leq j \leq n - 1, \quad k \neq j,$$
$$b_{kk} = a Z(x_k) - \frac{b}{\pi Y'(x_k)} \int_{-1}^{1} \frac{Y(t)dt}{Z(t)(t - x_k)^2}, \quad 1 \leq k \leq n - 1,$$
$$b_{kn} = b \mu, \quad 1 \leq k \leq n.$$
PROPOSITION 4.6. For any polynomial $\phi(t)$ of degree $\leq n - 2$, the matrix $B$ in (4.15) satisfies

$$\sum_{j=1}^{n-1} b_{kj} \phi(x_j) = V \left( \frac{\phi(x)}{Z(x)} \right) |_{x=x_k}, \text{ for } 1 \leq k \leq n.$$ 

By using Lemma 3.2, we know that the inverse of $A$ in (4.14) is $\frac{1}{n!} B$.

Note: The collocation points are not necessarily the FIRST $n - 1$ quadrature nodes. In fact, they can be arbitrary $n - 1$ nodes. If we take $n$ collocation points, we will obtain a system of $n + 1$ equations of $n$ unknowns. The rank of the coefficient matrix is $n$. We can use the regularization method to find it's solution.

4. Minimum Norm-Least Square Scheme.

Let \( \{x_1, x_2, \ldots, x_n\} = \{y_1, y_2, \ldots, y_n\} = \{\xi_1, \xi_2, \cdots, \xi_n\} \), then \( X(x) = Y(x) = p_n(x) \). We can construct a system of $n$ equations of $n$ unknowns only from the integral equation \( U(Z(x)\phi(x)) = f(x) \) without considering the additional condition \( \frac{1}{2} \int_{-1}^{1} Z(x)\phi(x) = c \). We are going to show that the rank of the coefficient matrix is $n - 1$ and to find a closed form of it's generalized inverse.

Denote the discrete system as

$$A \mathbf{\tilde{\phi}} = \mathbf{\tilde{f}},$$

where

$$\mathbf{\tilde{\phi}} = (\tilde{\phi}(\xi_1), \tilde{\phi}(\xi_2), \cdots, \tilde{\phi}(\xi_n))^T,$$

$$\mathbf{\tilde{f}} = (f(\xi_1), f(\xi_2), \cdots, f(\xi_n))^T,$$

$$A = (a_{kj})_{n \times n},$$

$$a_{kj} = \frac{b}{\pi p_n'(\xi_j)} \int_{-1}^{1} \frac{Z(t)p_n(t)dt}{(t-\xi_j)(t-\xi_k)}, \text{ for } 1 \leq k, j \leq n, \quad k \neq j,$$

$$a_{kk} = aZ(\xi_k) + \frac{b}{\pi p_n'(\xi_k)} \int_{-1}^{1} \frac{Z(t)p_n(t)dt}{(t-\xi_k)^2}, \text{ for } 1 \leq k \leq n.$$  \hspace{1cm} (4.16)

Similarly, we introduce a matrix $B$,

$$B = (b_{kj})_{n \times n},$$

$$b_{kj} = -\frac{b}{\pi p_n'(\xi_j)} \int_{-1}^{1} \frac{p_n(t)dt}{Z(t)(t-\xi_j)(t-\xi_k)} \text{ for } 1 \leq k, j \leq n, \quad k \neq j,$$

$$b_{kk} = \frac{a}{Z(\xi_k)} - \frac{b}{\pi p_n'(\xi_k)} \int_{-1}^{1} \frac{p_n(t)dt}{Z(t)(t-\xi_k)^2}, \text{ for } 1 \leq k \leq n.$$  \hspace{1cm} (4.17)

PROPOSITION 4.7. For any polynomial $\phi(t)$ of degree $\leq 2n$, the matrix $A$ in (4.16) satisfies

$$\sum_{j=1}^{n} a_{kj} \phi(\xi_j) = U(Z(x)\phi(x))|_{x=\xi_k}, \text{ for } 1 \leq k \leq n.$$ 

PROPOSITION 4.8. For any polynomial $\phi(t)$ of degree $\leq 2n$, the matrix $B$ in (4.17) satisfies

$$\sum_{j=1}^{n} b_{kj} \phi(\xi_j) = V \left( \frac{\phi(x)}{Z(x)} \right) |_{x=\xi_k}, \text{ for } 1 \leq k \leq n.$$ 

The proofs are the same as that of Proposition 4.5; however, we can not use the general Lemma 3.1 and Lemma 3.2 anymore.
Denote
\[ P = \begin{pmatrix} p_0(\xi_1) & p_1(\xi_1) & \cdots & p_{n-1}(\xi_1) \\ p_0(\xi_2) & p_1(\xi_2) & \cdots & p_{n-1}(\xi_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_0(\xi_n) & p_1(\xi_n) & \cdots & p_{n-1}(\xi_n) \end{pmatrix} \]
and
\[ Q = \begin{pmatrix} 0 & q_0(\xi_1) & \cdots & q_{n-2}(\xi_1) \\ 0 & q_0(\xi_2) & \cdots & q_{n-2}(\xi_2) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & q_0(\xi_n) & \cdots & q_{n-2}(\xi_n) \end{pmatrix} \]
We have
\[ AP = \mu Q. \]
Since \( P \) is nonsingular, \( \text{rank}(A) = \text{rank}(Q) = n - 1 \).
Denote
\[ \hat{P} = \begin{pmatrix} p_1(\xi_1) & p_2(\xi_1) & \cdots & p_{n-1}(\xi_1) & 0 \\ p_1(\xi_2) & p_2(\xi_2) & \cdots & p_{n-1}(\xi_2) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_1(\xi_n) & p_2(\xi_n) & \cdots & p_{n-1}(\xi_n) & 0 \end{pmatrix} \]
and
\[ \hat{Q} = \begin{pmatrix} q_0(\xi_1) & q_1(\xi_1) & \cdots & q_{n-1}(\xi_1) \\ q_0(\xi_2) & q_1(\xi_2) & \cdots & q_{n-1}(\xi_2) \\ \vdots & \vdots & \ddots & \vdots \\ q_0(\xi_n) & q_1(\xi_n) & \cdots & q_{n-1}(\xi_n) \end{pmatrix} \]
We have
\[ B\hat{Q} = \mu \hat{P}. \]
Since \( \hat{Q} \) is nonsingular, \( \text{rank}(A) = \text{rank}(\hat{p}) = n - 1 \).
Introduce
\[ \hat{B} = \frac{1}{\mu^2} B, \]
we have

**Proposition 4.9.** \( \hat{B} \) is the \( M - N \) generalized inverse of \( A \) in (4.16), that is \( \hat{B} \) and \( A \) satisfy

\[ A\hat{B}A = A, \quad (4.18.1) \]
\[ \hat{B}A\hat{B} = \hat{B}, \quad (4.18.2) \]
\[ (\hat{A}\hat{B})^T M = M(\hat{A}\hat{B}), \quad (4.18.3) \]
\[ (\hat{B}A)^T N = N(\hat{B}A), \quad (4.18.4) \]
where \( M = (QQ^T)^{-1} \) and \( N = (PP^T)^{-1} \).

**Proof.** Let
\[ J = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}, \]
where \( I_{n-1} \) is an identity matrix of order \( n - 1 \). It is obvious that \( JT \) and \( J \) satisfy

\[ JJ^T J = J, \quad (4.19.1) \]
\[ J^T J J^T = J^T, \quad (4.19.2) \]
\[ (JJ^T)^T = JJ^T, \quad (4.19.3) \]
\[ (J^T J)^T = J^T J. \quad (4.19.4) \]
Since $\hat{P} = pJ^T$ and $Q = \hat{Q}J$, we have

$$A = \mu \hat{Q}JP^{-1},$$

$$\hat{B} = \frac{1}{\mu}PJ^T\hat{Q}^{-1}.$$

Therefore

$$A\hat{B}A = \mu(\hat{Q}JP^{-1})(PJ^T\hat{Q}^{-1})(\hat{Q}JP^{-1}) = \mu(\hat{Q}JJ^TJP^{-1}) = A,$$

$$\hat{B}A\hat{B} = \frac{1}{\mu}(PJ^T\hat{Q}^{-1})(\hat{Q}JP^{-1})(PJ^T\hat{Q}^{-1}) = \frac{1}{\mu}PJTJ^T\hat{Q}^{-1} = \hat{B},$$

$$(AB)^T M = (\hat{Q}JP^{-1}PJT\hat{Q}^{-1})^{-1}(\hat{Q}Q^T)^{-1} = \hat{Q}^{-T}J^TJ\hat{Q}^{-1} = M(AB),$$

$$(BA)^T N = (PJT\hat{Q}^{-1}QJP^{-1})^T(PP^T)^{-1} = (P^{-T}JJP^{-1})(P^{-T}P^{-1}) = \hat{N}(BA)^T.$$

According to C. R. Rao and S. K. Mitra [13] (pp. 52), $\hat{B}$ is the $M - N$ generalized inverse of $A$. Q.E.D.

**PROPOSITION 4.10.** $\tilde{x} = \hat{B}\tilde{f}$ is the minimum $N$-norm $M-$ least square solution of the $n \times n$ system of equations $A\hat{\phi} = \tilde{f}$. That is, the $\tilde{x}$ satisfies

$$\| \tilde{x} \|_N = \min \| \tilde{y} \|_N,$$

where $\tilde{y}$ makes

$$\| A\tilde{y} - \tilde{f} \|_M = \min_{\tilde{x} \in \mathbb{R}^n} \| A\tilde{x} - \tilde{f} \|_M.$$

The $\| \tilde{x} \|_M$ and $\| \tilde{x} \|_n$ are defined as $\sqrt{\tilde{x}^T M \tilde{x}}$ and $\sqrt{\tilde{x}^T N \tilde{x}}$ respectively.

5. NUMERICAL EXAMPLE

We have considered a numerical example for the equation (1.1) with $a = 0$, $b = 1$, $k = 0$ and $f(x) = \frac{1}{(x-c)^2+c^2}$. As $c$ is quite small (e.g. $c = 0.01$), $f(x)$ has a peak value $10^4$ at $x = c$. The exact solution is given by

$$\phi(x) = \frac{\frac{1}{2} (x - c) \cos \frac{\theta}{2} + c \sin \frac{\theta}{2}}{c \left( (x - c)^2 + c^2 \right)},$$

where

$$\tan \theta = 2c^2.$$

It has also a pair of maximum and minimum in the vicinity of 0. Since the derivative is very large, the values of $f(x)$ at Chebyshev collocation points $\{s_j\}$ can not represent the feature of $f(x)$ and the classical Gauss-Chebyshev scheme fails to provide a satisfactory approximate solution. The curve (dot line) in the following figure shows unacceptable errors.

Another way is to choose collocation points flexibly. For example, if we take $y = -0.01$ as collocation point instead of $y = s_2 = 0$, we obtain a general quadrature collocation scheme. The curve (dash line ) in the figure shows that it gives a much better approximation to the exact solution (points labeled by small triangles in the figure) than the Gauss-Chebyshev scheme does.

As $n$ increases, Chebyshev nodes become dense. Since the Lebesgue constant of the interpolation at Chebyshev nodes are bounded, a more satisfying approximation may be obtained. However, under the restriction of the number $n$, the flexible choice of collocation nodes can lead to a better result at a lower computational cost.
Figure 1. The curve of $g(t)$ with general nodes $y_m = -0.01$.

REFERENCES