



## Powers of modified Bessel functions of the first kind<sup>☆</sup>

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Dedicated to my children Boróka and Koppány

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### ABSTRACT

In this short note we consider the modified Bessel function of the first kind  $I_\nu$ , and we present an alternative derivation of the MacLaurin series expansion of the power  $I_\nu^r$ , where  $r$  is an arbitrary real (or complex) number. The key tool in our proof is a very old formula of L. Euler deduced in 1748.

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For  $\nu$  an unrestricted real (or complex) number, let  $I_\nu$  be the modified Bessel function of the first kind of order  $\nu$ , defined by [1, p. 77]

$$I_\nu(x) = \sum_{n \geq 0} \frac{(x/2)^{2n+\nu}}{n! \Gamma(\nu + n + 1)},$$

which occurs frequently in problems of electrical engineering, finite elasticity, quantum billiards, wave mechanics, mathematical physics and chemistry, etc. Here  $x$  is an arbitrary real (or complex) number, and as usual,  $\Gamma$  denotes the Euler gamma function. As the authors of [2] remarked, the products of Bessel and of modified Bessel functions of the first kind appear frequently in problems of statistical mechanics and plasma physics [3]. Motivated by the importance of products of modified Bessel functions in [2] the authors deduced explicit representations for powers of these functions. In this note we would like to point out that although little is known about the explicit expression for the MacLaurin series of powers of modified Bessel functions of the first kind, the general theory is very old and is well developed.

In [2, Eq. 30] the authors deduced an explicit recurrence formula which determines the coefficients in the MacLaurin series expansion of powers of modified Bessel functions of the first kind, i.e. they proved that

$$[I_\nu(x)]^r = \sum_{n \geq 0} \frac{a_n(r)(x/2)^{2n+r\nu}}{n!(\nu+n)!(\nu!)^{r-1}}, \quad (1)$$

where the polynomials  $a_n(r)$  are determined recursively by [2, Eq. 31]

$$a_n(r) = r \frac{\nu+n}{\nu+1} a_{n-1}(r) + \sum_{k=0}^n b_k(\nu) \frac{(\nu+1)! C_{\nu+n}^k}{n!(\nu+k+1)!} a_{n-1}(r), \quad (2)$$

and the integer sequence  $b_k(\nu)$  is identified by expanding the right-hand side of

$$\sum_{k \geq 0} \frac{b_k(\nu)x^k}{(\nu+k)!(k-1)!} = \frac{x}{(\nu+1)!} \left[ \frac{\sqrt{x} I_\nu(2\sqrt{x})}{(\nu+1) I_{\nu+1}(2\sqrt{x})} - 2 \right].$$

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Here, as usual,

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

denotes the binomial coefficient. Although in [2] the ranges of validity of  $\nu$  and  $r$  are not specified, due to the notation tacitly it is assumed that  $\nu$  is a natural number and  $r$  is an arbitrary real number. In this short note we show that in fact (1) can be deduced easily for  $\nu$  and  $r$  real (or complex) numbers by using a very old formula of Euler; moreover instead of (2) we propose another recurrence formula for the coefficients which is more convenient for direct computations. To this end let  $r$  be an arbitrary real (or complex) number and consider the power series  $f(x) = \sum_{n \geq 0} c_n x^n$  and  $[f(x)]^r = \sum_{n \geq 0} d_n x^n$ . Thus [4, p. 754]

$$d_0 = c_0^r, \quad c_0 \neq 0, \quad d_n = \frac{1}{nc_0} \sum_{k=1}^n [k(r+1) - n] c_k d_{n-k}. \tag{3}$$

This recurrence formula was deduced first in 1748 by the genius of the “teacher of all mathematicians” whose iconic name is Leonhard Euler, in his famous *Introductio in Analysin Infinitorum*. However, this basic recurrence formula is not as widely known as it should be, and it has been rediscovered several times. For more details on the history of relation (3) and related coefficient problems in multiplying power series the interested reader is referred to [5] and to the references therein.

Now consider the power series

$$J_\nu(x) = 2^\nu \Gamma(\nu + 1) x^{-\nu/2} I_\nu(x^{1/2}) = \sum_{n \geq 0} \frac{(x/4)^n}{n!(\nu + 1)_n},$$

which is sometimes called the normalized modified Bessel function of the first kind. Here  $(\nu + 1)_n$  for  $\nu \neq -1, -2, \dots$  denotes the well-known Pochhammer (or Appell) symbol defined in terms of Euler’s gamma function, i.e.  $(\nu + 1)_n = (\nu + 1)(\nu + 2) \cdots (\nu + n) = \Gamma(\nu + n + 1)/\Gamma(\nu + 1)$ . For convenience we denote by  $\alpha_n$  the general coefficient of the above power series, i.e. let  $\alpha_n = [4^n n!(\nu + 1)_n]^{-1}$  for all  $n \geq 0$  integer. Moreover we assume that the power series of  $[J_\nu(x)]^r$  has the form  $\sum_{n \geq 0} \beta_n(r) x^n$ . Observe that  $\alpha_0 = 1$  and hence  $\beta_0(r) = 1$ . Then by using Euler’s formula (3) we easily get

$$\beta_n(r) = \frac{1}{n} \sum_{k=1}^n [k(r+1) - n] \alpha_k \beta_{n-k}(r) \tag{4}$$

for all  $n \geq 1$  integer. In particular, the first five polynomials  $\beta_n(r)$  are the following:

$$\begin{aligned} \beta_0(r) &= 1, \\ \beta_1(r) &= r\alpha_1, \\ \beta_2(r) &= \frac{1}{2}r(r-1)\alpha_1^2 + r\alpha_2, \\ \beta_3(r) &= \frac{1}{6}r(r-1)(r-2)\alpha_1^3 + r(r-1)\alpha_1\alpha_2 + r\alpha_3, \\ \beta_4(r) &= \frac{1}{24}r(r-1)(r-2)(r-3)\alpha_1^4 + \frac{1}{2}r(r-1)(r-2)\alpha_1^2\alpha_2 + r(r-1)\alpha_1\alpha_3 + \frac{1}{2}r(r-1)\alpha_2^2 + r\alpha_4. \end{aligned}$$

If we consider the polynomial  $\gamma_n(r)$  defined for all  $n \geq 0$  integer and  $r$  real by  $\beta_n(r) = \alpha_n \gamma_n(r)$ , then clearly we have

$$[J_\nu(x)]^r = \sum_{n \geq 0} \frac{\gamma_n(r) (x/2)^{2n+r\nu}}{n!(\nu + 1)_n [\Gamma(\nu + 1)]^r}, \tag{5}$$

which holds for all  $x, \nu$  and  $r$  real (or complex) numbers, such that  $\nu \neq -1, -2, \dots$ . Clearly  $\gamma_0(r) = 1$  and by using (4) we obtain that for all  $n \geq 1$  natural numbers for the polynomial  $\gamma_n(r)$  the following recurrence relation is valid:

$$\gamma_n(r) = \frac{1}{n} \sum_{k=1}^n [k(r+1) - n] \left[ \frac{\alpha_k \alpha_{n-k}}{\alpha_n} \right] \gamma_{n-k}(r),$$

which can be rewritten as follows:

$$\gamma_n(r) = \frac{1}{n} \sum_{k=1}^n [k(r+1) - n] C_n^k \frac{(\nu + k + 1)_{n-k}}{(\nu + 1)_{n-k}} \gamma_{n-k}(r). \tag{6}$$

It is important to note here that when  $\nu$  is a natural number, then the polynomials  $a_n(r)$  and  $\gamma_n(r)$  are the same even if the relations (2) and (6) are different. For this, just compare (1) with (5) when  $\nu$  is a natural number. This is in agreement with the fact that the first five polynomials  $a_n(r)$  computed in [2, Eq. 21] coincide with the first five polynomials  $\gamma_n(r) = \beta_n(r)/\alpha_n$ . Moreover, the recurrence relation (6) is much more convenient in computations than the relation (2), because only the corresponding coefficients appear in it, while in (2) we need to first identify the coefficients  $b_k(\nu)$ .

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