The semigroup of incline Hall matrices∗

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Abstract

This paper investigates the semigroup of incline Hall matrices. An incline Hall matrix means an incline matrix which is greater than or equal to an incline invertible matrix. It is proved that any incline Hall matrix has index. Some necessary and sufficient conditions for an incline Hall matrix to be regular in the semigroup of incline Hall matrices are given and Green’s relations on the such semigroup are described. Also the sandwich semigroup of incline Hall matrices is considered. The main results in the present paper generalize some previous results on Boolean matrices, fuzzy matrices and lattice matrices.

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1. Introduction

The notion of inclines and their applications were described in Cao et al. [1] comprehensively. Recently, Kim and Roush [11] also made a survey of the results obtained in the literature [1] and some applications. Inclines are the additively

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idempotent semirings in which the products are less than or equal to either factor. Thus inclines generalize Boolean algebra, fuzzy algebra and distributive lattice. And Boolean matrices, fuzzy matrices and lattice matrices (matrices over distributive lattices) are the prototypical examples of incline matrices (matrices over inclines).

Kim [9] described the structural and combinatorial properties of the semigroup of Hall relations in terms of Green’s relations. Cho [2] studied the regular matrices in the semigroup of Boolean Hall matrices. Cho [3] considered fuzzy Hall matrices (fuzzy matrices of permanent 1). And Tan [12] discussed Hall matrices over a complete and completely distributive lattice, thus generalizing the corresponding previous results. Meanwhile, Han and Li [5] gave a necessary and sufficient condition for an incline matrix to have index and established some sufficient conditions for an incline matrix to converge in finite steps. Han and Li [6] studied some necessary and sufficient conditions for an incline matrix to be invertible and presented Cramer’s rule over inclines.

The aim of this paper is to extend the results of the Refs. [2,3,9,12] to the setting of incline matrices. We prove that any incline Hall matrix has index and give some necessary and sufficient conditions for an incline Hall matrix to be regular in the semigroup of incline Hall matrices. Also we describe Green’s relations on the such semigroup and consider the sandwich semigroup of incline Hall matrices. The main results in the present paper are the generalizations of the previous results in the references.

2. Preliminaries and some lemmas

In this section, we give some definitions and prove the fundamental lemmas which will be used in the next sections.

**Definition 2.1** [1]. A nonempty set $L$ with two binary operations $+$ and $\cdot$ is called an incline if it satisfies the following conditions:

1. $(L, +)$ is a semilattice,
2. $(L, \cdot)$ is a commutative semigroup,
3. $x(y + z) = xy + xz$ for all $x, y, z \in L$,
4. $x + xy = x$ for all $x, y \in L$.

In an incline $L$, define a relation $\leq$ by $x \leq y \iff x + y = y$. Obviously, $xy \leq x$ for all $x, y \in L$.

The Boolean algebra $([0, 1], \lor, \land)$ is an incline. The fuzzy algebra $([0, 1], \lor, T)$ is also an incline, where $T$ is a t-norm. And distributive lattices are a kind of inclines.

**Definition 2.2.** An incline $L$ is called a complete semilattice-ordered incline if it satisfies the following conditions:
(1) \((L, +)\) is a complete semilattice,
(2) \(x (\sum_{\alpha \in \Omega} y_\alpha) = \sum_{\alpha \in \Omega} xy_\alpha\) for any \(\{y_\alpha | \alpha \in \Omega\} \subseteq L\),

where \(\Omega\) is arbitrary nonempty set of subscripts.

For any positive integer \(n\), \(\mathbb{Z}_n\) always stands for the set \(\{1, 2, \ldots, n\}\) and \([n]\) denotes the least common multiple of the integers 1, 2, \ldots, \(n\).

Throughout this paper, \(L\) always denotes any given incline with the additive identity 0 and the multiplicative identity 1. It follows that 0 is the least element and 1 is the greatest element in \(L\).

Denote by \(M_n(L)\) the set of all \(n \times n\) matrices over \(L\).

Given \(A = (a_{ij}) \in M_n(L)\) and \(B = (b_{ij}) \in M_n(L)\), the product \(A \cdot B \in M_n(L)\) is defined by

\[
A \cdot B := \left( \sum_{k \in \mathbb{Z}} a_{ik} b_{kj} \right).
\]

For \(A = (a_{ij}) \in M_n(L)\) and positive integer \(l\), denote by \(a_{ij}^{(l)}\) the \((i, j)\)-entry of \(A^l\), i.e., \(A^l = (a_{ij}^{(l)})\).

Denote by \(I_n\) the \(n \times n\) identity matrix over \(L\), i.e., \(I_n = (\delta_{ij}) \in M_n(L)\) and

\[
\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad \text{for } i, j \in \mathbb{Z}.
\]

For the sake of convenience, \(I_n\) is sometimes written as \(I\).

The set \(M_n(L)\) forms a partially ordered monoid with respect to the matrix multiplication.

**Definition 2.3** [1]. A matrix \(A \in M_n(L)\) is said to be invertible if \(AB = BA = I\) for some \(B \in M_n(L)\). The matrix \(B\), denoted by \(A^{-1}\), is called an inverse of \(A\).

Denote by \(G_n(L)\) the set of all \(n \times n\) invertible matrices over \(L\). It is clear that \(G_n(L)\) forms a subgroup of the monoid \(M_n(L)\).

**Definition 2.4** [1]. A matrix \(P \in M_n(L)\) is called a permutation matrix if only one entry of its every row and every column is 1 and the other entries are 0.

Denote by \(P_n(L)\) the set of all \(n \times n\) permutation matrices over \(L\). It is obvious that \(P_n(L)\) forms a subgroup of \(G_n(L)\) and \(P_n(L) \cong S_n\), where \(S_n\) is the symmetric group of degree \(n\).

**Definition 2.5.** A matrix \(A \in M_n(L)\) is called a Hall matrix if there exists an invertible matrix \(C \in G_n(L)\) such that \(A \geq C\).

A matrix \(A \in M_n(L)\) is said to be reflexive if \(A \succeq I\) [5]. Thus incline Hall matrices are a generalization of incline reflexive matrices.
Denote by $H_n(L)$ the set of all $n \times n$ Hall matrices over $L$. Then it is evident that $H_n(L)$ is a subsemigroup of $M_n(L)$ and

$$P_n(L) \subseteq G_n(L) \subseteq H_n(L) \subseteq M_n(L).$$

**Example 2.1.** Let $L = \langle \{1, 2, 3, 4, 6, 12\}, +, \cdot \rangle$, where $a + b = \text{l.c.m.}\{a, b\}$ and $a \cdot b = \text{g.c.d.}\{a, b\}$. Then $L$ is a distributive lattice. Obviously,

$$P_2(L) = \left\{ \begin{pmatrix} 12 & 1 \\ 1 & 12 \end{pmatrix}, \begin{pmatrix} 1 & 12 \\ 12 & 1 \end{pmatrix} \right\}.$$

From the proof of Theorem 5.3 in Han and Li [6], we can see that

$$G_2(L) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| a + b = 12, \ a, b \in L \right\} = \left\{ \begin{pmatrix} 12 & 1 \\ 1 & 12 \end{pmatrix} \cdot \begin{pmatrix} 1 & 12 \\ 12 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \right\}.$$

Hence we have that

$$H_2(L) = \left\{ \begin{pmatrix} 12 & a \\ b & 12 \end{pmatrix} \cdot \begin{pmatrix} a & 12 \\ 12 & b \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot \begin{pmatrix} y & x \\ w & z \end{pmatrix} \middle| a|12, \ b|12, \ 4|x|12, \ 3|y|12, \ 3|z|12, \ 4|w|12 \right\}.$$

Therefore, $P_2(L) \subseteq G_2(L) \subseteq H_2(L) \subseteq M_2(L)$.

**Example 2.2.** Let $L$ be the lattice $\{0, a_1, a_2, a_3, b, 1\}$ where $0 < a_i < b < 1$ for $1 \leq i \leq 3$ and $a_i \land a_j = 0, a_i \lor a_j = b$ for $i \neq j$. Then $L$ is not a distributive lattice. Now let $T_0$ be the binary operation on $L$ defined by

$$T_0(x, y) := \begin{cases} x \land y, & x = 1 \text{ or } y = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $T_0$ is a $\lor$-distributive t-norm on $L$ (see [8]). And $(L, \lor, T_0)$ forms an integral incline. By Theorem 5.2 in Han and Li [6], for any $n \geq 2$, $G_n(L) = P_n(L)$ and so $H_n(L) = \{ A \in M_n(L) \mid \exists Q \in P_n(L), \ A \geq Q \}$. In particular,

$$G_2(L) = P_2(L) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

and

$$H_2(L) = \left\{ \begin{pmatrix} 1 & r \\ s & 1 \end{pmatrix}, \begin{pmatrix} r & 1 \\ 1 & s \end{pmatrix} \middle| r, s \in L \right\}.$$

For a matrix $A \in H_n(L)$, denote $S_A := \{ C \in G_n(L) \mid C \leq A \}$. 
Lemma 2.1 [6]. If $A \in M_n(L)$, then the following statements are equivalent each other:

1. $A$ is invertible;
2. $AA^T = I$;
3. $A^TA = I$;
4. $A^k = I$ for some positive integer $k$.

Lemma 2.2 [6]. If $H$ is a subsemigroup of $G_n(L)$, then $H$ is a subgroup of $G_n(L)$.

Lemma 2.3. Let $A \in M_n(L)$ and $B, C \in G_n(L)$. If $BAC \leq A$ or $BAC \geq A$, then $BAC = A$.

Proof. By Lemma 2.1, $B^k = C^l = I$ for some positive integers $k$ and $l$. Put $p = k \cdot l$. Then $B^p = C^p = I$. If $BAC \leq A$, then

$$A \geq BAC \geq B^2AC^2 \geq \cdots \geq B^pAC^p = A.$$ 

If $BAC \geq A$, then

$$A \leq BAC \leq B^2AC^2 \leq \cdots \leq B^pAC^p = A.$$ 

This completes the proof. □

Definition 2.6 [6]. Let $A = (a_{ij}) \in M_n(L)$. The permanent $\text{per}(A)$ of $A$ is defined by

$$\text{per}(A) := \sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)},$$

where $S_n$ is the symmetric group of degree $n$.

Definition 2.7 [6]. Let $A \in M_n(L)$. An adjoint matrix $\text{adj}(A) \in M_n(L)$ of $A$ is the matrix whose $(i, j)$-entry is $\text{per}(A(j|i))$ for all $i, j \in n$. Here $A(j|i)$ denotes the $(n-1) \times (n-1)$ submatrix of $A$ obtained from $A$ by deleting the $j$th row and the $i$th column of $A$.

Lemma 2.4 [6]. If $A = (a_{ij}) \in M_n(L)$, then for any $i \in n$,

$$\text{per}(A) = \sum_{j \in n} a_{ij} \cdot \text{per}(A(i|j)).$$

Lemma 2.5 [6]. If $C \in G_n(L)$, then the following hold:

1. $\text{per}(C) = 1$,
2. $\text{adj}(C) = C^{-1}.$
Lemma 2.6 [5]. If \( A \in M_n(L) \) is a reflexive matrix, then \( I \leq A \leq A^2 \leq \cdots \leq A^{n-1} = A^n \).

Lemma 2.7. If \( A \in M_n(L) \) is a reflexive matrix, then the following hold:

(1) \( \text{adj}(A) = A^{n-1} \),
(2) \( \text{adj}(A) \) is idempotent,
(3) \( \text{Adj}(A) = \text{adj}(A)A = \text{adj}(A) \).

Proof. (1) From Lemma 2.4, we see that for any \( i, j \in \mathbb{N} \), \( \text{per}(A(j|i)) \) is the permanent of the matrix \( A_{ji} \) obtained from \( A \) by replacing \( a_{ji} \) with 1 and all other \( a_{jk} \) \((k \neq i)\) with 0.

Any term of \( \text{per}(A_{ji}) \) is of the form
\[
    a_{k_1}a_{k_2}\cdots b_{j_{k_j}}\cdots a_{kn},
\]
where
\[
    b_{j_{k_j}} = \begin{cases} 1, & k_j = i, \\ 0, & k_j \neq i. \end{cases}
\]

On the other hand, if \( k_j = i \), then the permutation
\[
    \sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}
\]
includes a cycle \((i \ t_1 \ \cdots \ t_r \ j)\). Hence for any nonzero term of \( \text{per}(A_{ji}) \), we have that
\[
    a_{i_1}a_{i_2}\cdots b_{j_{k_j}}\cdots a_{kn} \leq a_{i_1}a_{i_2}\cdots a_{i_{n-j}}
\]
by Lemma 2.6. Thus \( \text{per}(A(j|i)) = \text{per}(A_{ji}) \leq a_{i_j}^{(n-1)} \) and so \( \text{adj}(A) \leq A^{n-1} \).

Now we show that \( A^{n-1} \leq \text{adj}(A) \). Obviously, \( a_{i_j}^{(n-1)} \leq 1 = \text{per}(A(j|i)) \). Let \( i \neq j \). We have that
\[
    a_{i_j}^{(n-1)} = \sum_{i_1,i_2,\ldots,i_{n-2} \in \mathbb{N}} a_{i_1}a_{i_2}\cdots a_{i_{n-2}j}.
\]
Consider any term \( b = a_{i_1}a_{i_2}\cdots a_{i_{n-2}j} \). If \( n \) subscripts \( i, i_1, i_2, \ldots, i_{n-2}, j \) are distinct, then \( b \) is a term of \( \text{per}(A(j|i)) \). Otherwise, there exist \( r \) and \( s \) such that \( i_r = i_s \) and \( 0 \leq r < s \leq n-1 \) (putting \( i_0 = i \) and \( i_{n-1} = j \)). Then
\[
    c = \prod_{i \in \{0,1,\ldots,n-2\}\setminus\{r,\ldots,s-1\}} a_{i_0,i_{i+1}} \geq b.
\]
Assume that all subscripts in \( c \) have been made distinct by repeated deletions. For all \( z \in \mathbb{N} \) different from the subscripts in \( C \), we can insert the factors \( a_{zz} (=1) \) without changing the value of \( c \), thus yielding a term of \( \text{per}(A(j|i)) \). Hence \( a_{i_j}^{(n-1)} \leq \text{per}(A(j|i)) \) for \( i \neq j \). So \( A^{n-1} \leq \text{adj}(A) \).

(2) \( \text{(adj}(A))^2 = A^{2(n-1)} = A^{n-1} = \text{adj}(A) \) by (1) and Lemma 2.6. Hence \( \text{adj}(A) \) is idempotent.
By (1) and Lemma 2.6, 
\[ A \text{adj}(A) = A^n = A^{n-1} = \text{adj}(A). \]
Similarly, \( \text{adj}(A)A = \text{adj}(A) \). This completes the proof. □

**Lemma 2.8.** If \( A \in M_n(L) \) and \( P, Q \in G_n(L) \), then \( \text{adj}(PAQ) = Q^T \text{adj}(A)P^T \).

**Proof.** We first verify that \( \text{adj}(PA) = \text{adj}(A)P^T \). Let \( A = (a_{ij}), \ P = (p_{ij}), \ B = (b_{ij}) = PA, \ C = (c_{ij}) = \text{adj}(A)P^T \) and \( D = (d_{ij}) = \text{adj}(PA) \). Then for any subscripts \( i, j \in n \), we have that
\[
d_{ij} = \text{per}(B(j|i)) = \sum_{\sigma \in B(j,i), r \neq j} \prod_{r \neq j} b_{r \sigma(r)}
\]
\[
= \sum_{\sigma \in B(j,i), r \neq j} \prod_{r \neq j} (p_{r1}a_{1 \sigma(r)} + p_{r2}a_{2 \sigma(r)} + \cdots + p_{rn}a_{n \sigma(r)}),
\]
where \( B(j,i) \) is the set of all bijective mappings of \( n \setminus \{j\} \) to \( n \setminus \{i\} \). Upon expanding, this expression can be written as follows:
\[
d_{ij} = \sum_{\sigma \in B(j,i)} \sum_{\tau \in m(j)} \left( \prod_{r \neq j} p_{r \tau(r)} \prod_{r \neq j} a_{\tau(r) \sigma(r)} \right),
\]
where \( m(j) \) is the set of all mappings of \( n \setminus \{j\} \) to \( n \).

Suppose that \( \tau(u) = \tau(v) \) for some distinct \( u \) and \( v \). Since \( P \) is invertible, \( \prod_{r \neq j} p_{r \tau(r)} = 0 \) by (2) of Lemma 2.1. Thus
\[
d_{ij} = \sum_{\sigma \in B(j,i)} \sum_{\tau \in B(j)} \left( \prod_{r \neq j} p_{r \tau(r)} \prod_{r \neq j} a_{\tau(r) \sigma(r)} \right),
\]
where \( B(j) \) is the set of all injective mappings of \( n \setminus \{j\} \) to \( n \). Furthermore, we have that
\[
d_{ij} = \sum_{\sigma \in B(j,i)} \sum_{k \in n} \sum_{\tau \in B(j,k)} \left( \prod_{r \neq j} p_{r \tau(r)} \prod_{r \neq j} a_{\tau(r) \sigma(r)} \right)
\]
\[
= \sum_{k \in n} \sum_{\tau \in B(j,k), r \neq j} p_{r \tau(r)} \left( \sum_{\sigma \in B(j,i), r \neq j} a_{\tau(r) \sigma(r)} \right)
\]
\[
= \sum_{k \in n} \sum_{\tau \in B(j,k), r \neq j} p_{r \tau(r)} \text{per}(A(k|i))
\]
\[
= \sum_{k \in n} \text{per}(A(k|i)) \left( \sum_{\tau \in B(j,k), r \neq j} p_{r \tau(r)} \right)
\]
\[
= \sum_{k \in n} \text{per}(A(k|i)) \text{per}(P(j|k))
\]
\[
= \sum_{k \in n} \text{per}(A(k|i)) p_{jk}
\]
by (2) of Lemma 2.5. On the other hand, 
\[ c_{ij} = \sum_{k \in \mathbb{N}} \text{per}(A(k|i)) p_{jk} \text{ for all } i, j \in \mathbb{N}. \]

Hence \( \text{adj}(PA) = \text{adj}(A)P^T \). Similarly, we prove that \( \text{adj}(AQ) = Q^T\text{adj}(A) \). Therefore, \( \text{adj}(PAQ) = \text{adj}(AQ)P^T = Q^T\text{adj}(A)P^T \). □

**Definition 2.8** [7]. Let \( A \in M_n(L) \). If there exist two positive integers \( k \) and \( d \) such that \( A^k = A^{k+d} \), then the least such integers \( k \) and \( d \) are called the index and the period of \( A \), and denoted by \( i(A) \) and \( p(A) \), respectively. In this case, we say that \( A \) has index or that \( A \) is an incline matrix with index.

**Lemma 2.9** [5]. Let \( A \in M_n(L) \). Then \( A \) has index if and only if there exist positive integers \( p \) and \( r \) such that \( p < r \) and \( A^p \leq A^r \). And in this case, \( i(A) \leq p + (n-1)(r-p) \) and \( p(A) \mid (r-p) \).

**Theorem 2.1.** If \( A \in H_n(L) \), then the following hold:

1. \( A \) has index,
2. \( i(A) \leq 1 + (n-1)\min\{p(C) \mid C \in S_A\} \) and \( p(A) \mid (p(C)|\{p(C)|C \in S_A\}) \).

**Proof.** If \( C \in S_A \), then there is a positive integer \( k \) such that \( C^k = I \) by Lemma 2.1. Hence \( C^{k+1} = C \) and so \( C \) has index 1. Thus \( C^{1+p(C)} = C \). Since \( C \) is invertible, \( C^{p(C)} = I \). We now obtain that \( A^{p(C)+1} = A^{p(C)} A \geq C^{p(C)} A = A \), i.e., \( A^{p(C)+1} \geq A \). By Lemma 2.9, \( A \) has index. Furthermore, \( i(A) \leq 1 + (n-1)p(C) \) and \( p(A) \mid p(C) \). This completes the proof. □

**Definition 2.9.** Let \( A, B \in M_n(L) \). A is said to be equivalent to \( B \) if \( B = PAQ \) for some \( P, Q \in G_n(L) \). In particular, if \( B = PAP^T \) for some \( P \in G_n(L) \), then \( A \) is said to be similar to \( B \).

**Lemma 2.10.** If \( A \in H_n(L) \), then \( A \) is equivalent to a reflexive matrix in \( H_n(L) \).

**Proof.** Suppose that \( A \geq P \) for some \( P \in G_n(L) \). By (4) of Lemma 2.1, there is a positive integer \( l \) such that \( l \geq 2 \) and \( P^l = I \). Thus \( AP^{l-1} \geq P^l = I \) and \( P^{l-1} \in G_n(L) \). Hence \( A \) is equivalent to a reflexive matrix \( AP^{l-1} \). □

3. Regular matrices in \( H_n(L) \)

In this section, we give criteria for an incline Hall matrix to be regular in the semigroup of incline Hall matrices in terms of idempotent matrices and adjoint matrices.

**Definition 3.1.** Let \( S \) be a semigroup. An element \( a \in S \) is said to be regular in \( S \) if \( aga = a \) for some \( g \in S \). In this case, \( g \) is called a generalized inverse of \( a \).
If $a \in S$ is regular in $S$, then there is an element $g' \in S$ such that $ag'a = a$ and $g'ag' = g'$. In fact, we can take the element $gag$ as an example of $g'$. Such an element $g'$ is called a semiinverse of $a$.

Lemma 3.1. If $A, G \in H_n(L)$ and $AGA = A$, then $APA = A$ and $P^2 \in S_A$ for all $P \in SG$.

Proof. If $Q \in S_A$, then $APQ \leq APA \leq AGA = A$. Since $PQ \in G_n(L)$, $A = APQ$ by Lemma 2.3. This implies that $APA = A$. If $Q_1, Q_2 \in S_A$, then $Q_1PQ_2 \leq APA = A$ and $Q_1PQ_2 \in G_n(L)$. Thus $Q_1PQ_2 \in S_A$ and so $(PQ_1)(PQ_2) \in P \cdot S_A = \{PQ \mid Q \in S_A\}$. Hence $P \cdot S_A$ is a subsemigroup of $G_n(L)$. By Lemma 2.2, $P \cdot S_A$ is a subgroup of $G_n(L)$. Therefore, $I \in P \cdot S_A$ and $PQ = I$ for some $Q \in S_A$. By Lemma 2.1, $P^2 = P^{-1} = Q \in S_A$. This completes the proof. □

Definition 3.2. A matrix $A \in M_n(L)$ is said to be idempotent if $A^2 = A$.

Theorem 3.1. If $A \in H_n(L)$, then the following statements are equivalent:

1. $A$ is regular in $H_n(L)$;
2. $A$ is equivalent to an idempotent matrix in $M_n(L)$.

Proof. (1) ⇒ (2) Let $G \in H_n(L)$ be a generalized inverse of $A$. Then $AGA = A$ and so $APA = A$ for all $P \in SG$ by Lemma 3.1. Hence $PAPA = PA$ and $PAI = PA$ is an idempotent matrix. Thus $A$ is equivalent to an idempotent matrix.

(2) ⇒ (1) Suppose that $(PAPA)^2 = PAPA$ for some $P, Q \in G_n(L)$. Since $P$ and $Q$ are invertible, we obtain that $AQPA = A$ and $QP \in G_n(L) \subseteq H_n(L)$. Hence $A$ is regular in $H_n(L)$. □

A matrix $A \in M_n(L)$ is said to be transitive if $A^2 \leq A$ [5].

Lemma 3.2. If $A \in H_n(L)$, then the followings hold:

1. if $A$ is transitive, then it is reflexive and $\text{adj}(A) = A$,
2. $\text{adj}(A) \in H_n(L)$ and it is regular in $H_n(L)$.

Proof. (1) If $A$ is transitive, then for any $P_1, P_2 \in S_A$, $P_1P_2 \leq AA \leq A$ and $P_1P_2 \in G_n(L)$. Hence $P_1P_2 \in S_A$ and so $S_A$ is a subsemigroup of $G_n(L)$. By Lemma 2.2, $S_A$ is a subgroup of $G_n(L)$ and $I \in S_A$. Thus $A$ is reflexive and so $A \leq A^2$ by Lemma 2.6. Hence $A = A^2$ and $\text{adj}(A) = A^{n-1} = A$ by Lemma 2.7.

(2) Let $P \in S_A$, since $I = P^2P \leq P^2A$ by Lemma 2.1, $P^2A$ is reflexive and so $\text{adj}(P^2A)$ is idempotent by (2) of Lemma 2.7. By Lemma 2.8, $\text{adj}(P^2A) = \text{adj}(A)P$. Hence $\text{adj}(A) = \text{adj}(P^2A)$ is equivalent to an idempotent matrix $\text{adj}(P^2A)$. On the other hand, $\text{adj}(P^2A) \geq \text{adj}(I) = I$. Hence $\text{adj}(A) = \text{adj}(P^2A)P^2 \geq P^2$, and so $\text{adj}(A) \in H_n(L)$. Therefore, $\text{adj}(A)$ is regular in $H_n(L)$ by Theorem 3.1. □
From the proof of (1) in Lemma 3.2, we obtain the following.

**Corollary 3.1.** A ∈ Hn(L) is transitive if and only if it is idempotent.

**Theorem 3.2.** If A ∈ Hn(L) is regular in Hn(L), then adj(A) is a semiinverse of A in Hn(L).

**Proof.** Since A is regular in Hn(L), by Theorem 3.1 there exist P, Q ∈ Gn(L) such that D = PAQ is idempotent. By (1) of Lemma 3.2, \( \text{adj}(D) = D \). Hence by Lemmas 2.1 and 2.8,

\[
\text{adj}(A)A = (P^TDQ^T)(Q\text{adj}(D)P)(P^TDQ^T) = P^TDQ^T = A
\]

and

\[
\text{adj}(A)\text{adj}(A) = (Q\text{adj}(D)P)(P^TDQ^T)(Q\text{adj}(D)P) = Q\text{adj}(D)D\text{adj}(D)P = Q\text{adj}(D)P = \text{adj}(A).
\]

On the other hand, by (2) of Lemma 2.5, \( \text{adj}(A) \geq \text{adj}(C) = C^T \) for any \( C \in S_A \). Hence \( \text{adj}(A) \in H_n(L) \) and so \( \text{adj}(A) \) is a semiinverse of A in Hn(L).

**Theorem 3.3.** If A ∈ Hn(L), then the following statements are equivalent:

1. A is regular in Hn(L);
2. \( \text{adj(} \text{adj}(A) = A \).

**Proof.** (1) ⇒ (2) If A is regular in Hn(L), then by Theorem 3.1 there exist P, Q ∈ Gn(L) such that D = PAQ is idempotent. Then D ∈ Hn(L) and \( \text{adj}(D) = D \) by (1) of Lemma 3.2. Hence A = \( P^TDQ^T \) and by Lemma 2.8, we have that

\[
\text{adj}(\text{adj}(A)) = \text{adj}(Q\text{adj}(D)P) = \text{adj}(QDP) = P^TDQ^T = A.
\]

(2) ⇒ (1) By (2) of Lemma 3.2, \( \text{adj(} \text{adj}(A) \in H_n(L) \) and it is regular in Hn(L). Since A = \( \text{adj(} \text{adj}(A) \), A is also regular in Hn(L).

**Theorem 3.4.** Let L be a complete semilattice-ordered incline. If A ∈ Hn(L) is regular in Hn(L), then \( \text{adj}(A) \) is the unique semiinverse of A in Hn(L).

**Proof.** Put \( S = \{ X \in H_n(L) | AXA = A \} \). Since L is a complete semilattice-ordered incline, we can define the sum \( G = \sum_{X \in S} X \in H_n(L) \). Then \( AGA = A \) (\( \sum_{X \in S} X \)) A = \( \sum_{X \in S} AXA = A \). Hence G is the greatest generalized inverse of A in Hn(L).

Since A(GAG)A = (AGA)GA = AGA = A, we obtain \( GAG \leq G \). Hence \( G \geq GAG \geq G PQ \) for each \( P \in S_A \) and \( Q \in S_G \). But \( PQ \in G_n(L) \) and so \( G = GPQ = GAG \) by Lemma 2.3. Therefore, G is also the greatest semiinverse of A in Hn(L).
Now consider any semiinverse \( H \) of \( A \) in \( H_n(L) \). Since \( H \leq G \), for any \( P \in S_H \) and \( Q \in S_A \), we have that \( G = GAG \geq HAH \geq PAP = PAGAP \geq (PQ)G(QP) \). Since \( PQ, QP \in G_n(L) \), by Lemma 2.3, \( G = (PQ)G(QP) = HAH = H \), i.e., \( G = H \). This shows that \( A \) has the unique semiinverse in \( H_n(L) \). Therefore, the conclusion follows from Theorem 3.2. □

Theorems 3.1, 3.2 and 3.4 generalize Theorem 2.3 in Cho [2], Theorem 2.2 in Cho [3] and Theorem 3.4 in Tan [12]. Theorems 3.2–3.4 generalize and develop Theorem 3.3 in Cho [2], Theorem 2.5 in Cho [3] and Theorem 3.6 in Tan [12].

4. Green’s relations on \( H_n(L) \)

In this section, we characterize Green’s relations on the semigroup of incline Hall matrices.

Let \( S \) be an arbitrary monoid. We write that

\[
\mathcal{L} = \{(a, b) \in S \times S | Sa = Sb\},
\]
\[
\mathcal{R} = \{(a, b) \in S \times S | aS = bS\},
\]
\[
\mathcal{J} = \{(a, b) \in S \times S | SaS = SbS\}
\]

and \( \mathcal{D} = \mathcal{L} \cap \mathcal{R} \). Then \( \mathcal{L}, \mathcal{R}, \mathcal{J} \) and \( \mathcal{D} \) are equivalence relations on \( S \). Furthermore, since \( \mathcal{L} \) and \( \mathcal{R} \) commute, the relation \( \mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} \) is also an equivalence relation on \( S \). These are called Green’s relations on \( S \). Since \( \mathcal{L} \subseteq \mathcal{J} \) and \( \mathcal{R} \subseteq \mathcal{J} \), we have \( \mathcal{D} \subseteq \mathcal{J} \). In general, \( \mathcal{D} \neq \mathcal{J} \). For Green’s relations on general semigroup, see Clifford et al. [4].

From the definition of Green’s relations, we have the following.

**Lemma 4.1.** Let \( S \) be a monoid and \( a, b \in S \). Then the following hold:

1. \( a \mathcal{L} b \) if and only if \( xa = b \) and \( yb = a \) for some \( x, y \in S \),
2. \( a \mathcal{R} b \) if and only if \( ax = b \) and \( by = a \) for some \( x, y \in S \),
3. \( a \mathcal{J} b \) if and only if \( xay = b \) and \( zb = a \) for some \( x, y, z, w \in S \).

**Theorem 4.1.** Let \( A, B \in H_n(L) \). Then the following hold:

1. \( A \mathcal{L} B \) if and only if \( A = PB \) for some \( P \in G_n(L) \),
2. \( A \mathcal{R} B \) if and only if \( A = BQ \) for some \( Q \in G_n(L) \),
3. \( A \mathcal{D} B \) if and only if \( A = PBQ \) for some \( P, Q \in G_n(L) \).

**Proof.** (1) If \( A \mathcal{L} B \), then \( A = YB \) and \( B = XA \) for some \( X, Y \in H_n(L) \) by (1) of Lemma 4.1. For any \( P \in S_Y \) and \( Q \in S_X \), we have that \( A = YB \geq PB = PXA \geq PQA \). Hence \( A = PQA = PB \) by Lemma 2.3. Conversely, if \( A = PB \) for some
$P \in G_n(L)$, then $P^2 = P^{-1} \in G_n(L)$ by Lemma 2.1 and $B = P^2 A$. Hence $A \not\preceq B$ follows from (1) of Lemma 4.1.

(2) It is verified similarly to (1).

(3) It follows (1),(2) and $\mathcal{D} = \mathcal{D} \circ \mathcal{R}$. □


Theorem 4.2. Let $A, B \in H_n(L)$. Then the following hold:

1. $A \mathcal{H} B$ if and only if $PB = A = BQ$ for some $P, Q \in G_n(L)$,
2. $A \not\preceq B$.

Proof. (1) It follows from (1) and (2) of Theorem 4.1 and the fact that $\mathcal{H} = \mathcal{D} \cap \mathcal{R}$.

(2) It suffices to verify the necessity. If $A \not\preceq B$, then $A = XBY$ and $B = ZAW$ for some $X, Y, Z, W \in H_n(L)$ by (3) of Lemma 4.1. For any $P \in S_X, Q \in S_Y, R \in S_Z$ and $S \in S_W$, we have that $A = XBY \geq PBQ = PZAWQ \geq (PR)A(SQ)$ and $PR, SQ \in G_n(L)$. Hence $A = (PR)A(SQ) = PBQ$ by Lemma 2.3. So $A \not\preceq B$ by (3) of Theorem 4.1. □

Theorem 4.2 generalizes Corollary 4.3 in Tan [12].

5. Sandwich semigroups of incline Hall matrices

In this section, as an application of the preceding sections, we study the sandwich semigroups of incline Hall matrices.

Definition 5.1. Let $A \in H_n(L)$. We denote by $H_n(A)$ the semigroup $(H_n(L), \ast_A)$ such that

$$X \ast_A Y := XAY \quad \text{for all } X, Y \in H_n(L).$$

It is evident that $H_n(A)$ forms a partially ordered semigroup.

Theorem 5.1. Let $A \in H_n(L)$ and $B = PAQ$ for some $P, Q \in G_n(L)$. Then the mapping

$$f : H_n(A) \rightarrow H_n(B); \ X \mapsto Q^T XP^T$$

has the following properties:

1. $f$ is an isomorphism of $H_n(A)$ to $H_n(B)$ in viewpoint of partially ordered semigroups,
2. $f(\text{adj}(A)) = \text{adj}(B)$. 
Proof. (1) Obviously, $f$ is a bijective mapping. Let $X, Y \in H_n(A)$. Then $X \preceq Y$ if and only if $f(X) \preceq f(Y)$. Also, we have that

$$f(X *_A Y) = f(XAY) = Q^T(XAY)P^T$$
$$= (Q^TXP^T)(PAQ)(Q^TP^T)$$
$$= f(X)B f(Y) = f(X) B f(Y).$$

Thus $f$ is an isomorphism in viewpoint of partially ordered semigroups.

(2) By Lemma 2.8, we have that

$$f(\text{adj}(A)) = Q^T\text{adj}(A)P^T = \text{adj}(PAQ) = f(\text{adj}(B)).$$

This completes the proof. □

Lemma 5.1. If $A \in H_n(L)$, then $\text{adj}(A)$ is an idempotent matrix in $H_n(A)$.

Proof. By Lemma 2.10 and Theorem 5.1, it suffices to consider the case that $A$ is a reflexive matrix in $H_n(L)$. By (3) and (2) of Lemma 2.7, we have that

$$\text{adj}(A)*_A \text{adj}(A) = \text{adj}(A)\text{adj}(A) = \text{adj}(A).$$

This completes the proof. □

A subset $M$ of a semigroup $S$ is called a submonoid of $S$ if $M$ is a subsemigroup of $S$ with an identity element.

Theorem 5.2. Let $L$ be a complete semilattice-ordered incline. If $A \in H_n(L)$, then the following hold:

1. $\text{adj}(A)$ is the smallest idempotent matrix in $H_n(A)$.
2. $D *_A \text{adj}(A) = \text{adj}(A) *_A D = D$ for any idempotent matrix $D$ in $H_n(A)$.
3. $\{H \in H_n(A) | H *_A \text{adj}(A) = \text{adj}(A) *_A H = H\}$ is the greatest submonoid of $H_n(A)$.

Proof. By Lemma 2.10 and Theorem 5.1, we will assume that $A$ is a reflexive matrix. Put $D_n(A) := \{D \in H_n(A) | D *_A D = D\}$.

(1) By Lemma 5.1, we see that $\text{adj}(A) \in D_n(A)$. Let $D$ be any element of $D_n(A)$. Then $\text{adj}(A) *_A D = \text{adj}(A) *_A D = D$ and so $D$ is regular in $H_n(L)$ and $A$ is a generalized inverse of $D$ in $H_n(L)$. By the proof of Theorem 3.4, $\text{adj}(D)$ is the greatest generalized inverse of $D$ in $H_n(L)$. This means that $A \preceq \text{adj}(D)$. Hence $\text{adj}(A) \preceq \text{adj}(\text{adj}(D)) = D$ by Theorem 3.3. Thus $\text{adj}(A)$ is the smallest idempotent matrix in $H_n(A)$.

(2) For any $D \in D_n(A)$, $\text{adj}(A) \preceq D$ by (1). Since $I \preceq \text{adj}(A)$, we have that $D \preceq D \text{adj}(A) \preceq DAD = D *_A D = D$. Thus $D *_A \text{adj}(A) = D \text{adj}(A) = D$. Similarly, $\text{adj}(A) *_A D = D$.

(3) If $D \in D_n(A)$, then the set $M(D) = \{H \in H_n(A) | H *_A D = D *_A H = H\}$ is a submonoid of $H_n(A)$ with identity element $D$ since
for all $H, T \in M(D)$. It is obvious that $M(D)$ is the greatest submonoid of $H_n(A)$ whose identity element is $D$. Since $\text{adj}(A) \in D_n(A)$, $M = M(\text{adj}(A))$ is a submonoid of $H_n(A)$. Suppose that $W$ is any other submonoid of $H_n(A)$ with the identity element $F$. Then $F * A F = F$, i.e., $F \in D_n(A)$. Since $\text{adj}(A)$ is the smallest element in $D_n(A)$ by (1), $\text{adj}(A) \subseteq F$. Let $H \in W$. Since $\text{adj}(A)$ is reflexive, we have that $H \leq \text{adj}(A) = H * A \text{adj}(A) \leq H * A F = H$. Thus $H * A \text{adj}(A) = H$. Similarly, $\text{adj}(A) * A H = H$. Hence $H \in M$. Therefore, $M$ is the greatest submonoid of $H_n(A)$. □

Theorem 5.2 generalizes Theorem 3.5 in Cho [3] and Theorem 5.2 in Tan [12].

6. Conclusions

The present paper studied the semigroup of Hall matrices over an commutative incline.

From the proofs in this paper, we easily see that Theorems 2.1, 3.1, 4.1, 4.2 and (1) of Theorem 5.1 can be further generalized to the setting of noncommutative inclines.

References