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# Noncommutative gauge field theories: a no-go theorem

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## Abstract

Studying the general structure of the noncommutative (NC) local groups, we prove a no-go theorem for NC gauge theories. According to this theorem, the closure condition of the gauge algebra implies that: (1) the local NC  $u(n)$  algebra only admits the irreducible  $n \times n$  matrix-representation. Hence the gauge fields are in  $n \times n$  matrix form, while the matter fields *can only be* in fundamental, adjoint or singlet states; (2) for any gauge group consisting of several simple-group factors, the matter fields can transform nontrivially under *at most two* NC group factors. In other words, the matter fields cannot carry more than two NC gauge group charges. This no-go theorem imposes strong restrictions on the NC version of the Standard Model and in resolving the standing problem of charge quantization in noncommutative QED.

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## 1. Introduction

During the past two years, there has been a lot of work devoted to the theories formulated on the noncommutative space–time [1]. Apart from the string theory interests, the field theories on noncommutative space–times (Moyal plane) have their own attractions. To obtain a noncommutative version of the action for any given field theory one should replace the usual product of the functions (fields) with the

★-product:

$$\begin{aligned} (f \star g)(x) &= \exp\left\{\frac{i}{2}\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}\right\}f(x)g(y)\Big|_{x=y} \\ &= f(x)g(x) + \frac{i}{2}\theta_{\mu\nu}\partial_\mu f\partial_\nu g + \mathcal{O}(\theta^2), \end{aligned} \quad (1.1)$$

where  $\theta_{\mu\nu} = -\theta_{\nu\mu}$ . In a more mathematical way, the fields of a noncommutative field theory should be chosen from the  $C^*$ -algebra of functions with the above ★-product.

Although in the noncommutative case the Lorentz symmetry is explicitly broken, one can still realize the same representations as in the commutative case where, depending on the space–time (Lorentz) repre-

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sensation, the fields can be scalars, Dirac fields, vector bosons, etc. This can be done noting the notion of “trace” in the corresponding C\*-algebra. This “trace” is basically the integration over the whole space–time, which is already there in the usual definition of the action. Therefore, the notion of “trace” implies that the  $\star$ -product in the quadratic terms of the actions can be removed. In other words, only the interaction terms in the action receive some corrections due to the  $\star$ -product [2–5]. However, in order to quantize the theory, besides the quadratic parts of the action, we should specify the Hilbert space of the theory (or, equivalently, the measure in the path integral quantization), as well as the conjugate momentum of any field. For the space-like noncommutativity,  $\theta_{\mu\nu}\theta^{\mu\nu} > 0$ , this Hilbert space (or the path integral measure) can consistently be chosen the same as in the commutative case [6–8].

The next step in constructing a physical noncommutative model is to develop the concept of local gauge symmetry. Intuitively, because of the inherent non-locality induced by the  $\star$ -product (1.1), the notion of local symmetry in the noncommutative case should be handled with special care. As a result, the pure noncommutative  $U(1)$  theory, which we shall denote by  $U_\star(1)$ , behaves similarly to the usual non-Abelian gauge theories, but now the structure constants depend on the momenta of the fields [4]. We shall discuss this point in more detail later.

However, before turning to more physical questions, one should develop the noncommutative groups underlying the gauge theories, as well as their representations. In general, as discussed in [9,10], it is not trivial to define the noncommutative version of usual simple local groups, as the  $\star$ -product will destroy the closure condition. For example, let  $g_1$  and  $g_2$  be two traceless hermitian  $x$ -dependent  $n \times n$  matrices (elements of the usual  $su(n)$ ). It is very easy to check that  $g_1 \star g_2 - g_2 \star g_1$  is not traceless anymore. Consequently, the only group which admits a minimal noncommutative extension is  $U(n)$  (we denote this extension by  $U_\star(n)$ ). The noncommutative extensions of the other groups are not trivially obtained by the insertion of the  $\star$ -product. However, the noncommutative  $SO$  and  $USp$  algebras have been constructed in a more involved way [9,10].

Besides the simple noncommutative group  $U_\star(n)$  and its representations, we also need to define the

direct product of noncommutative groups, such as  $U_\star(n) \times U_\star(m)$ . In this case, since the group elements are matrix valued functions, in general the usual definition of the direct product of group elements does not work. We will show that this fact imposes strong restrictions on the matter fields (those which are in fundamental representations).

This work is organized as follows. In the next section, reviewing the pure noncommutative Yang–Mills theories, we discuss the gauge invariance issue in more detail and show that the only possible representation for the  $u_\star(n)$  algebra is the one generated through  $n \times n$  hermitian matrices. In Section 3, we consider the matter fields and their gauge invariance. This will lead to the charge quantization for noncommutative QED [11]. Then we proceed with the cases in which our gauge group consists of a direct product of some simple  $U_\star(n)$  factors. We show that group theory considerations (closure condition) will restrict our matter fields to be charged at most under two  $U_\star(n)$  factors of our gauge group. We close this work with discussions and conclusions.

## 2. Pure $U_\star(n)$ gauge theories

To define the pure  $U_\star(n)$  Yang–Mills theory, we start with introducing the  $U_\star(n)$  group and the corresponding algebra. The  $u_\star(n)$  algebra is generated by  $n \times n$  hermitian matrices whose elements (which are complex valued functions) are multiplied by the  $\star$ -product (1.1) [12]. If we denote the usual  $n \times n$   $su(n)$  generators by  $T^a$ ,  $a = 1, 2, \dots, n^2 - 1$ , normalized as  $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ , by adding  $T^0 = \frac{1}{\sqrt{2n}}\mathbf{1}_{n \times n}$  we can cover all  $n \times n$  hermitian matrices<sup>1</sup> [13]. Then any element of  $u_\star(n)$  can be expanded as

$$f = \sum_{A=0}^{n^2-1} f^A(x) T^A, \tag{2.1}$$

<sup>1</sup> We note that the normalization factor  $1/\sqrt{2n}$  for  $T^0$  is chosen conveniently, so that  $d^{ABC} = 2 \text{Tr}(\{T^A, T^B\}T^C)$  is totally symmetric. As it is expected, the renormalizability of the gauge theory does not depend on the relative normalization for the  $su(n)$  and  $u(1)$  generators. We are grateful to L. Bonora and M. Salizzoni for a discussion on this point.

and the  $u_*(n)$  Lie-algebra is defined with the star-matrix bracket:

$$[f, g]_* = f \star g - g \star f, \quad f, g \in u_*(n). \quad (2.2)$$

Evidently, the above bracket closes on the  $u_*(n)$  algebra. For the case of  $n = 1$ , the  $u_*(1)$  case, the above bracket reduces to the so-called Moyal bracket.

The  $U_*(n)$  gauge theory is described by the  $u_*(n)$  valued gauge fields

$$G_\mu = \sum_{A=0}^{n^2-1} G_\mu^A(x) T^A. \quad (2.3)$$

It is straightforward to show that the field strength

$$G_{\mu\nu} = \partial_{[\mu} G_{\nu]} + ig[G_\mu, G_\nu]_*, \quad (2.4)$$

under the infinitesimal  $u_*(n)$  gauge transformations:

$$G_\mu \rightarrow G'_\mu = G_\mu + i\partial_\mu \lambda + g[\lambda, G_\mu]_*, \quad (2.5)$$

$$\lambda \in u_*(n),$$

transforms covariantly:

$$G_{\mu\nu} \rightarrow G'_{\mu\nu} = G_{\mu\nu} + ig[\lambda, G_{\mu\nu}]_*. \quad (2.6)$$

To construct the gauge invariant action we need to define a “trace” in the  $C^*$ -algebra of the functions (elements of  $n \times n$  matrices). It can be shown that the integration over the space-time can play the role of this trace; it enjoys the cyclic permutation symmetry and can be normalized. Hence, the action we are looking for is

$$S = -\frac{1}{4\pi} \int d^D x \operatorname{Tr}(G_{\mu\nu} \star G^{\mu\nu}), \quad (2.7)$$

where the trace is taken over the  $n \times n$  matrices.

The first peculiar feature of the pure  $U_*(n)$  gauge theory we would like to mention here is that, fixing the number of gauge field degrees of freedom (which is  $n^2$ ) the dimension of the matrix representation is automatically fixed, i.e., the gauge fields must be in the  $n \times n$  matrix form. This is a specific property dictated by noncommutativity and in particular the fact that the algebra bracket (2.2) also involves the  $\star$ -product. To make it clear, let us consider a particular example of  $u_*(2)$  and take the  $3 \times 3$  representation for the matrix part, which we denote by  $\Sigma^i$ ,  $i = 1, 2, 3$ , and  $\mathbf{1}_{3 \times 3}$ . It is easy to see that in order to close the algebra with the star-matrix bracket (2.2), in fact besides the  $\Sigma^i$ 's

we need all the other six  $3 \times 3$  hermitian matrices. Therefore, the algebra is not  $u_*(2)$  anymore (it is what we call  $u_*(3)$ ). The above argument for  $u_*(2)$  can be generalized to the  $u_*(n)$  case. Let us start with an irreducible  $N \times N$  representation ( $N \geq n$ ). The enveloping algebra of  $u(n)$  for this representation closes in  $u(N)$  (and not  $u(n)$ ), unless  $N = n$  or otherwise our representation is reducible. Therefore, this irreducible  $N \times N$  representation ( $N > n$ ) is not forming a proper basis for  $u_*(n)$  gauge fields.

The finite  $U_*(n)$  gauge transformations are generated by the elements of the group (in the adjoint representation) which are obtained by star-exponentiation of the elements of the algebra:

$$U = (e_\star)^{i\lambda} = 1 + i\lambda - \frac{1}{2}\lambda \star \lambda \dots, \quad (2.8)$$

$$U \in U_*(n).$$

Then under finite gauge transformations  $G_\mu$  should transform as

$$G_\mu \rightarrow G'_\mu = U \star G_\mu \star U^{-1} + \frac{i}{g} U \star \partial_\mu U^{-1}. \quad (2.9)$$

It can be easily checked that  $G_{\mu\nu} \rightarrow U \star G_{\mu\nu} \star U^{-1}$  and hence the action (2.7) remains invariant.

### 3. Matter fields

Now that we have introduced the pure  $U_*(n)$  gauge theory and the adjoint representation of  $U_*(n)$  group we are ready to add the matter fields, which are in the fundamental representation of the group. Hence, if we denote the matter fields by  $\psi$ , under gauge transformations [11]

$$\psi \rightarrow \psi' = U \star \psi. \quad (3.1)$$

Of course, the anti-fundamental representation is also possible:

$$\chi \rightarrow \chi' = \chi \star U^{-1}. \quad (3.2)$$

For the fermionic (Dirac) matter fields, it is straightforward to show that the action for the  $\psi$ -field, defined as [11]

$$S_f = \int d^D x \bar{\psi} \gamma^\mu D_\mu \star \psi, \quad (3.3)$$

$$D_\mu = \partial_\mu + igG_\mu,$$

is invariant under the above gauge transformations. We also note that  $\psi$  and the anti-fundamental matter field,  $\chi$ , are related by the noncommutative version of charge conjugation [14].

### 3.1. Charge quantization

Before proceeding with the more complicated gauge groups we would like to point out a peculiar property of the  $U_\star(1)$  theory with matter fields, which may be called NCQED. It is well known that in the non-Abelian gauge theories the corresponding “charge” is fixed by specifying the representation of the fields (like the  $SU(2)$  weak charges in the usual electro-weak Standard Model). The noncommutative  $U_\star(1)$  theory in many aspects behaves like a non-Abelian gauge theory whose group structure constants depend on the momenta of the particles [2,4,5]. So, one expects to see the charge quantization emerging also in the NCQED. In fact this has been shown by Hayakawa [11]: the noncommutative fermions can carry charge +1 for  $\psi$ -type fields, -1 for  $\chi$ -type fields and zero for  $\phi$ -type fields ( $\phi \rightarrow \phi' = U \star \phi \star U^{-1}$ ) [15,16]. We would like to mention that the latter ( $\phi$ -type field), although is not carrying any  $U_\star(1)$  charge, similarly to noncommutative photons, carries the corresponding *dipole* moment [4,14,16].

### 3.2. The case with more than one group factor

So far we have only discussed the gauge groups which were consisting of a simple noncommutative group  $U_\star(n)$ . However, for building a physical model, it is necessary and important to consider noncommutative groups which are semi-simple, i.e., composed of some simple  $U_\star(n)$  group factors. A study on this point with similar partial results has been previously performed in [17]. To study these cases we have to develop the direct product of groups in the noncommutative case. In order to show the obstacle, let us first review the direct product of groups in the commutative case. Let  $G_1$  and  $G_2$  be two local gauge groups. Then, the group  $G = G_1 \times G_2$  is defined through the relations:

$$g = g_1 \times g_2, \quad g' = g'_1 \times g'_2, \\ g_i, g'_i \in G_i, \quad g, g' \in G,$$

$$g \cdot g' = (g_1 \times g_2) \cdot (g'_1 \times g'_2) \\ \equiv (g_1 \cdot g'_1) \times (g_2 \cdot g'_2), \tag{3.4}$$

where the “.” corresponds to the relevant group multiplication. Now, let us turn to the noncommutative case and consider  $G_1 = U_\star(n)$  and  $G_2 = U_\star(m)$ . Since both the  $U_\star(n)$  and  $U_\star(m)$  products, besides the matrix multiplication also involve the  $\star$ -product, one cannot re-arrange the group elements and therefore it is not possible to generalize Eq. (3.4) to the noncommutative case. As a consequence of the above argument we cannot have matter fields which are in fundamental representation of both  $U_\star(n)$  and  $U_\star(m)$  factors. However, still we have another possibility left: a matter field,  $\Psi$ , can be in the fundamental representation of one group (e.g.,  $U_\star(n)$ ) and anti-fundamental representation of the other, i.e.,

$$\Psi \rightarrow \Psi' = U \star \Psi \star V^{-1}, \\ U \in U_\star(n), \quad V \in U_\star(m). \tag{3.5}$$

For the most general case where the gauge group contains  $N$   $U_\star(n_i)$  factors,  $G = \prod_{i=1}^N U_\star(n_i)$ , the matter fields can at most be charged under two of the  $U_\star(n_i)$  factors, while they must be singlets under the rest of them. Hence, we have  $N$  types of matter fields which are charged only under one  $U_\star(n_i)$  factor and  $\frac{1}{2}N(N - 1)$  types of them which are carrying two different charges. Therefore, altogether we can have  $\frac{1}{2}N(N + 1)$  kinds of matter fields.

We would also like to make two other remarks:

- (i) there are also  $N$  different  $\phi$ -type fields, which only carry dipole moments under each group factors and no net charges;
- (ii) for the gauge bosons we do not face any further problem when the gauge group has more than one simple  $U_\star(n)$  factor. This is because the gauge fields are always carrying *only* one type of charge (or/and dipole moment), i.e., they are singlets under the remaining group factors.

## 4. Discussion

In this Letter, elaborating more on the structure of noncommutative local groups, we have uncovered some facts about these groups and their representations. We show that the closure condition on the repre-

representations of  $u_*(n)$  algebra restricts these representations only to the one realized through  $n \times n$  hermitian matrices, i.e., higher irreducible representations for the  $u_*(n)$  algebra do not exist.

We have discussed that the concept of a direct product of local gauge groups in the noncommutative case cannot be obtained by a simple generalization of the commutative case. Therefore, the matter fields (which are, in general, in the fundamental representation of the group(s)) cannot carry more than two different charges. More explicitly, the matter fields are either non-singlet under *only* one single  $U_*(n)$  factor of the semi-simple gauge group or they are in fundamental representation of one factor, while in anti-fundamental representation of another factor, as indicated in (3.5).

Although in our group theoretical arguments we have considered the  $\star$ -product (1.1), our discussions and results are independent of the specific form of the space-time noncommutativity and hold for a general noncommutative product between functions.

We would like to note that  $U_*(n)$  has various interesting subgroups. As some examples, one can define  $O_*(n)$  and  $Usp_*(n)$  as subgroups of  $U_*(n)$  [10]. As for other examples, it is straightforward to check that for  $\lambda \in u_*(n)$ ,  $\text{Tr} \lambda$  forms a  $u_*(1)$  subalgebra of  $u_*(n)$ . Along the same line, star-exponentiation of  $\text{Tr} \lambda$  defines a  $U_*(1)$  sub-group.

It is important that this  $U_*(1)$  subgroup can be extended to a nontrivial invariant subgroup of  $U_*(n)$ . Its physical significance is essential: besides the matter fields studied so far, one can consider new fields which are charged only under this subgroup and not the whole  $U_*(n)$ . In fact it turns out that such fields are indeed necessary for building physical noncommutative models [18].

In the present work we have mostly focused on the general properties of the noncommutative gauge theories. As for concrete physical models, the construction of a realistic noncommutative version of the Standard Model, i.e., the  $SU_c(3) \times SU_L(2) \times U(1)$  gauge theory together with its specific matter content, which has not the problem of charge  $e = 0, \pm 1$  quantization, can be the main goal. Such a theory can be constructed uniquely thanks to the present no-go theorem [18]. Furthermore as a direct consequence of our no-go theorem, since in the construction of grand unified theories (GUTs) the matter fields should take representa-

tions of the group other than the fundamental or anti-fundamental, the GUTs generally do not admit simple noncommutative extensions.

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## References

- [1] N. Seiberg, E. Witten, JHEP 9909 (1999) 032, hep-th/9908142.
- [2] T. Filk, Phys. Lett. B 376 (1996) 53.
- [3] M. Chaichian, A. Demichev, P. Prešnajder, Nucl. Phys. B 567 (2000) 360, hep-th/9812180.
- [4] M.M. Sheikh-Jabbari, JHEP 9906 (1999) 015, hep-th/9903107.
- [5] C.P. Martin, D. Sanchez-Ruiz, Phys. Rev. Lett. 83 (1999) 476, hep-th/9903077; T. Krajewski, R. Wulkenhaar, Int. J. Mod. Phys. A 15 (2000) 1011, hep-th/9903187.
- [6] N. Seiberg, L. Susskind, N. Toumbas, JHEP 0006 (2000) 044, hep-th/0005015.
- [7] J. Gomis, T. Mehen, Nucl. Phys. B 591 (2000) 265, hep-th/0005129.
- [8] M. Chaichian, A. Demichev, P. Prešnajder, A. Tureanu, hep-th/0007156, Eur. Phys. J. C, in press.
- [9] L. Bonora, M. Schnabl, M.M. Sheikh-Jabbari, A. Tomasiello, Nucl. Phys. B 589 (2000) 461, hep-th/0006091; B. Jurčo, S. Schraml, P. Schupp, J. Wess, Eur. Phys. J. C 17 (2000) 521, hep-th/0006246.
- [10] I. Bars, M.M. Sheikh-Jabbari, M.A. Vasiliev, hep-th/0103209, Phys. Rev. D, in press.
- [11] M. Hayakawa, hep-th/9912167; M. Hayakawa, Phys. Lett. B 478 (2000) 394, hep-th/9912094.
- [12] A. Armoni, Nucl. Phys. B 593 (2001) 229, hep-th/0005208.
- [13] L. Bonora, M. Salizzoni, Phys. Lett. B 504 (2001) 80, hep-th/0011088.
- [14] M.M. Sheikh-Jabbari, Phys. Rev. Lett. 84 (2000) 5265, hep-th/0001167.
- [15] J.M. Gracia-Bondia, C.P. Martin, Phys. Lett. B 479 (2000) 321, hep-th/0002171.
- [16] L. Alvarez-Gaume, J.L.F. Barbon, Int. J. Mod. Phys. A 16 (2001) 1123, hep-th/0006209.
- [17] S. Terashima, Phys. Lett. B 482 (2000) 276, hep-th/0002119.
- [18] M. Chaichian, P. Prešnajder, M.M. Sheikh-Jabbari, A. Tureanu, hep-th/0107055.