Journal of Pure and Applied Algebra 149 (2000) 251–266

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Outer automorphism groups of metabelian groups ☆

Rüdiger Göbel[a,*], Agnes T. Paras[b]

[a] Fachbereich 6, Mathematik und Informatik, Universität Essen, 45117 Essen, Germany

[b] Department of Mathematics, University of the Philippines at Diliman, 1101 Quezon City, Philippines

Received 4 March 1998; received in revised form 21 October 1998

Communicated by A. Blass

Abstract

Recall that the outer automorphism group of a group G, denoted $\operatorname{Out} G$, is the quotient group $\operatorname{Aut} G/\operatorname{Inn} G$. If M is any group, then there exists a torsion-free, metabelian group G with trivial center such that $\operatorname{Out} G \cong M$. This answers a problem in the Kourovka Notebook (Mazurov, Khukhro, Unsolved problems in group theory; the Kourovka Notebook, Russian Academy of Science, Novosibirsk, 1992).

MSC: Primary 08A35; 20C07; 20F29; secondary 04A20; 20K20

1. Introduction

The automorphism groups of metabelian groups have long been the object of study of several authors. In particular, the papers [1–4,23] describe the automorphism groups of finite rank, free metabelian groups.

Let $F(n)$ denote the free group of rank n and $B(n)=F(n)/F(n)''$ the free metabelian group of rank n. If G is a group, define $IA(G)$ to be the normal subgroup of $\operatorname{Aut} G$ which consists of automorphisms inducing the identity on the abelian quotient G/G'. Any $g \in G$ induces an inner automorphism $g^* \in \operatorname{Aut} G$ with $xg^* = x^g = g^{-1}xg$. Clearly $g^* \in IA(G)$ and the normal subgroup $\operatorname{Inn} G = \{g^* \colon g \in G\}$ of $\operatorname{Aut} G$ becomes a subgroup

☆ This work was done at the University of Essen, where the first author was supported by the German–Israeli Foundation for Scientific Research and Development Project No. G-0294-081.06/93, and the second author was supported by the German Academic Exchange Service (DAAD).

* Corresponding author.

E-mail addresses: r.goebel@uni-essen.de (R. Göebel), agnes@math01.cs.upd.edu.ph (A.T. Paras).

0022-4049/00/$ - see front matter
PII: S0022-4049(99)00022-5

of $IA(G)$. Bachmuth used the Magnus representation of free metabelian groups to yield a faithful representation

$$IA(B(n)) \hookrightarrow GL_n(\mathbb{Z}[F(n)/F(n)']) \tag{1}$$

in [1]. Through this representation, he showed that $IA(B(2)) \leq \operatorname{Inn} B(2)$ and any automorphism of $B(2)$ is induced by an automorphism of $F(2)$. In [2,4], it was shown that there exists $\phi \in IA(B(3))$ such that ϕ is not induced by an automorphism of $F(3)$, and $\operatorname{Aut} B(3)$ is not finitely generated. This culminated in [3], where it was shown, using the above identification (1) and results on matrix groups over integral Laurent polynomial rings, that for all $n \geq 4$

$$IA(F(n)) \to IA(B(n)) \to 1$$

is a canonical epimorphism. Thus proving that for all $n \geq 4$, every automorphism of $B(n)$ is induced by one of $F(n)$ and, hence, $\operatorname{Aut} B(n)$ is finitely generated.

Another approach to the problem has been undertaken in [8,10,13,14,24], wherein the question of which groups can be realized as the automorphism groups of metabelian groups is considered.

In [8], it is shown that any group H can be realized as

$$\operatorname{Aut} G/\operatorname{Stab} G \cong H$$

for some torsion-free, nilpotent group G of class 2 (hence metabelian with nontrivial center $Z(G)$), where

$$\operatorname{Stab}(G) = \{\alpha \in \operatorname{Aut}(G)\colon \alpha \text{ induces the identity on } Z(G) \text{ and } G/Z(G)\}.$$

Zalesskii's example, using upper triangular matrices over a ring, of a torsion-free, nilpotent group of rank 3 and class 2 with no outer automorphisms, was adapted, and the group H was realized as the automorphism group of a ring. In this setting, $\operatorname{Inn}(G) \leq \operatorname{Stab}(G)$ and $\operatorname{Stab}(G)$ is abelian. A similar result was arrived at in [10] using the Baer–Lazard theorem, which provides a correspondence between nilpotent groups of class 2 and alternating bilinear maps. This was refined to show that $\operatorname{Stab}(G)/\operatorname{Inn}(G)$ is isomorphic to a direct sum of $|G|$-copies of the cyclic group $\mathbb{Z}/2\mathbb{Z}$ of order 2.

If a group G has trivial center, then G automatically embeds as $\operatorname{Inn} G$ in $\operatorname{Aut} G$. This led the first named author to the question, which can be found in the Kourovka Notebook [22] (Problem 11.26, 11th ed., 1990),

> For which groups H does there exist a metabelian group G with trivial center such that $\operatorname{Out} G = H$?

A classification of all finite, metabelian groups with $\operatorname{Out} G = 1$ was given in [13]. Robinson considered infinite soluble groups with $\operatorname{Out} G = 1$ in [24] using homological methods. In [14], it was shown that a free metabelian group B of rank λ ($3 \leq \lambda < 2^{\aleph_0}$) could be embedded in a torsion-free, metabelian group G with $\operatorname{Out} G = 1$, $|G'| = 2^{\aleph_0}$ and $G/G' \cong B/B'$. Moreover, there exist $2^{2^{\aleph_0}}$ non-isomorphic extensions G for a given B. The extension $G = H \cdot B$ involved the construction of a group H such that $B' < H < \widehat{B'}$ ($\widehat{B'}$ is the p-adic completion of the free abelian group B'), H is B-invariant and $\operatorname{Aut} H \leq \mathbb{Z}[B/B']$. Note that G here is not a semi-direct product. It was also shown that every abelian group and every unique product group can be realized as the outer automorphism group of a metabelian group with trivial center and torsion part isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Recall that a group K is a *unique product group* if, given any two non-empty finite subsets A and B of K, there exists at least one element x of K that has a unique representation in the form $x = ab$ with $a \in A$ and $b \in B$ (see [17], p. 269). Free groups, and more generally, right ordered groups are examples of unique product groups, but groups with non-trivial torsion elements are not. This realization was obtained through a semi-direct product construction applying known facts about endomorphism rings of torsion-free abelian groups.

In this paper, we proceed differently. Rather than using realization theorems of rings as endomorphism rings [5–7], we will carry over methods from this area and apply them directly to non-commutative groups. This way we succeed in proving the following main result.

Theorem. *Every group can be realized as the outer automorphism group of some torsion-free, metabelian group with trivial center.*

The size of the torsion-free, metabelian group can be any cardinal λ with $\lambda = \lambda^{\aleph_0}$ dominating the cardinality of the prescribed group. Hence there is a proper class of such metabelian groups.

The main result of this paper has predecessors for several classes of groups. Proving that a prescribed group M is the outer automorphism group of a group G from a particular class $\mathscr{C}$ of groups depends strongly on $\mathscr{C}$. Generally it can be said that such a proof becomes much more complicated if $\mathscr{C}$ is very restricted. This can be seen in the case when $\mathscr{C}$ is the class of metabelian groups, whereas it is easier to allow arbitrary groups. The reader may want to compare our theorem with [9,11,18,21]. In answering a problem of P. Hall, it was proved in [11] that any countable group is the outer automorphism group of some locally finite p-group. In [9], it was shown that any group is the outer automorphism group of some torsion-free, locally solvable group. Hence our result, using metabelian groups, strengthens [9] substantially. Our theorem can also be viewed as a measure of the complexity of the class of metabelian groups. Transferring the term 'endo-wild' from the representation theory of modules to this setting, it shows that metabelian groups are 'outer-wild'.

As indicated, the construction here must bear similarities to that in [14] in the sense that ideas from abelian group theory, properties of group rings and the Magnus representation of a free metabelian group are applied. But the reader will note that the new construction is based on a combinatorial idea due to Shelah [25], hence the free metabelian groups considered have cardinality at least $2^{\aleph_0}$. This so-called Shelah's Black Box has proved very useful in the investigation of endomorphism rings of torsion-free, abelian groups. It is interesting to note that this prediction principle applies in the context of outer automorphisms of metabelian groups as well as in various other contexts of algebra.

2. Representation of free metabelian groups

We include, for convenience, a special case of the Magnus representation and a corollary in [14].

Suppose F is a non-cyclic free group with basis $\mathscr{A}=\{x_i\colon i\in I\}$. If $g=x_i\in\mathscr{A}$, let

$$s_g=s_i=gF',\qquad a_g=a_i=gF''\quad\text{and}\quad t_g=t_i.$$

Let $\{s_i=x_iF'\colon i\in I\}$ and $\{a_i=x_iF''\colon i\in I\}$ be generators of F/F' and F/F'', respectively. Let $\bigoplus_{i\in I}\mathbb{Z}[F/F']t_i$ be a free $\mathbb{Z}[F/F']$-module of rank $|I|$. The set of matrices

$$\begin{bmatrix} F/F' & 0\\ \bigoplus_{i\in I}\mathbb{Z}[F/F']t_i & 1\end{bmatrix}=\left\{\begin{pmatrix} g & 0\\ \sum_{i\in I} r_it_i & 1\end{pmatrix}: g\in F/F',\ \sum_{i\in I} r_it_i\in\bigoplus_{i\in I}\mathbb{Z}[F/F']t_i\right\}$$

forms a group under formal matrix multiplication.

Lemma 2.1 (Fox [12], Magnus [19]). *The map*

$$a_j\to\begin{bmatrix} s_j & 0\\ t_j & 1\end{bmatrix}\quad (j\in I)\tag{2}$$

extends to an injective homomorphism

$$\psi: F/F''\to\begin{bmatrix} F/F' & 0\\ \bigoplus_{i\in I}\mathbb{Z}[F/F']t_i & 1\end{bmatrix}.\tag{3}$$

If B is a metabelian group, then $\bar{B}=B/B'$ acts on B' via conjugation. Hence there exists a homomorphism $\varphi\colon\bar{B}\to\operatorname{Aut}(B')$. This extends to a ring homomorphism

$$\varphi:\mathbb{Z}[\overline{B}]\to\operatorname{End}(B'),\tag{4}$$

and so B' can be viewed as a $\mathbb{Z}[\overline{B}]$-module. When B is free metabelian, the Magnus representation enables us to see that each nonzero element of $\varphi(\mathbb{Z}[\overline{B}])<\operatorname{End}(B')$ is a monomorphism. We express this in terms of modules as

Corollary 2.2 (Göbel and Paras [14]). *Let F be a free group and $B=F/F''$ a free metabelian group. Then* (4) *makes B' a torsion-free $\mathbb{Z}[\overline{B}]$-module.*

Proof. Using the Magnus representation (3), we identify B with $\psi(B)$ and notice that B' embeds in

$$C=\begin{bmatrix} 1 & 0\\ \bigoplus_{i\in I}\mathbb{Z}[F/F']t_i & 1\end{bmatrix}.$$

If $a = \left(\begin{smallmatrix} x & 0 \\ y & 1 \end{smallmatrix}\right) \in B$ and $Z = \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right) \in B'$, then conjugation of Z by a is $Z^a = \left(\begin{smallmatrix} 1 & 0 \\ zx & 1 \end{smallmatrix}\right)$. Moreover, if

$$b = \sum_{j \in I} n_j \overline{\begin{pmatrix} u_j & 0 \\ v_j & 1 \end{pmatrix}} \in \mathbb{Z}[\overline{B}],$$

then $Z^b = \left(\begin{smallmatrix} 1 & 0 \\ z \cdot \sum n_j u_j & 1 \end{smallmatrix}\right)$, where $n_j \in \mathbb{Z}$, $u_j \in F/F'$ and $^- : B \to B/B'$ is the canonical epimorphism. Let $z = \sum_{i \in I} b_i t_i$, where $b_i \in \mathbb{Z}[F/F']$. Since $\mathbb{Z}[F/F']$ is an integral domain (see [16], p. 41),

$$\left(\sum b_i t_i\right) \cdot \left(\sum n_j u_j\right) = 0 \text{ iff for each } i,\ b_i = 0 \quad \text{or} \quad \sum n_j u_j = 0.$$

Hence $Z^b = 1$ implies $Z = 1$ or b acts as the identity on all of B'. If $Z \neq 1$, b is $1 \in \mathbb{Z}[\overline{B}]$ by Lemma 2.1. Thus B' is a torsion-free $\mathbb{Z}[\overline{B}]$-module. □

Identifying B' with a subgroup of $\bigoplus_{i \in I} \mathbb{Z}[\overline{B}]t_i$, we consider the general form of the elements $b_i \in \mathbb{Z}[\overline{B}]$ for $\sum b_i t_i \in B'$. A characterization of the b_i's is given in [1], but for our purposes it is enough to know, for each $b_i = \sum n_g g \in \mathbb{Z}[\overline{B}]$, that $\sum n_g = 0$.

In order to prove the latter statement, we first recall from the preceding proof that the action of $b \in \mathbb{Z}[\overline{B}]$ on $z = \sum b_i t_i \in B'$ is defined by $z^b = \sum b b_i t_i$. Since the commutator equality

$$[uv, w] = [u, w]^v [v, w] \tag{5}$$

holds for elements of any group, it suffices to verify that the free generators a, x of B ($a, x \in \{x_i F'' \colon i \in I\}$) satisfy the last claim. Now

$$[x, a] = (s_a - 1)t_x + (1 - s_x)t_a. \tag{6}$$

In this case the desired coefficients are 1 and -1, and their sum is 0.

We now set up the algebraic preliminaries for constructing metabelian groups with prescribed outer automorphism group M. Let M be a group and T be a set with at least two elements, which we will define to be a tree in the next section. Consider the free metabelian group B_M with free generating set

$$\{m_\tau \colon (m, \tau) \in M \times T\}.$$

The group M acts naturally on B_M via right regular representation, i.e.,

$$m_\tau^x = (m \cdot x)_\tau, \text{ where } (m, \tau) \in M \times T \text{ and } x \in M.$$

Hence $M \subseteq \operatorname{Aut} B_M$ and each $B_\tau = \langle m_\tau \colon m \in M \rangle$ is M-invariant. Since $\operatorname{Inn} B_M$ is a normal subgroup of $\operatorname{Aut} B_M$ and $M \cap \operatorname{Inn} B_M$ is trivial, we have a semi-direct product $\operatorname{Inn} B_M \rtimes M \subseteq \operatorname{Aut} B_M$. Since $\overline{B_M}$ acts on B'_M and B'_M is an M-invariant abelian group, then B'_M is a $\mathbb{Z}[\overline{B_M} \rtimes M]$-module, where $\overline{B_M} = B_M / B'_M$.

From now on, we adopt the following notation.

$S = \mathbb{Z}[\overline{B_M}]$

S^* is the group of multiplicative units of S

$R=\mathbb{Z}[\overline{B_M}\rtimes M]$

Rx is the R-submodule generated by the element x

In contrast to Corollary 2.2, B'_M as an R-module is not torsion-free, e.g., when M has nontrivial torsion elements. However we have the following property.

Proposition 2.3. *B'_M is a faithful R-module.*

Proof. Let τ and μ be distinct elements of T and

$$d=\sum n_{mg}mg\in R \quad (n_{mg}\in\mathbb{Z},\ m\in M,\ g\in\overline{B_M})$$

such that $a^d=1$ for all $a\in B'_M$. In particular, $[e_\tau,e_\mu]^d=1$, where e is the identity element of M. Applying the Magnus representation (3) and the respective actions of $\overline{B_M}$ and M on B'_M, we obtain

$$\begin{aligned}0&=((s_{e_\mu}-1)t_{e_\tau}+(1-s_{e_\tau})t_{e_\mu})^d=\sum_{m\in M,g\in\overline{B_M}}n_{mg}g((s_{m_\mu}-1)t_{m_\tau}+(1-s_{m_\tau})t_{m_\mu})\\&=\sum_{m\in M}\sum_{g\in\overline{B_M}}n_{mg}g(s_{m_\mu}-1)t_{m_\tau}+\sum_{m\in M}\sum_{g\in\overline{B_M}}n_{mg}g(1-s_{m_\tau})t_{m_\mu}.\end{aligned}$$

Thus, $\sum_{g\in\overline{B_M}}n_{mg}g(s_{m_\mu}-1)=0$ for all $m\in M$. Since $\overline{B_M}$ is torsion-free, $\mathbb{Z}[\overline{B_M}]$ is an integral domain (see [16], p. 41). Hence $\sum_{g\in\overline{B_M}}n_{mg}g=0$ for all $m\in M$. The definition of equality in a group ring forces $n_{mg}=0$ for all $m\in M$ and $g\in\overline{B_M}$. Therefore $d=0$. $\square$

3. Prescribing automorphisms

An abelian group A is said to be *p-reduced* if $\bigcap_{n\in\omega}p^nA=0$, for some prime number p. If an abelian group A is p-reduced and torsion-free, we denote its *completion* relative to the p-adic topology by $\widehat{A}$. However, we denote the p-adic completion of the group of integers by J_p. Let B be a free metabelian group. Note that conjugation of elements of B' by $b\in B$ extends uniquely to $\widehat{B'}$. We also call such an extension to a subgroup of $\widehat{B'}$ conjugation by b and do not distinguish these maps. Suppose $B'\le H\le\widehat{B'}$ and H is B-invariant, i.e., closed under conjugation by elements of B. The set $G=H\cdot B$ of elements of the form $h\cdot b$ naturally forms a group under the operation

$$(h\cdot b)(g\cdot c)=hg^{b^{-1}}\cdot bc \quad \text{where } h,g\in H,\ b,c\in B.$$

Note however that representation of elements of G in the form $h\cdot b$ is not unique, i.e., G is not a semi-direct product.

Given B_M, as defined after Corollary 2.2, we set out to construct a group H such that $B'_M<H<\widehat{B'_M}$, H is an R-module, $\operatorname{Aut}H\le R$ and $\operatorname{Out}(H\cdot B_M)\cong M$. The construction of a group H with $\operatorname{Aut}H\le R$ finds its motivation in abelian group theory, where a given torsion-free ring is realized as the endomorphism ring of a torsion-free abelian group (see [5,6]). We apply a combinatorial principle called Shelah's Black Box to obtain the described group H. This combinatorial foundation is adapted from [5] to fit the representation of the derived subgroup of a free metabelian group in Section 2.

From now on, let M be a group and λ be a cardinal such that $\lambda^{\aleph_0} = \lambda$ and $|M| \leq \lambda$. It follows from König's Theorem (see [15], p. 45) that $cf(\lambda) > \omega$.

Definition 3.1. Define the *tree* $T = {}^{\omega>}\lambda$ to be the set of all functions

$$\tau\colon n \to \lambda \quad (n < \omega)$$

ordered by set-theoretical containment, i.e., $\sigma \leq \tau$ if and only if $\sigma \subseteq \tau$. The *length* of an element $\tau \in T$ is defined to be the natural number $l(\tau) = \mathrm{dom}(\tau)$. Let B_M be the free metabelian group with free generating set $\{m_\tau\colon (m,\tau) \in M \times T\}$.

Let x^φ denote the image of the element x under the homomorphism φ, and let

$$\langle A\rangle_* = \{x \in \widehat{B'_M}\colon x^n \in A \text{ for some non-zero } n \in \mathbb{Z}\}$$

be the *pure subgroup of* $\widehat{B'_M}$ *generated by* A, for some $A \leq \widehat{B'_M}$.

The group ring S is a unique factorization domain (see [16], p. 106) and canonically a subring of R. By the mapping (3) in Lemma 2.1, we identify the derived group B'_M of B_M with a submodule of the free S-module

$$B'_M \hookrightarrow \bigoplus_{(m,\tau)\in M\times T} St_{m_\tau}. \tag{7}$$

As in Section 2, for each $\tau \in T$, let B_τ be the free metabelian group with free generating set $\{m_\tau\colon m \in M\}$ and $\langle B'_\tau\rangle_R$ be the R-submodule generated by B'_τ. Note that $\langle B'_\tau\rangle_R$ is pure, and $\langle B'_\tau\rangle_R = \bigoplus_{m\in M} St_{m_\tau} \cap B'_M$ since B'_τ is M-invariant. From the definition of the p-adic completion, the elements of $\widehat{B'_M}$ may be identified with sums

$$g = \sum_{(m,\tau)\in M\times T} g_{m_\tau} t_{m_\tau}, \tag{8}$$

where $g_{m_\tau} \in \widehat{S}$, with the property that, for each $n \in \mathbb{N}$, $g_{m_\tau} \in p^n\widehat{S}$ for almost all $(m,\tau) \in M \times T$.

The T-*support*, or simply the *support* of the element g of the form (8) is the set

$$[g] = \left\{\tau \in T\colon \sum_{m\in M} g_{m_\tau} t_{m_\tau} \neq 0\right\},$$

which is always at most countable; for a subset X of $\widehat{B'_M}$, we write $[X] = \bigcup_{g\in X}[g]$. We define the notion of a *norm* on $\widehat{B'_M}$, by fixing a continuous, strictly increasing function $\rho\colon cf(\lambda)+1 \to \lambda+1$ such that $\rho(cf(\lambda)) = \lambda$. Then norms of elements and subsets of $\widehat{B'_M}$ are defined by

$$||g|| = \min\{\nu \subseteq cf(\lambda)\colon [g] \subseteq {}^{\omega>}\rho(\nu)\},\ ||X|| = \sup\{||g||\colon g \in X\}.$$

Definition 3.2. For a subset X of T and an ordinal $v < cf(\lambda)$, define the *part of X to the right of v* to be

$$_{v}X = \{\tau \in X: ||\tau|| > v\}.$$

This notation will be applied to supports of elements described in the next definition.

Definition 3.3. For $n < \omega$, an *n-chain* is a sequence $(x^k)_{k \geq n}$ indexed by the natural numbers $k \geq n$, with the property that there exists an ordinal $v < ||x^n||$ such that for all $k \geq n$,

$$x^k \in \widehat{B'_M}, \quad x^k - px^{k+1} \in B'_M, \quad {}_v[x^k] \subseteq [x^n].$$

Lemma 3.4. *Every element of $\widehat{B'_M}$ extends to an n-chain.*

Proof. Since B'_M is free abelian, there exists a set E of free generators of B'_M and $\widehat{\bigoplus_{d \in E} \mathbb{Z}d} = \widehat{B'_M}$.

Let $g \in \widehat{B'_M}$ and $g = g^n$ $(n < \omega)$. In order to extend g^n to an n-chain, it suffices to define g^{n+1}. By the preceding observation and the definition of the p-adic completion, we can represent g^n as $g^n = \sum_{d \in E} \eta_d d$, where, $\eta_d \in J_p$ and, given $k \in \mathbb{N}$, $\eta_d \in p^k J_p$ for almost all $d \in E$. Let $E_0 = \{d \in E : \eta_d \notin pJ_p\}$, which is a finite set. Let $E_1 = E \setminus E_0$. For each $d \in E_1$, there exists a unique $\xi_d \in J_p$ such that $\eta_d = p\xi_d$. Define $g^{n+1} = \sum_{d \in E_1} \xi_d d$. Clearly $g^{n+1} \in \widehat{B'_M}$, $g^n - pg^{n+1} = \sum_{d \in E_0} \eta_d d \in B'_M$ and $[g^{n+1}] \subseteq [g^n]$. $\square$

A *branch* v of T is a linearly ordered sequence $v = \{v_n \in T: n \in \omega\}$ with $l(v_n) = n$ for all $n \in \omega$. We also identify v with a map $v : \omega \to \lambda$. Note that $v_n = v \upharpoonright n$. The set of all branches of T contained in a subset X of T will be denoted by $\mathrm{Br}(X)$.

The next definition provides a sequence of elements from $\prod_{(m,\tau) \in M \times T} St_{m_\tau}$ and is analogous to the one constructed in [5], where the module under consideration is a full direct sum. In contrast, we consider here B'_M, which is a proper submodule of the direct sum $\bigoplus_{(m,\tau) \in M \times T} St_{m_\tau}$ since it does not contain any t_{m_τ} (cf. [5], p. 453). Hence the components of the proper submodule have no distinguished elements like a ring identity to be used in the definition of a branch element. In order for the sequence of elements to belong to $\widehat{B'_M}$, we make the following modification.

Definition 3.5. Given a branch $v \in \mathrm{Br}(T)$, $k < \omega$, define

$$v^k = \sum_{v_i \in v, i \geq k} p^{i-k} [e_{v_i}, e_{v_{i+1}}],$$

where e is the identity element of M, e_{v_i} is the corresponding element in B_{v_i} and $[e_{v_i}, e_{v_{i+1}}] = (s_{e_{v_{i+1}}} - 1)t_{e_{v_i}} + (1 - s_{e_{v_i}})t_{e_{v_{i+1}}} \in B'_M$ is given by (6).

The next lemma follows easily from Definition 3.3 and Proposition 2.3.

Lemma 3.6. *The sequence $(v^k)_{k\geq n}$ in Definition* 3.5 *is an n-chain and each v^k is R-torsion-free.*

Proof. Suppose $(v^k)^d = 0$, for some $d = \sum_{m\in M} s_m m \in S[M] = R$. By definition, $v^k = \sum_{i\geq k} b_i t_{e_{v_i}}$, where each $b_i \in S$ is non-zero, and $(v^k)^d = \sum_{i\geq k}\sum_{m\in M} s_m b_i^m t_{m_{v_i}}$. This means $s_m b_i^m t_{m_{v_i}} = 0$, and hence $s_m b_i^m = 0$ for each $i \geq k$ and $m \in M$. Since $s_m, b_i^m \in S$, $b_i^m \neq 0$ and S is an integral domain, it is clear that $d = 0$. $\square$

We shall now apply the Black Box, which we include here for completeness. Let B_M be as in Definition 3.1 and identify each $\tau \in T$ with an arbitrary non-zero element of B'_τ.

Definition 3.7. (i) A *canonical submodule of B'_M* is an R-module of the form $\langle B'_{T_0}\rangle_R$, where $B_{T_0} = \langle B_\tau \colon \tau \in T_0\rangle$ for some countable subset T_0 of T.

(ii) A *trap* is a triple (f, P, φ), where $f : {}^{\omega>}\omega \to {}^{\omega>}\lambda = T$ is a tree embedding, P is a canonical submodule of B'_M and $\varphi \in \operatorname{End}\widehat{P}$ such that the following four conditions are satisfied:

(a) *$Im\, f \subseteq P$*;

(b) $[P] \subseteq P$, and $[P]$ is a subtree of T, i.e., $\sigma \leq \tau$, $\tau \in [P]$ implies $\sigma \in [P]$;

(c) $cf(||P||) = \omega$;

(d) $||v|| = ||P||$ whenever $v \in \mathrm{Br}(\mathrm{Im}\, f)$.

Let $\eta < \lambda$ be an ordinal. A branch $w = w(\eta)$ is said to be a *constant branch* if, $w : \omega \to \{\eta\}$. The norm of the constant branch $w(\eta)$ is a discrete or isolated ordinal. From parts (c) and (d) of the definition of a trap, the norm of each $v \in \mathrm{Br}(\mathrm{Im}\, f)$ is a limit ordinal. Hence $\mathrm{Br}(\mathrm{Im}\, f)$ contains no constant branches.

Theorem 3.8 (The Black Box)**.** *For some ordinal λ^*, there exists a transfinite sequence of traps $(f_\alpha, P_\alpha, \varphi_\alpha)$ $(\alpha < \lambda^*)$ such that, for $\alpha, \beta < \lambda^*$,*(i) $\beta < \alpha \Rightarrow ||P_\beta|| \leq ||P_\alpha||$;

(ii) $\beta \neq \alpha \Rightarrow \mathrm{Br}(\mathrm{Im}\, f_\alpha) \cap \mathrm{Br}(\mathrm{Im}\, f_\beta) = \emptyset$;

(iii) $\beta + 2^{\aleph_0} \leq \alpha \Rightarrow \mathrm{Br}(\mathrm{Im}\, f_\alpha) \cap \mathrm{Br}([P_\beta]) = \emptyset$;

(iv) *for any subset $X \subset \widehat{B'_M}$ with $|X| \leq \aleph_0$, and for any $\varphi \in \operatorname{End}\widehat{B'_M}$, there exists $\alpha < \lambda^*$ such that*

$$X \subseteq \widehat{P_\alpha}, \quad ||X|| < ||P_\alpha||, \quad \varphi \restriction P_\alpha = \varphi_\alpha.$$

A proof of Theorem 3.8 is given in the appendix of [5], which goes back to [25]. The Black Box replaces Jensen's $\diamondsuit$-prediction principle, which follows from $V{=}L$ (the constructible universe), whereas Theorem 3.8 holds in ordinary set theory ZFC. The following application is patterned after [5]. However we consider here a proper S-submodule B'_M instead of the free S-module $\bigoplus_{(m,\tau)\in M\times T} St_{m_\tau}$; and we want an S-module H such that $B'_M < H < \widehat{B'_M}$, $\operatorname{Aut} H \leq R$, and H is at the same time an R-module. We begin with the construction of the desired submodule H.

Choose a transfinite sequence $(f_\alpha, P_\alpha, \varphi_\alpha)_{\alpha<\lambda^*}$ satisfying the conclusion of the Black Box. Let ∞ denote a fixed element not in $\widehat{B'_M}: \infty \notin \widehat{B'_M}$. Recall the notation $R = \mathbb{Z}[\overline{B_M} \rtimes M]$.

Let $\mu \le \lambda^*$ and assume we have found an ascending chain of R–submodules H_α $(\alpha < \mu)$ of $\widehat{B_M}'$ and elements b_β $(\beta + 1 < \mu)$ of $\widehat{B'_M} \cup \{\infty\}$ such that for $\alpha < \mu$

$$(\mathrm{I}_\alpha) \qquad b_\beta \notin H_\alpha(\beta < \alpha).$$

If $\mu = 0$, put

$$(\mathrm{II}_0) \qquad H_0 = B'_M.$$

If μ is a limit ordinal, take

$$(\mathrm{II}_\mu) \qquad H_\mu = \bigcup_{\alpha<\mu} H_\alpha.$$

When $\mu = \alpha + 1$, we have the following cases.

(i) Suppose it is possible to choose a branch $v_\alpha \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$, an element g_α in $\widehat{P_\alpha}$, $H_{\alpha+1}$ and b_α in such a way that $(I_{\alpha+1})$ and each of the following are satisfied:

$$(\mathrm{II}_{\alpha+1}) \qquad H_{\alpha+1} = \langle H_\alpha, Rg_\alpha \rangle_*$$

$$(\mathrm{III}_\alpha) \qquad ||g_\alpha - v_\alpha^1|| < ||v_\alpha||$$

$$(\mathrm{IV}_\alpha) \qquad \text{either (strong version) } b_\alpha = g_\alpha^{\varphi_\alpha},$$

$$\text{or (weak version) } b_\alpha = \infty.$$

We then make a choice, using the strong version of (IV_α), whenever this is possible, and call α *strong*. Otherwise call α *weak*.

(ii) If (i) does not occur, call α *useless* and take $H_{\alpha+1} = H_\alpha$, $g_\alpha = 0$, $b_\alpha = \infty$. Theorem 3.9 shows that this case actually does not occur.

In both cases (I_μ) is clearly satisfied, and the μth step is completed. Therefore the recursion proceeds for all $\mu \le \lambda^*$ and yields a submodule H_{λ^*} satisfying (I_{λ^*}). Clearly,

$$H_{\lambda^*} = \langle B'_M, Rg_\alpha \colon \alpha < \lambda^* \rangle_*.$$

Let $H = H_{\lambda^*}$.

Theorem 3.9. *Let $\alpha < \lambda^*$ and $\nu < ||P_\alpha||$. For each $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$, let $(g_v^k)_{k \ge n}$ be an n-chain such that ${}_\nu[g_v^k - v^k] = \emptyset$. For each $\beta < \alpha$, assume $g_\beta \in \widehat{P_\beta}$ and the existence of $\nu_\beta < ||P_\beta||$ and $v_\beta \in \mathrm{Br}(\mathrm{Im}\, f_\beta)$ such that ${}_{\nu_\beta}[g_\beta - v_\beta^1] = \emptyset$. Then there exists $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ such that if*

$$b_\beta \notin H = \langle B'_M, Rg_\beta \colon \beta < \alpha \rangle_* \quad (\beta < \alpha),$$

then $b_\beta \notin \langle H, Rg_v^n \rangle_* =: H(v)$, $(\beta < \alpha)$.

Proof. Suppose the conclusion does not hold. Then for each $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$, there exists $\beta = \beta(v) < \alpha$ such that $b_\beta \in H(v) \setminus H$ for some $b_\beta \in \widehat{P_\beta}$. That means there exists a non-zero $s \in \mathbb{Z}$ and a non-zero $a = a_v \in R$ (since $b_\beta \notin H$) with $b_\beta^s - g_v^a \in H$ (for simplicity, let $g_v = g_v^n$). Since ${}_v[g_v - v^n] = \emptyset$, we have ${}_v[g_v] =_v [v^n]$. The support ${}_v[v^n]$ must be an infinite subset of v, since $v < ||P_\alpha||$ and $||v|| = ||P_\alpha||$. Now for some $h \in B'_M$, $g_{\beta_i} \in \widehat{P_{\beta_i}}$ and $r_{\beta_i} \in R$,

$$b_\beta^s = h + \sum_{i=1}^{n} g_{\beta_i}^{r_{\beta_i}} + g_v^a;$$

and by the Black Box, ${}_v[v^n] \cap {}_{v_{\beta_i}}[g_{\beta_i}]$ is at most finite. By our assumption on the supports of the g_β's, an infinite subset of v has to be contained in $[b_\beta] \subseteq [P_\beta]$. Since $[P_\beta]$ is a subtree of T, $v \subseteq [P_\beta]$. Hence $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha) \cap \mathrm{Br}([P_\beta])$, and by the Black Box, $\alpha < \beta + 2^{\aleph_0}$. Therefore, if $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$, there exists an ordinal $\beta(v)$, $a_v \in R$, $s(v) \in \mathbb{Z}$ such that $\beta(v) < \alpha < \beta(v) + 2^{\aleph_0}$ and $b_{\beta(v)}^{s(v)} - g_v^{a_v} \in H$. Let β_0 be the least ordinal with $\beta_0 < \alpha < \beta_0 + 2^{\aleph_0}$. Then $\beta_0 \le \beta(v) < \beta_0 + 2^{\aleph_0}$ and so $\beta(v)$ assumes $< 2^{\aleph_0} = |\mathrm{Br}(\mathrm{Im}\, f_\alpha)|$ values. Hence there must exist distinct branches $v, w \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ such that $\beta(v) = \beta(w) = \beta$. It follows that $g_v^{a_v s(w)} - g_w^{a_w s(v)} \in H$. The hypothesis on the supports of the g_β's and condition (ii) of the Black Box implies that an infinite subset of v is contained in w or vice versa. This yields a contradiction since v and w are almost disjoint branches of $\mathrm{Im}\, f_\alpha$. $\square$

Theorem 3.9 shows that no ordinal in the construction of H_{λ^*} is useless. An abelian group A is said to be *cotorsion-free* if it does not contain a copy of $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$ or the group of p-adic integers J_p, for any prime p. Equivalently, $\mathrm{Hom}\,(J_p, A) = 0$ for any prime p. A ring R is called *cotorsion-free* if the additive structure $(R, +)$ is a cotorsion-free group. In particular, all integral group rings are cotorsion-free rings since their additive groups are free abelian.

Theorem 3.10. *Suppose that each g_β is defined as in Theorem 3.9 ($\beta < \alpha < \lambda^*$). Then $H_\alpha = \langle B'_M, Rg_\beta \colon \beta < \alpha \rangle_*$ is cotorsion-free.*

Proof. Suppose that H_α is not cotorsion-free, i.e., there exists a non-zero homomorphism $\varphi : J_p \to H_\alpha$. Let $1^\varphi = g \in H_\alpha \subseteq \widehat{B'_M}$. By continuity, $r^\varphi = g^r$ and $[g^r] \subseteq [g]$ for each $r \in J_p$. Since B'_M is free abelian, it is cotorsion-free. If $g \in B'_M$, then $\varphi \in \mathrm{Hom}(J_p, B'_M) = 0$, since the only elements of H_α with finite support are the elements of B'_M. This contradicts cotorsion-freeness of B'_M. Hence, $g \notin B'_M$. By the definition of H_α, there exists a unique $\delta < \alpha < \lambda^*$ such that $g \in H_{\delta+1} \setminus H_\delta$, where $H_{\delta+1} = \langle B'_M, Rg_\beta : \beta < \delta + 1 \rangle_*$. For some non-zero $s \in \mathbb{Z}$ and some non-zero $a \in R$ (since $g \notin H_\delta$), $g^s - g_\delta^a \in H_\delta$. Since ${}_{v_\delta}[g] = {}_{v_\delta}[g^s] \subseteq v_\delta$ for some $v_\delta < ||g||$, the set $[g]$ cannot contain infinitely many elements of any branch v_β ($\beta > \delta$). Hence $g^r \in H_{\delta+1} \setminus H_\delta$ and $g^{rs_0} - g_\delta^{a_0} \in H_\delta$, for some non-zero $s_0 \in \mathbb{Z}$ and non-zero $a_0 \in R$. Now $g_\delta^{a_0 s} - g_\delta^{a r s_0} = g_\delta^{a_0 s - a r s_0} \in H_\delta$. Its support is contained in some finite union of supports of elements in H_δ and so its intersection with the branch v_δ is finite. Since ${}_{v_\delta}[g_\delta - v_\delta^1] = \emptyset$, $a_0 s = a r s_0 \in R \cap J_p[\overline{B_M} \rtimes M] s_0 = R s_0$. Let $a_0 s = a_1 s_0$ for some non-zero $a_1 \in R$. Then $(g^{rs_0} - g_\delta^{a_0})^s = (g^{rs} - g_\delta^{a_1})^{s_0} \in H_\delta$ and the purity of H_δ imply $g^{rs} - g_\delta^{a_1} \in H_\delta$. By

Lemma 3.6, if $s \in \mathbb{Z}$ is fixed, then, for each $r \in J_p$, the element $a_1 \in R$ satisfying $g^{rs} - g_\delta^{a_1} \in H_\delta$ is uniquely determined. It is easy to check that the map $\psi : J_p \to R$ defined by $\psi(r) = a_1$ is a non-trivial group homomorphism. This contradicts the fact that R is cotorsion-free. Therefore the defined group H_α is cotorsion-free. $\square$

Theorem 3.11. *Let H be a cotorsion-free, pure subgroup of $\widehat{B'_M}$ such that $B'_M \le H < \widehat{B'_M}$ and H is an R-submodule. Let $\varphi \in \operatorname{Aut}(\widehat{B'_M}) \setminus R$. Then there exists a canonical submodule P such that $\widehat{P}^{\varphi s - a} \nsubseteq H$, for all $(a,s) \in R \times (\mathbb{Z} \setminus \{0\})$ such that $s = 1$ or $a \notin sR$.*

Proof. Suppose the conclusion is false. Let P be a canonical submodule such that $P \cap B'_\tau$ and $P \cap B'_\mu$ are non-empty, for some distinct $\tau, \mu \in T$. Then there exist $a \in R$ and non-zero $s \in \mathbb{Z}$ such that $\widehat{P}^{\varphi s - a} \subseteq H$. Since H is cotorsion-free, there exists $x \in \widehat{B'_M}$ such that $x^{\varphi s - a} \notin H$. Let P_0 be a canonical submodule such that $\widehat{P_0} \supseteq \langle P, Rx\rangle_*$. By assumption, there exist $a_1 \in R$ and non-zero $s_1 \in \mathbb{Z}$ such that $\widehat{P_0}^{\varphi s_1 - a_1} \subseteq H$. Since $P \subseteq \widehat{P_0}$, $\widehat{P}^{a s_1 - a_1 s} \subseteq H$. The cotorsion-free property of H and Proposition 2.3 imply $a s_1 = a_1 s$. Hence, $x^{\varphi s_1 - a_1} \in H$ implies $x^{(\varphi s - a)s_1} \in H$. The purity of H implies that $x^{\varphi s - a} \in H$, which is a contradiction. $\square$

Theorem 3.12. *Suppose $\varphi \in \operatorname{Aut}(\widehat{B'_M}) \setminus R$ and $H = \langle B'_M, Rg_\beta : \beta < \alpha\rangle_*$, with each g_β defined as in Theorem 3.9. Then there exists $x \in \widehat{B'_M}$ such that $x^\varphi \notin \langle H, Rx\rangle_*$.*

Proof. Let P be as in Theorem 3.11. Choose an ordinal $\eta < \lambda$ such that

$$\max\{||P||, ||P^\varphi||\} < ||\eta||.$$

Let $w = w(\eta)$ be a constant branch of norm $||w|| = ||\eta||$. If $(w^1)^\varphi \notin \langle H, Rw^1\rangle_*$, then we are done. Otherwise, suppose $(w^1)^{\varphi s - r} \in H$ for some non-zero $s \in \mathbb{Z}$ and $r \in R$. By Theorem 3.11, there exists $z \in \widehat{P}$ such that $z^{\varphi s - r} \notin H$. We claim that $(w^1 + z)^\varphi \notin \langle H, R(w^1 + z)\rangle_*$. Suppose the claim is false. Then $(w^1 + z)^{\varphi s_0 - r_0} \in H$, for some non-zero $s_0 \in \mathbb{Z}$ and $r_0 \in R$. Without loss of generality, $(w^1 + z)^{\varphi s s_0 - r_0} \in H$, i.e., $(w^1)^{\varphi s s_0} - (w^1)^{r_0} + z^{\varphi s s_0 - r_0} \in H$. Since $(w^1)^{\varphi s - r} \in H$, $(w^1)^{r s_0} - (w^1)^{r_0} + z^{\varphi s s_0 - r_0} \in H$. The norm of (w^1) is equal to $||\eta||$ and the norm of $z^{\varphi s s_0 - r_0}$ is less than $||\eta||$. The elements of H do not contain infinite subsets of constant branches in their supports, by definition of H and condition (d) of Definition 3.7. Hence $r s_0 = r_0$ and $z^{\varphi s s_0 - r s_0} \in H$. Since H is pure, $z^{\varphi s - r} \in H$, which is a contradiction. $\square$

Theorem 3.13. *Let $H = H_{\lambda^*} = \langle B'_M, Rg_\beta : \beta < \lambda^*\rangle_*$ as constructed before Theorem 3.9. If $\varphi \in \operatorname{Aut} H$, then $\varphi \in R$.*

Proof. Suppose $\varphi \in \operatorname{Aut} H \setminus R$. From Theorem 3.12, there exists $x \in \widehat{B'_M}$ such that $x^\varphi \notin \langle H, Rx\rangle_*$. By the Black Box, there exists $\alpha < \lambda^*$ such that $x, x^\varphi \in P_\alpha$, $||P_\alpha|| > ||x||, ||x^\varphi||$ and $\varphi_\alpha = \varphi \restriction \widehat{P_\alpha}$. It suffices to show that α is strong, for then $g_\alpha^\varphi = g_\alpha^{\varphi_\alpha} \notin H$. This contradicts the assumption that $\varphi \in \operatorname{Aut} H$.

We now show that α is strong, i.e., there exists g_α in $\widehat{P_\alpha}$ and $v_\alpha \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ such that

$$||g_\alpha - v_\alpha^1|| < ||v_\alpha|| = ||P_\alpha||;\ g_\alpha^{\varphi_\alpha}, g_\beta^{\varphi_\beta} \notin \langle H_\alpha, Rg_\alpha\rangle_* \ (\beta < \alpha).$$

Let $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ be distinct from v_β $(\beta < \alpha)$. We claim that there exists $\varepsilon \in \{0,1\}$ such that $(v^1 + \varepsilon x)^\varphi \notin \langle H_\alpha, R(v^1 + \varepsilon x)\rangle_*$. Suppose otherwise, i.e., $(v^1 + x)^{\varphi s - r}$ and $(v^1)^{\varphi s_0 - r_0}$ are both in H_α for some $r, r_0 \in R$ and non-zero $s, s_0 \in \mathbb{Z}$. Applying s_0 on the first and s on the second term, and subtracting the resulting terms, we obtain $(v^1)^{-rs_0 + r_0 s} + x^{\varphi s s_0 - r s_0} \in H_\alpha$. The norm of v^1 is equal to $||P_\alpha||$, while the norm of $x^{\varphi r_0 s - r s_0}$ is less than $||P_\alpha||$. Since v is distinct from the branches v_β $(\beta < \alpha)$, $(v^1)^{s s_0 - r s_0} = 0$ and so $x^{(\varphi s - r) s_0} \in H_\alpha$. By the purity of H_α, $x^{\varphi s - r} \in H_\alpha < H$, which contradicts $x^\varphi \notin \langle H, Rx\rangle_*$. So $g_\alpha \in \{v^1 + \varepsilon x : \varepsilon = 0, 1\}$, and we clearly have $||g_\alpha - v^1|| < ||v||$. In addition, by Theorem 3.9, there exists a branch $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ such that $g_\beta^{\varphi_\beta} \notin \langle H_\alpha, Rg_\alpha\rangle_*$ $(\beta < \alpha)$. □

4. Outer automorphism groups

Using the group $H = H_{\lambda^*}$ constructed in Section 3, and the free metabelian group B_M from Section 2, we define the extension $G = H \cdot B_M$, which is a torsion-free, metabelian group. In this section we show that $\mathrm{Out}\, G$ is indeed M.

Lemma 4.1 (Göbel and Paras [14]). *Suppose $G = H \cdot B$, where B is free metabelian of rank at least two, $B' \le H < \widehat{B'}$ and H is B-invariant. If A is a normal, abelian subgroup of G, then $A \le H$. Hence H is the largest normal, abelian subgroup of G and so is characteristic in G.*

Proof. We first observe using Corollary 2.2, that if $b \in B$ and $x \in B'$ with $x^b = x$, then either $b \in B'$ or $x = 1$, i.e., conjugation by elements $b \in B \setminus B'$ does not leave non-trivial elements of B' fixed. By the continuity of homomorphisms on H and the B-invariance of H, it follows that conjugation by $b \in B \setminus B'$ does not leave non-trivial elements of H fixed.

Suppose there exists $x \in A$ such that $x = h \cdot b$, $h \in H$ and $b \in B \setminus B'$. Since A is normal, abelian in G, $x^c \in A$ and $x^c x^{-1} = x^{-1} x^c \in A$ for all $c \in B$, i.e., $[c, x^{-1}] = [x, c]$. Hence $[c,x]^{x^{-1}} = [c,x]$. Since $[c,x] \in H$ and $h \in H$, $[c,x]^{b^{-1}} = [c,x]$. Taking $c = b$, we get $[b,h]^b = [b,h]$. This implies $[b,h] = 1$, i.e., $h^b = h$. Since $b \in B \setminus B'$, it follows that $h = 1$. So $x = b \in (B \setminus B') \cap A$ and $b^c \in A$ for all $c \in B$. Since $b^c b^{-1} \in A \cap B'$, $b \cdot b^c b^{-1} = b^c b^{-1} \cdot b = b^c$. This means $(b^c)^{b^{-1}} = b^c$, and so, by our first observation, $b^c = 1$ for all $c \in B$. But this occurs only if $b = 1$, thus giving us a contradiction. □

Since $G/H \cong B/B'$, it follows that $G' \le H$.

Lemma 4.2 (Göbel and Paras [14]). *Suppose $G = H \cdot B$, as defined in Lemma 4.1. Then no automorphism of G induces inversion on H, i.e.,*
if $\varphi \in \mathrm{Aut}(G)$ then $\varphi \restriction H \neq -1 \cdot id_H$.

Proof. Suppose $\varphi \in \mathrm{Aut}(G)$ and $\varphi \restriction H = -1 \cdot id_H$. Let $a \in H$ and $b \in B$. Then $b^{-1}a^{-1}b = (b^{-1}ab)^{\varphi} = (b^{-1})^{\varphi}a^{-1}b^{\varphi}$ implies that $b^{\varphi}b^{-1}$ commutes with every element of H. Hence $b^{\varphi}b^{-1} \in H$. Since b is an arbitrary element of B, this means that φ induces the identity on G/H.

Let $b, c \in B$ and suppose $b^{\varphi} = h \cdot b$, $c^{\varphi} = k \cdot c$, for some $h, k \in H$. Then $b^{\varphi^2} = h^{\varphi} \cdot b^{\varphi} = h^{-1}h \cdot b = b$. Hence $\varphi^2 = id_G$. Now $[b,c]^{\varphi} = [h \cdot b, k \cdot c] = [h,c]^b[b,c][b,k]^c$. Since $[h,c], [b,k] \in [G,H]$, it follows that $[b,c]^{\varphi} = [b,c] \bmod [G,H]$. Hence $[b,c]^2 \in [G,H]$. By commutator calculus (see [20], p. 293), $[b,c]^2 = [b,c^2] \bmod [G,G']$. Since $G' \leq H$, $[b,c]^2 = [b,c^2] \bmod [G,H]$. Thus $[b,c^2] \in [G,H]$. This yields a contradiction when b and c are chosen to be free generators of B. □

Proposition 4.3. *Suppose $G = H \cdot B_M$, where $H = H_{\lambda^*}$.*
If $\varphi \in \mathrm{Aut}\, G$ such that $\varphi \restriction H = id_H$, then $\varphi \in \mathrm{Inn}\, G$.

Proof. Suppose $\varphi \in \mathrm{Aut}\, G$ such that $\varphi \restriction H = id_H$. Then we claim that φ induces $id_{G/H}$. Let $b \in B_M$ and $h \in H$. Now $b^{-1}hb = (b^{-1}hb)^{\varphi} = (b^{-1})^{\varphi}hb^{\varphi}$ implies that $b^{\varphi}b^{-1}$ commutes with every $h \in H$. By Lemma 4.1, $b^{\varphi}b^{-1} \in H$. The claim follows.

Let $b, c \in B_M$, $b^{\varphi} = hb$ and $c^{\varphi} = kc$ for some $h, k \in H$. Then $[hb, kc] = [b,c]^{\varphi} = [b,c]$ implies that $h^{\bar{b}(\bar{c}-1)} = k^{\bar{c}(\bar{b}-1)}$. If $\bar{b} = \bar{c}$, then $h = k$. Without loss of generality, assume $\bar{b} \neq \bar{c}$.

Note that by the action of S on H, the element h has finite support if and only if k has finite support. The construction of H guarantees that an element has finite support if and only if it belongs to B'_M. Suppose h has finite support and $h = \sum \rho_i t_i$, $k = \sum \sigma_i t_i$, for some $\rho_i, \sigma_i \in S$ and $t_i \in \{t_{m_\tau} : (m,\tau) \in M \times T\}$. The action of S on H yields $\sum \bar{b}(\bar{c}-1)\rho_i t_i = \sum \bar{c}(\bar{b}-1)\sigma_i t_i$. So $\bar{b}(\bar{c}-1)\rho_i = \bar{c}(\bar{b}-1)\sigma_i$ for all i. Since S is a unique factorization domain (see [16], p. 106) and $\bar{b} \neq \bar{c}$, then, for each i, $\rho_i = \rho'_i(\bar{b}-1)$ and $\sigma_i = \sigma'_i(\bar{c}-1)$ for some $\rho'_i, \sigma'_i \in S$. If we let $h_0 = \sum \rho'_i t_i$, which is clearly in B'_M, then $h = h_0^{(\bar{b}-1)}$ and $k = h_0^{\bar{b}\bar{c}^{-1}(\bar{c}-1)}$. Thus φ is conjugation by the element h_0^{-b}.

Suppose h has infinite support and n is a non-zero integer such that $h^n, k^n \in \langle B'_M, Rg_\beta : \alpha < \lambda^* \rangle$. Then $h^n = h_0 + \sum g_{\alpha_i}^{a_i}$ and $k^n = k_0 + \sum g_{\alpha_i}^{r_i}$, for some $a_i, r_i \in R$, $h_0, k_0 \in B'_M$ and $g_{\alpha_i} \in \widehat{P_{\alpha_i}}$ chosen such that there exist an ordinal ν_{α_i} and a branch $v_{\alpha_i} \in \mathrm{Br}(\mathrm{Im}\, f_{\alpha_i})$ with ${}_{\nu_{\alpha_i}}[g_{\alpha_i} - v^1_{\alpha_i}] = \emptyset$ and $||P_{\alpha_i}|| \leq ||P_{\alpha_j}||$ for $\alpha_i < \alpha_j$. Since $h^{n\bar{b}(\bar{c}-1)} = k^{n\bar{c}(\bar{b}-1)}$, it follows that $\sum g_{\alpha_i}^{r_i\bar{c}(\bar{b}-1) - a_i\bar{b}(\bar{c}-1)} = h_0^{\bar{b}(\bar{c}-1)} - k_0^{\bar{c}(\bar{b}-1)} \in B'_M$. By the choice of the supports of the g_{α_i}'s,

$$h_0^{\bar{b}(\bar{c}-1)} = k_0^{\bar{c}(\bar{b}-1)} \quad \text{and} \quad r_i\bar{c}(\bar{b}-1) = a_i\bar{b}(\bar{c}-1) \quad \text{for all } i.$$

As was shown in the finite support case, $h_0 = h_1^{(\bar{b}-1)}$ and $k_0 = k_1^{(\bar{c}-1)}$, for some $h_1, k_1 \in B'_M$. Let $r_i = \sum n_{mg} mg$, $a_i = \sum s_{mg} mg$. So $\sum n_{mg} mg\bar{c}(\bar{b}-1) = \sum s_{mg} mg\bar{b}(\bar{c}-1)$ implies $(\sum n_{mg} g)\bar{c}(\bar{b}-1) = (\sum s_{mg} g)\bar{b}(\bar{c}-1)$ for each m. Since all the terms in the latter equation are elements of S and $\bar{b} \neq \bar{c}$, it follows that $r_i = r_i'(\bar{c}-1)$ and $a_i = a_i'(\bar{b}-1)$. We now have $h^n = (h_1 + \sum g_{\alpha_i}^{a_i'})^{(\bar{b}-1)}$ and $k^n = (k_1 + \sum g_{\alpha_i}^{r_i'})^{(\bar{c}-1)}$. Let $x' = h_1 + \sum g_{\alpha_i}^{a_i'}$ and $y' = k_1 + \sum g_{\alpha_i}^{r_i'}$. Clearly $x' \in H$, $h^n = (x')^{(\bar{b}-1)}$ and $k^n = (y')^{\bar{b}\bar{c}^{-1}(\bar{c}-1)}$. Since H is S-torsion-free, the equation $((x')^{(\bar{b}-1)})^{\bar{b}(\bar{c}-1)} = ((y')^{(\bar{c}-1)})^{\bar{c}(\bar{b}-1)}$ implies that $(x')^{\bar{b}} = (y')^{\bar{c}}$. Thus $h^n = (x')^{(\bar{b}-1)}$ and $k^n = (x')^{\bar{b}\bar{c}^{-1}(\bar{c}-1)}$. By the purity of H and the action of S on H, there exists $x \in H$ such that $h = x^{(\bar{b}-1)}$ and $k = x^{\bar{b}\bar{c}^{-1}(\bar{c}-1)}$. Hence $b^{\varphi} = x^{(\bar{b}-1)}b = b^{x^{-b}}$ and $c^{\varphi} = x^{\bar{b}\bar{c}^{-1}(\bar{c}-1)}c = c^{x^{-b}}$. It follows that φ is conjugation by $x^{-b} \in H$. □

An easy consequence of Proposition 4.3 is that two automorphisms of G which agree on H are congruent modulo $\operatorname{Inn} G$.

Theorem 4.4. *Suppose $G = H \cdot B_M$ is defined as in Proposition* 4.3. *Then* $\operatorname{Out} G = M$. *More precisely*, $\operatorname{Aut} G = \operatorname{Inn} G \rtimes M$.

Proof. Let $\varphi \in \operatorname{Aut} G$. By Lemma 4.1, $\varphi \restriction H$ is an automorphism of H; and by Theorem 3.13, $\varphi \restriction H \in R$. Hence the restriction $\varphi \restriction H = u \in R^*$. Let $h \in H$ and $b \in B$. Then $h^{u\bar{b}^{\varphi}} = (b^{-1}hb)^{\varphi} = (b^{-1}hb)^{u} = h^{\bar{b}u}$ for all $h \in H$. By Lemma 2.3, $u\bar{b}^{\varphi} = \bar{b}u$, when viewed as elements of R^*. Let $u = \sum_m \sigma_m m \in S[M] = R$. Without loss of generality, assume $\sigma_e \neq 0$, where e is the identity element in M (otherwise, multiply u by m^{-1} if $\sigma_m \neq 0$). Now $\sum_m \sigma_m m \cdot \bar{b}^{\varphi} = \bar{b} \cdot \sum_m \sigma_m m$, and by the multiplication in a semi-direct product, $\sum_m \sigma_m \bar{b}^{\varphi m^{-1}} m = \sum_m \bar{b}\sigma_m m$. This means $\sigma_m \bar{b}^{\varphi m^{-1}} = \bar{b}\sigma_m$ for all $m \in M$ and $\bar{b} \in \overline{B_M}$. Since S is an integral domain, $\bar{b}^{\varphi m^{-1}} = \bar{b}$ for all $\bar{b} \in \overline{B_M}$ and for all m such that $\sigma_m \neq 0$. In particular, $\sigma_e \neq 0$ and so $\bar{b}^{\varphi} = \bar{b}$ for all $\bar{b} \in \overline{B_M}$, i.e., φ induces the identity on $\overline{B_M}$. Suppose $\sigma_{m_0} \neq 0$ for some $m_0 \neq e$. Then there must exist $\bar{b}_0 \in \overline{B_M}$ such that $\bar{b}_0^{m_0} \neq \bar{b}_0$ (e.g., take $\bar{b}_0 = \bar{e}_\tau$). Hence $(\bar{b}_0^{\varphi})^{m_0^{-1}} \neq \bar{b}_0$ gives a contradiction. This shows that the M-support of u as an element of the group ring $S[M]$ is a singleton, i.e., $u \in S^*$. Thus, in general, $u \in \overline{B_M} \rtimes M$ by Lemma 4.2. Therefore, by Proposition 4.3, $\varphi \in \operatorname{Inn} G \rtimes M$ and $\operatorname{Aut} G \leq \operatorname{Inn} G \rtimes M$. The reverse inclusion is clear by construction. □

Note added in proof

In a recent paper (R. Göbel, A. Paras, Realizing automorphism groups of metabelian groups, to appear in Proceedings of the Dublin Conference on abelian groups and modules 1998, Birkhäuser Verlag, Basel, 1999) we will take care of the countable case and sharpen our main theorem: if M is any group of cardinality $\kappa < 2^{\aleph_0}$, then there is a torsion-free metabelian group G of cardinality $|G| = \max\{\kappa, \aleph_0\}$ with out $G \cong M$.

References

[1] S. Bachmuth, Automorphisms of free metabelian groups, Trans. Amer. Math. Soc. 118 (1965) 93–104.
[2] S. Bachmuth, H.Y. Mochizuki, The non-finite generation of Aut(G), G free metabelian of rank 3, Trans. Amer. Math. Soc. 270 (1982) 693–700.

[3] S. Bachmuth, H.Y. Mochizuki, $\mathrm{Aut}(F) \to \mathrm{Aut}(F/F'')$ is surjective for free group F of rank ≥ 4, Trans. Amer. Math. Soc. 292 (1985) 81–101.
[4] O. Chein, IA-automorphisms of free and free metabelian groups, Comm. Pure Appl. Math. 21 (1968) 605–629.
[5] A.L.S. Corner, R. Göbel, Prescribing endomorphism algebras – A unified treatment, Proc. London Math. Soc. (3) 50 (1985) 447–479.
[6] M. Dugas, R. Göbel, Every cotorsion-free algebra is an endomorphism algebra, Math. Z. 181 (1982) 451–470.
[7] M. Dugas, R. Göbel, Every cotorsion-free ring is an endomorphism ring, Proc. London Math. Soc. (3) 45 (1982) 319–336.
[8] M. Dugas, R. Göbel, Torsion-free nilpotent groups and E-modules, Arch. Math. 54 (1990) 340–351.
[9] M. Dugas, R. Göbel, Outer automorphisms of groups, Illinois J. Math. 35 (1991) 27–46.
[10] M. Dugas, R. Göbel, Automorphisms of torsion-free nilpotent groups of class 2, Trans. Amer. Math. Soc. 332 (1992) 633–646.
[11] M. Dugas, R. Göbel, On locally finite p-groups and a problem of Philip Hall's, J. Algebra 159 (1993) 115–138.
[12] R.H. Fox, Free differential calculus I, Ann. Math. (2) 57 (1953) 547–560.
[13] R.M. Gagen, D.J.S. Robinson, Finite metabelian groups with no outer automorphisms, Arch. Math. 32 (1979) 417–423.
[14] R. Göbel, A.T. Paras, Automorphisms of metabelian groups with trivial center, Illinois J. Math. 42 (1998) 333–346.
[15] T. Jech, Set Theory, Academic Press, New York, 1978.
[16] G. Karpilovsky, Commutative group algebras, Marcel Dekker, New York, 1983.
[17] G. Karpilovsky, Unit groups of classical rings, Clarendon Press, Oxford, 1988.
[18] S. Kojima, Isometry transformation of hyperbolic 3-manifolds, Topology Appl. 29 (1988) 297–307.
[19] W. Magnus, On a theorem of Marshall Hall, Ann. Math. (2) 40 (1939) 764–768.
[20] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory, Wiley-Interscience, New York, 1966.
[21] T. Matumoto, Any group can be represented by an outer automorphism group, Hiroshima Math. J. 19 (1989) 209–219.
[22] V.D. Mazurov, E.I. Khukhro, Unsolved problems in group theory; the Kourovka Notebook, Russian Academy of Science, Novosibirsk, 1992.
[23] H.Y. Mochizuki, Automorphisms of solvable groups, Part II, C.M. Campbell, E.F. Robertson (Eds.), London Math. Soc., vol. 121, Cambridge Univ. Press, Cambridge, 1987, pp. 15–29.
[24] D.J.S. Robinson, Infinite soluble groups with no outer automorphisms, Rend. Sem. Mat. Univ. Padova 62 (1980) 281–294.
[25] S. Shelah, A combinatorial theorem and endomorphism rings of abelian groups II, in: Abelian Groups and Modules, Proc. 1984 Udine Conf., CISM Courses and Lectures, vol. 287, pp. 37–86.