Theoretical Computer Science 412 (2011) 3370–3377

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

The free process algebra generated by δ, ϵ and τ

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ARTICLE INFO

For Jan Bergstra, on the occasion of his sixtieth birthday, in friendship and gratitude

Keywords:
Process algebra
Bisimulation
Heyting algebra
Empty process
Silent step
Infinite sums and products

ABSTRACT

We establish the structure of the initial process algebra with additive and multiplicative identity elements and no article silent step.

1. Introduction

In process algebra, as in arithmetic, there are basic operations $+$ and $\cdot$; both theories also employ a neutral element for $+$, called 0 in arithmetic, and δ in our formulation of process algebra. Apart from their use in streamlining theory, these notions have a concrete interpretation: penury in arithmetic, and perceived impass in process algebra. Now, further streamlining is achieved in arithmetic by the introduction of a neutral element 1 for $\cdot$; and there were heated discussions in the nineteen-eighties on whether process algebra should also have such an element (to be called ϵ). The arguments to the contrary prevailed, at the time. They are twofold. First, there is no 1 in the most general formulation of ring theory either, because you do not want to miss applications in which there is no such identity element. The second argument is peculiar to process algebra: what is a communication with nothing, a process without duration, to be?

In spite of these difficulties, the possibilities of ϵ were investigated, notably in papers by Baeten and Van Glabbeek [2] and by one of us [10]. Moreover, formulations with ϵ abound in the textbook by Baeten and Weijland [4]; and they have yet gained in prominence in Baeten's new book [1]. (Indeed, ϵ and δ are written 1 and 0 there.)

A rather odd result was announced at a meeting in 1985. An exploration of processes without visible actions, constructed from just impass (δ), the silent step τ, and ϵ, indicated that they have the structure of the free Heyting algebra on a single generator. This algebra is a catalogue of the statements constructible from one primitive statement that are distinct in intuitionistic logic; these are infinitely many, in contrast to classical logic where there are only four: p, $\neg p$, *true*, and *false*. It was discovered by Rieger [9], and rediscovered by Nishimura [8] a decade later.

We give a proof for this description of actionless processes below, with some further remarks. The deeper connection between hidden action and intuitionistic truth values remains an open question of philosophy.

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0304-3975/$ – see front matter
doi:10.1016/j.tcs.2011.03.019

2. Preliminaries

We denote sets-with-structure, such as algebras or relational structures, by bold type, e.g. **A**, and the underlying universe by the corresponding italic letter (in the example, A).

The *type* of an algebra **A** is the set of operation symbols (assuming arities fixed) interpreted in **A**. Terms are formed from variables and operation symbols as usual. A *term algebra* is a suitable set of terms with the application of the various operation symbols to form new terms as operations. A term not containing variables will be called a *ground* term.

An *ordering* of a set X is a binary relation on X that is reflexive, antisymmetric and transitive.

The diagonal Δ_X of a set X is the relation $\{\langle x, x\rangle \mid x \in X\}$.

If X is a set, then X^* is the set of finite sequences of elements of X, including the empty sequence.

Process algebras

A process algebra has two binary operations, $\cdot$ and $+$, (sequential composition or product, and alternative composition or sum); and satisfies

$$\begin{array}{ll} x+y=y+x & A1 \\ (x+y)+z=x+(y+z) & A2 \\ x+x=x & A3 \\ (x+y)\cdot z=(x\cdot z)+(y\cdot z) & A4 \\ (x\cdot y)\cdot z=x\cdot(y\cdot z) & A5. \end{array}$$

Product is considered to bind more strongly than sum, and the symbol $\cdot$ is often omitted.

$A1$–$A5$ were introduced by Bergstra and Klop [3] as the laws of *basic* process algebra.

Definition 2.1. Let **P** be a process algebra, and $A \subseteq P$. The *basic processes* of **P** over A are inductively defined by:

1. the elements of A are basic processes;
2. if $a \in A$ and t is a basic process, then at is a basic process;
3. if s and t are basic processes, then $s + t$ is a basic process.

Proposition 2.2. *Every basic process of* **P** *over A can be written in the form*

$$a_0t_0 + \cdots + a_{n-1}t_{n-1} + b_0 + \cdots + b_{k-1} \tag{1}$$

with $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{k-1} \in A$ and $t_0, \ldots, t_{n-1}$ basic processes.

Lemma 2.3. *Products and sums of basic processes are basic processes.*

Proof. If s and t are basic processes, then $s + t$ is a basic process by definition. That st is a basic process, is shown by induction on s, as follows: at is a basic process by clause (ii) of the definition; $(as_1)t = a(s_1t)$ by $A5$, and s_1t is a basic process by induction hypothesis, so $a(s_1t)$ is a basic process by clause (ii) of the definition; and $(s_1 + s_2)t = s_1t + s_2t$ by $A4$, s_1t and s_2t are basic processes by induction hypothesis, so $s_1t + s_2t$ is a basic process by clause (iii) of the definition. □

Proposition 2.4. *Let* **B** *be a process algebra, $A \subseteq B$. The following statements are equivalent:*

1. *all elements of* **B** *are basic processes over A;*
2. **B** *is generated by A.*

Proof. 1. $\Rightarrow$ 2. It is clear from Definition 2.1 that the basic processes are generated by A.
2. $\Rightarrow$ 1. By Definition 2.1 and Lemma 2.3. □

Process structures

Let A be a set. An *A-process structure* is a structure

$$\langle P, \langle R_a \mid a \in A\rangle, R_\epsilon, \surd\rangle$$

with binary relations R_a ($a \in A$) and R_ϵ, and a constant $\surd$. We allow process structures to be *partial*; that is, the constant $\surd$ may be undefined (have empty interpretation). For the rest of this section we fix an A-process structure **P**.

Suppose $\alpha \in A \cup \{\epsilon\}$. We write $p \xrightarrow{\alpha} q$ if $\langle p, q\rangle \in R_\alpha$. For $\sigma = \langle \alpha_1, \ldots, \alpha_n\rangle \in (A \cup \{\epsilon\})^*$ we write $p \overset{\sigma}{\twoheadrightarrow} q$ if $p_0, \ldots, p_n$ exist such that

$$p_0 \xrightarrow{\alpha_1} p_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} p_n$$

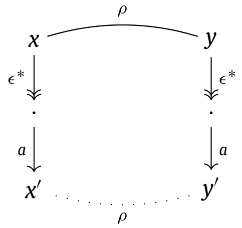
and $p = p_0$, $p_n = q$. We call σ a *trace* from p to q. We write

$$p \twoheadrightarrow q$$

if a trace σ exists such that $p \overset{\sigma}{\twoheadrightarrow} q$. We say that p terminates if $p \twoheadrightarrow \surd$.

Concatenation of two sequences σ_1 and σ_2 will be represented by juxtaposition: $\sigma_1\sigma_2$; coupling of a behind σ as σa. An arbitrary element of $\{\alpha\}^*$ will sometimes be indicated by α^*. Note that ϵ-steps are treated on a par with real actions (elements of A); they are not omitted in traces.

Definition 2.5. (Let **P** be an A-process structure.) A symmetric binary relation ρ on **P** is a *bisimulation* if, whenever $x\rho y$, if $a \neq \epsilon$ and $x \overset{\epsilon^* a}{\twoheadrightarrow} x'$, then $y' \in P$ exists such that $y \overset{\epsilon^* a}{\twoheadrightarrow} y'$ and $x'\rho y'$; and $x \overset{\epsilon^*}{\twoheadrightarrow} \surd$ implies $y \overset{\epsilon^*}{\twoheadrightarrow} \surd$.



We note that the endpoints of ϵ-steps need not be related. We say that p and q are *bisimilar*, notation $p \underline{\leftrightarrow} q$, if there exists a bisimulation ρ such that $p\rho q$.

Let for $x \in P$, $\mathbf{P}_x$ be the relative substructure of **P** with universe

$$\{y \in P \mid x \longrightarrow\!\!\!\!\rightarrow y\},$$

the set of nodes reachable from x. It is evident that $\mathbf{P}_p \cong \mathbf{P}_q$ implies that p and q are bisimilar: an isomorphism is a bisimulation.

The union $\mathbf{X} \cup \mathbf{Y}$ of relative substructures **X** and **Y** of **P** is

$$\langle X \cup Y, \langle R_a^{\mathbf{X}} \cup R_a^{\mathbf{Y}} \mid a \in A\rangle, R_\epsilon^{\mathbf{X}} \cup R_\epsilon^{\mathbf{Y}}, (\surd^{\mathbf{P}})\rangle,$$

in which $\surd$ is interpreted, as in **P**, if and only if $\surd^{\mathbf{P}}$ belongs to X or Y. (One could write the interpretation of $\surd$ as $\surd^{\mathbf{X}} \cup \surd^{\mathbf{Y}}$, a union of partial nullary functions.)

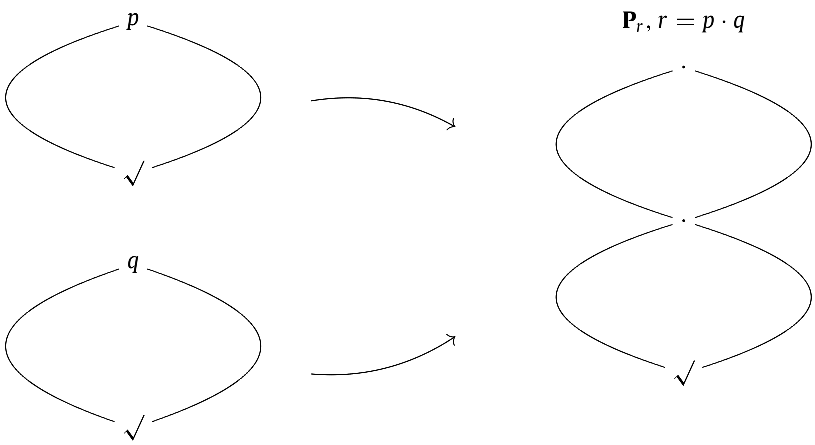
We write $r = p + q$ if $\mathbf{P}_r$ results from extending $\mathbf{P}_p \cup \mathbf{P}_q$ by

$$p \overset{\epsilon}{\longleftarrow} r \overset{\epsilon}{\longrightarrow} q,$$

or more precisely, if $\mathbf{X} = \mathbf{P}_p$ and $\mathbf{Y} = \mathbf{P}_q$ then

$$\mathbf{P}_r = \langle X \cup Y \cup \{r\}, \langle R_a^{\mathbf{X}} \cup R_a^{\mathbf{Y}} \mid a \in A\rangle, R_\epsilon^{\mathbf{X}} \cup R_\epsilon^{\mathbf{Y}} \cup \{\langle r, p\rangle, \langle r, q\rangle\}, \surd^{\mathbf{X}} \cup \surd^{\mathbf{Y}}\rangle,$$

and $r = p \cdot q$ if either p does not terminate and $r = p$, or $\mathbf{P}_r$ is isomorphic with the structure one obtains from the disjoint union of $\mathbf{P}_p$ and $\mathbf{P}_q$ by identifying $\surd^{\mathbf{P}_p}$ and q, and interpreting $\surd$ as $\surd^{\mathbf{P}_q}$. (The diagram below sketches the relation between $\mathbf{P}_p$, $\mathbf{P}_q$ and $\mathbf{P}_r$ in the latter case, with an embedding of $\mathbf{P}_q$ in $\mathbf{P}_r$ and an almost-embedding – $\surd$ is not preserved – of $\mathbf{P}_p$; if $\surd$ is not defined in $\mathbf{P}_q$, we intend it to be not defined in $\mathbf{P}_r$.)



The process structure **P** is *closed under composition* if for all $p, q \in P$, r and s exist such that $r = p + q$ and $s = p \cdot q$.

The diagonal Δ_P is a bisimulation; the inverse of a bisimulation is a bisimulation; and the composite of two bisimulations is a bisimulation. So $\underline{\leftrightarrow}$ is an equivalence relation on P.

Lemma 2.6. *Suppose $p_1 \underline{\leftrightarrow} p_2$ and $q_1 \underline{\leftrightarrow} q_2$. Then*

1. *if $r_i = p_i + q_i$ ($i = 1, 2$), then $r_1 \underline{\leftrightarrow} r_2$;*
2. *if $r_i = p_i \cdot q_i$ ($i = 1, 2$), then $r_1 \underline{\leftrightarrow} r_2$.*

Proof. Let β be a bisimulation of p_1 and p_2, and ρ a bisimulation of q_1 and q_2. Then p_1 terminates if and only if p_2 terminates.

1. One easily checks that $\{\langle r_1, r_2\rangle\} \cup \beta \cup \rho$ is a bisimulation of r_1 and r_2.
2. Since isomorphisms are bisimulations and composites of bisimulations are bisimulations, we may assume that β is a bisimulation of the copies of $\mathbf{P}_{p_1}$ and $\mathbf{P}_{p_2}$ that are part of $\mathbf{P}_{r_1}$ and $\mathbf{P}_{r_2}$ (in which $\surd$ has been replaced by, respectively, q_1 and q_2), and ρ a bisimulation of the copies of $\mathbf{P}_{q_1}$ and $\mathbf{P}_{q_2}$. Then $\beta \cup \rho$ is a bisimulation of r_1 and r_2. □

Consider the quotient $P/\underline{\leftrightarrow}$. The lemma implies that for any equivalence classes X and Y, there is at most one equivalence class that contains an element z for which there are $x \in X$ and $y \in Y$ such that $z = x + y$, and similarly for $\cdot$. So putting

$$X + Y = Z \Leftrightarrow \exists x \in X \exists y \in Y \exists z \in Z\ z = x + y,$$
$$X \cdot Y = Z \Leftrightarrow \exists x \in X \exists y \in Y \exists z \in Z\ z = x \cdot y,$$

we define a partial algebra with universe $P/\underline{\leftrightarrow}$. Let us denote it by $\mathbf{P}/\underline{\leftrightarrow}$.

Corollary 2.7. *If a process structure* $\mathbf{P}$ *is closed under composition, then* $\mathbf{P}/\underline{\leftrightarrow}$ *is an algebra of type* $\{+, \cdot\}$.

Now assume that $\mathbf{P}$ is closed under composition.

Alternative composition in $\mathbf{P}/\underline{\leftrightarrow}$ is evidently commutative. It is associative: let $s_1 = p + q$, $t_1 = s_1 + r$, $s_2 = q + r$, and $t_2 = p + s_2$. Then

$$\{\langle t_1, t_2\rangle\} \cup \Delta_{P_p} \cup \Delta_{P_q} \cup \Delta_{P_r}$$

is a bisimulation of t_1 and t_2.

For idempotency: let $q = p + p$. Put $X = P_p - \{p\}$. Then

$$\{\langle q, p\rangle\} \cup \Delta_X$$

is a bisimulation of q and p.

Right distributivity: take $s = p + q$, $t = sr$, $s_1 = pr$, $s_2 = qr$, and $u = s_1 + s_2$. Since we may assume that r is the same in s_1 and s_2, we may also assume that $s = u$. Similarly $(pq)r$ and $p(qr)$ may be assumed identical.

We conclude that $\mathbf{P}/\underline{\leftrightarrow}$ is a process algebra.

Action relations

Let $\mathbf{T}$ be the ground term algebra of type $\{+, \cdot\}$ over a set A of constant symbols ('atoms'); take $\surd \notin T$. The *action relations* are the binary relations $\xrightarrow{a}$, for $a \in A$, and $\xrightarrow{\epsilon}$, inductively defined by

$$\begin{aligned} a &\xrightarrow{a} \surd, \\ x + y \xrightarrow{\epsilon} x \quad &\text{and} \quad y + x \xrightarrow{\epsilon} x, \\ x \xrightarrow{a} x' \quad &\Rightarrow \quad xy \xrightarrow{a} x'y, \\ x \xrightarrow{a} \surd \quad &\Rightarrow \quad xy \xrightarrow{a} y. \end{aligned}$$

In this way the algebra $\mathbf{T}$ determines an A-process structure

$$(\langle T \cup \{\surd\}\rangle, \langle \xrightarrow{a} \rangle_{a \in A}, \xrightarrow{\epsilon}, \surd\rangle).$$

The *action graph* of $x \in T$ is the closure of $\{x\}$ under the action relations, with the restricted relations.

3. Order and lattice structure in process algebras

Let $\mathbf{B}$ be a process algebra with a constant δ satisfying

$$\begin{aligned} x + \delta &= x \qquad D1 \\ \delta x &= \delta \qquad D2 \end{aligned}$$

We write $x \leq y$, for $x, y \in B$, if $x + y = y$. From $A1$–3 it follows that $\leq$ is an ordering; by $D1$, δ is its least element.

Proposition 3.1. *If for all* $x \in B$, $(x]\ (= \{y \in B \mid y \leq x\})$ *is finite, then* $\langle B, \leq\rangle$ *is a lattice.*

Proof. By ($A1$–3) $\langle B, +\rangle$ is an (upper) semilattice. Define

$$x \wedge y = \Sigma((x] \cap (y]).$$

It is easy to see that

$$u \leq x, y \Leftrightarrow u \leq x \wedge y$$

and

$$x \wedge y = x \Leftrightarrow x \leq y \Leftrightarrow x + y = y.$$

So $\langle B, +, \wedge\rangle$ is a lattice. □

There is no reason why the lattice induced by a process algebra should be distributive:

Proposition 3.2. *Every* $+$*-semilattice can be expanded to a process algebra.*

Proof. Define: $xy = x$. □

4. The $\{\delta, \epsilon, \tau\}$-lattice

Still there are natural process algebras that are distributive lattices, even free Heyting algebras. The well-known constants δ, ϵ and τ satisfy the laws $D1$ and $D2$ stated above and:

$$\begin{array}{rl} \epsilon x = x = x\epsilon & E \\ \tau\tau = \tau & T1 \\ \tau + \epsilon = \tau & T2. \end{array}$$

One easily derives $\tau\tau x = \tau x$ (by $T1$ and $A5$) and

$$\tau x + x = \tau x \quad T2'$$

(by $T2$, $A4$ and E); these are the simplest of Milner's classical τ-laws, for the case that τ is the only proper step. More complex τ-laws are also derivable:

Proposition 4.1.

1. $\tau(\tau(x + y) + x) = \tau(x + y)$.
2. $\tau(x + \tau y) + \tau y = \tau(x + \tau y)$.

Proof.

1.

$$\begin{array}{rll} \tau(\tau(x+y)+x) & = \tau(\tau(x+y)+x+y+x) & \text{by } T2' \\ & = \tau(\tau(x+y)+x+y) & \text{by } A1\text{–}3 \\ & = \tau\tau(x+y) = \tau(x+y) & \text{by } T2' \text{ and } T1 \end{array}$$

2. $\tau(x + \tau y) \leq \tau(x + \tau y) + \tau y \leq \tau(x + \tau y) + x + \tau y = \tau(x + \tau y)$, by $T2'$. □

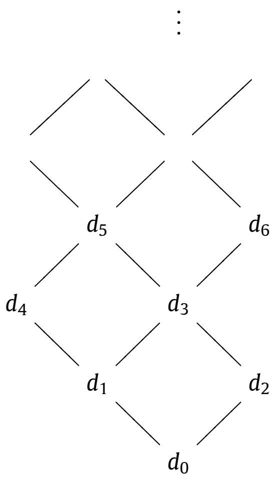
In the absence of actions, (1) is the Branching Law, and (2) the Third τ-Law.

Definition 4.2. Let $d_0 = \delta$, $d_1 = \epsilon$, $d_2 = \tau d_0$, $d_{2n+3} = d_{2n+1} + d_{2n+2}$, $d_{2n+4} = \tau d_{2n+1}$.

Lemma 4.3. *If* $n > 0$, *then* $d_{2n-1} \leq d_{2n+2}$ *and* $d_{2n+3} = d_{2n} + d_{2n+2}$.

Proof. Assume $n > 0$. Then $d_{2n+2} = \tau d_{2n-1}$, so by $T2'$ $d_{2n-1} \leq d_{2n+2}$, so $d_{2n+3} = d_{2n+1} + d_{2n+2} = d_{2n-1} + d_{2n} + d_{2n+2} = d_{2n} + d_{2n+2}$. □

It follows that the lattice structure of the processes $d_0, d_1, \ldots$ is a homomorphic image of



In fact this lattice contains *all* processes that are generated by δ, ϵ and τ. Indeed, by Proposition 2.4 all these processes are basic processes over δ, ϵ and τ. So it suffices to note that, obviously, the lattice is closed under alternative composition; and that $\delta d_n = d_0$, $\epsilon d_n = d_n$, $\tau d_0 = d_2$, $\tau d_{2n+1} = d_{2n+4}$, and $\tau d_{2n+2} = d_{2n+2}$.

5. $\epsilon\tau$-bisimulation

Now consider a process structure with only two relations, indexed τ and ϵ. The constant τ stands for a *silent step*; in essence, some activity of an undetermined nature. By the axiom $T1$, we cannot tell two such steps from a single one. Accordingly, we adapt Definition 2.5 as follows, in two steps.

Definition 5.1. Let **P** be a $\{\tau\}$-process structure.

1. A symmetric relation ρ on P is an $\epsilon\tau$-*bisimulation* if, whenever $x\rho y$, if $x \xrightarrow{\epsilon^*\tau} x'$, either $x'\rho y$, or y' exists such that $y \xrightarrow{\epsilon^*\tau} y'$ and $x'\rho y'$; and x terminates if and only if y terminates.
2. An $\epsilon\tau$-bisimulation ρ on P satisfies the *root condition* with respect to p and q if $p\rho q$ and $p \xrightarrow{\epsilon^*\tau} x$ implies that y exists such that $q \xrightarrow{\epsilon^*\tau} y$ and $x\rho y$, and similarly with p and q interchanged.

If an $\epsilon\tau$-bisimulation ρ satisfies the root condition with respect to p and q, we also say ρ is a *rooted* $\epsilon\tau$-*bisimulation* between p and q.

We say that p and q are $\epsilon\tau$-*bisimilar*, notation

$$p \underline{\leftrightarrow}_{\epsilon\tau} q,$$

if there exists an $\epsilon\tau$-bisimulation ρ such that $p\rho q$; and *rooted* $\epsilon\tau$-*bisimilar*, notation

$$p \underline{\leftrightarrow}_{\mathrm{r}\epsilon\tau} q,$$

if there exists a rooted $\epsilon\tau$-bisimulation between p and q.

Observe that if ρ_i, $i \in I$, are $\epsilon\tau$-bisimulations on P, then the union $\rho = \bigcup_i \rho_i$ is an $\epsilon\tau$-bisimulation, and if any ρ_i satisfies the root condition with respect to p and q, then so does ρ. Hence in any process structure, $\underline{\leftrightarrow}_{\epsilon\tau}$ is an $\epsilon\tau$-bisimulation, and $\underline{\leftrightarrow}_{\mathrm{r}\epsilon\tau}$ a rooted $\epsilon\tau$-bisimulation.

A bisimulation in the sense of Definition 2.5, with $A = \{\tau\}$, is an $\epsilon\tau$-bisimulation that satisfies the root condition with respect to any two points that it connects.

Lemma 5.2. *Let π and ρ be $\epsilon\tau$-bisimulations on a $\{\tau\}$-process structure* **P**. *Then $\pi \circ \rho$ is an $\epsilon\tau$-bisimulation.*

Proof. Suppose $w\pi x\rho y$. Then w terminates if and only if y terminates. Suppose $w \xrightarrow{\epsilon^*\tau} w'$. Then either $w'\pi x$, or x' exists such that $x \xrightarrow{\epsilon^*\tau} x'$ and $w'\pi x'$. In the first case,

$$w'(\pi \circ \rho)y. \qquad (*)$$

In the second case, either (a) $x'\rho y$, or (b) y' exists such that $y \xrightarrow{\epsilon^*\tau} y'$ and $x'\rho y'$. In case (a), we again have $(*)$. In case (b), $w'(\pi \circ \rho)y'$. $\square$

Corollary 5.3. *If $p \underline{\leftrightarrow}_{\mathrm{r}\epsilon\tau} q$ and $q \underline{\leftrightarrow}_{\mathrm{r}\epsilon\tau} r$, then $p \underline{\leftrightarrow}_{\mathrm{r}\epsilon\tau} r$.*

Hence the argument of Section 2, showing that bisimulation is a congruence of the A-process structure, and that the quotient is a process algebra, also proves that rooted $\epsilon\tau$-bisimilarity is a congruence of the $\{\tau\}$-process structure, and the quotient is a process algebra.

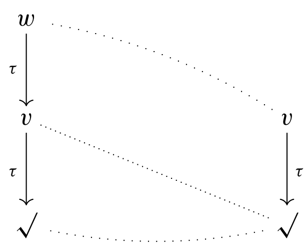
Fix a $\{\tau\}$-process structure **P** that is closed under composition. Let δ be an unrelated node ($\neq \surd$) in P; and a, for $a \in \{\epsilon, \tau\}$, a node that is connected, by R_a exclusively, to $\surd$.

The diagonal Δ_P is a rooted $\epsilon\tau$-bisimulation between $p + \delta$ and p; and $\delta \cdot p = \delta$. So $\mathbf{P}/\underline{\leftrightarrow}_{\mathrm{r}\epsilon\tau}$ is a model of $D1$ and $D2$. It is easy to see that $\mathbf{P}/\underline{\leftrightarrow}_{\mathrm{r}\epsilon\tau}$ is a model of E.

Let $v = \tau$, $w = \tau \cdot v$. Then

$$\{\langle w, v\rangle, \langle v, \surd\rangle, \langle \surd, \surd\rangle\}$$

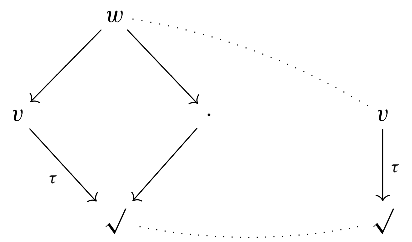
is a rooted $\epsilon\tau$-bisimulation between w and v.



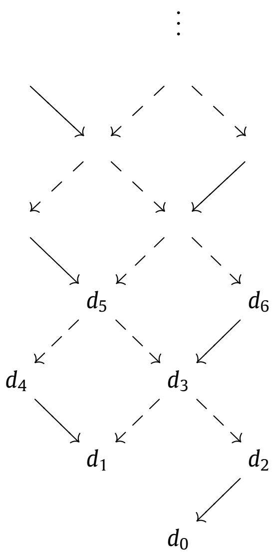
Let $v = \tau$, $w = v + \epsilon$. Then

$$\{\langle w, v\rangle, \langle \surd, \surd\rangle\}$$

is a rooted $\epsilon\tau$-bisimulation of w and v. In the figure below the ϵ-labels have been suppressed.



We conclude that $T1$ and $T2$ hold in $\mathbf{P}/\underline{\leftrightarrow}_{r\epsilon\tau}$. So $\mathbf{P}/\underline{\leftrightarrow}_{r\epsilon\tau}$ is a model of $A1$–5, $D1$–2, E and $T1$–2. Hence, to prove that the terms $d_0, d_1, \ldots$ have distinct meanings in our theory, it is enough that we find a $\{\tau\}$-process structure in which their representatives are not rooted $\epsilon\tau$-bisimilar. We obtain such a structure by substituting action relations for (almost all) the Hasse-lines in the +-lattice of Section 4. The result is shown below, with solid arrows representing τ-, and broken arrows ϵ-steps.



We first prove that d_m and d_n are $\epsilon\tau$-bisimilar only if either they are identical or they are connected by a τ-step. Let β be an $\epsilon\tau$-bisimulation.

1. d_0 and d_2 are not $\epsilon\tau$-bisimilar to any other node, for all other nodes terminate.
2. d_1 and d_4 are not $\epsilon\tau$-bisimilar to any d_{2k+3}, because $d_{2k+3} \xrightarrow{\epsilon^*\tau}\!\!\!\!\rightarrow d_0$. A fortiori they are not $\epsilon\tau$-bisimilar to any d_{2k+6}. For suppose $d_{2k+6}\beta d_4$. Since $d_{2k+6} \xrightarrow{\tau} d_{2k+3}$, we must have $d_{2k+3}\beta d_4$ or $d_{2k+3}\beta d_1$, both known to be impossible. Similarly $d_{2k+6}\beta d_1$ is impossible. Observe that $d_{2n+5} \xrightarrow{\epsilon^*\tau}\!\!\!\!\rightarrow d_i$ if and only if $i = 2m + 1$ for some $m \le n$ or $i = 0$; and $d_{2n+4} \xrightarrow{\epsilon^*\tau}\!\!\!\!\rightarrow d_i$ if and only if $i = 2n + 1$.
3. $d_{2n+5}\beta d_3$ requires, by $d_{2n+5} \xrightarrow{\epsilon^*\tau}\!\!\!\!\rightarrow d_1$, that either $d_1\beta d_3$ or $d_1\beta d_0$, known to be impossible by 2 and 1. If $d_{2n+8}\beta d_3$, then, by $d_{2n+8} \xrightarrow{\tau} d_{2n+5}$, either $d_{2n+5}\beta d_3$ or $d_{2n+5}\beta d_0$, impossible by what we just proved and by 1. If $d_{2n+5}\beta d_6$, then by $d_{2n+5} \xrightarrow{\epsilon^*\tau}\!\!\!\!\rightarrow d_1$, either $d_1\beta d_6$ or $d_1\beta d_3$, both known to be impossible by 2. If $d_{2n+8}\beta d_6$, then, by $d_{2n+8} \xrightarrow{\tau} d_{2n+5}$, either $d_{2n+5}\beta d_6$ or $d_{2n+5}\beta d_3$, both of which we have shown to be impossible.
4. For the remainder we proceed by induction on k. Assume that for all $j < k$ and for all i,

   d_i is $\epsilon\tau$-bisimilar to d_{2j+3} or d_{2j+6} only if $i \in \{2j + 3, 2j + 6\}$.

   Take any $n \ge k > 0$. If $d_{2n+5}\beta d_{2k+3}$, then, by $d_{2n+5} \xrightarrow{\epsilon^*\tau}\!\!\!\!\rightarrow d_{2k+1}$, we must have $d_{2k+1}\beta d_{2k+3}$ or $d_{2k+1}\beta d_{2m+1}$ for some $m < k$, or $d_{2k+1}\beta d_0$. The last alternative is ruled out by 1; the other alternatives contradict the induction hypothesis. If

$d_{2n+8}\beta d_{2k+3}$, then, by $d_{2n+8} \xrightarrow{\tau} d_{2n+5}$, we must have $d_{2n+5}\beta d_{2k+3}$ or $d_{2n+5}\beta d_{2m+1}$ for some $m < k$, or $d_{2n+5}\beta d_0$. We have shown above that the first alternative is impossible; the second contradicts the induction hypothesis; and the third contradicts 1. A fortiori $d_{2n+5}\beta d_{2k+6}$ is impossible since $d_{2n+5} \overset{\epsilon^*\tau}{\twoheadrightarrow} d_{2k+1}$; and this precludes $d_{2n+8}\beta d_{2k+3}$.

It follows that distinct d_m and d_n can only be rooted $\epsilon\tau$-bisimilar if $d_m \xrightarrow{\tau} d_n$. But then, rooted $\epsilon\tau$-bisimilarity requires that $d_n \overset{\epsilon^*\tau}{\twoheadrightarrow} d_k$ for some d_k $\epsilon\tau$-bisimilar to d_n; and such d_k do not exist.

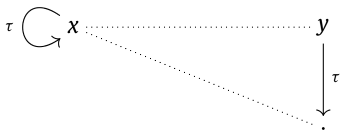
6. Concluding remarks

By the discussion above, infinitely many distinct processes are nameable by terms constructed from δ, ϵ and τ. One easily checks that on omission of τ, only δ and ϵ remain, since $\delta x = \delta$, $\delta + x = x + x = x$, and $\epsilon x = x\epsilon = x$. Similarly, omitting δ grounds us at $\{\epsilon, \tau\}$, and without ϵ we only have δ, τ and Δ $(=\tau\delta)$.

There are various definitions for bisimulation involving τ, but in the absence of real actions they do not seem to make much of a difference. The root condition is quite important though, as we have seen.

Further graphs

As set forth in [10], δ may be equated with the ϵ-cycle ϵ^ω. Under Definition 5.1, the τ-cycle τ^ω is rooted $\epsilon\tau$-bisimilar with Δ.



It has been argued that Δ, and not δ, deserves the appellation *deadlock*. If δ is among the options a process has, it is recognized, and can be avoided (as reflected by $D1$); a summand Δ, on the other hand, cannot in general be cancelled. Whereas δ is an observable lock that allows the choice of a different channel, Δ is a true dead-end.

De Jongh's model

The Rieger–Nishimura lattice is the lattice of propositions of a particular Kripke model, first discovered by De Jongh (it is described in an exercise to Section 5.4 of [5]). In process algebraic terms, it consists of the outer nodes of the process structure of the previous section — the ones with even index, except that d_1 replaces d_0. The proposition p expresses immediate termination; $\perp$ does not hold anywhere, and may be considered to correspond with δ (lock), which is not represented; $\neg p$ corresponds with Δ (deadlock). The accessibility relation is reachability in the process structure.

Infinite elements

The lattice of Section 4 is completed by $d_\omega = \Sigma_n d_n$ and $d_{\omega+1} = \tau d_\omega$; there it ends. Thus it is isomorphic to the completion of the Rieger–Nishimura lattice [7], $d_{\omega+1}$ corresponding to $\top$. The argument above, showing $\tau\delta = \tau^\omega$, applies to all infinite repetitions of elements d_ξ with $\xi > 1$: they are all equivalent to Δ. Indeed, any infinite product

$$\prod_{j=0}^{\infty} d_{\xi_j}$$

is Δ, except if the first $\xi_j \neq 1$ is 0, or all ξ_j are 1: then the result is δ.

Acknowledgements

A comment by one of the referees made us aware of a serious mistake in the first version of this paper.

References

[1] J.C.M. Baeten, T. Basten, M.A. Reniers, Process Algebra: Equational Theories of Communicating Processes, Cambridge, 2010.
[2] Jos Baeten, Rob van Glabbeek, Abstraction and empty process in process algebra, Fundamenta Informaticae 12 (1989) 221–242.
[3] J.A. Bergstra, J.W. Klop, Process algebra for synchronous communication, Information and Control 60 (1984) 109–137.
[4] J.C.M. Baeten, W.P. Weijland, Process Algebra, Cambridge, 1990.
[5] Dirk van Dalen, Logic and structure, Berlin, 1980.
[6] Robin Milner, Semantics of concurrent processes, in: J. van Leeuwen (Ed.), Handbook of Theoretical Computer Science, vol. B, Amsterdam, 1990, pp. 1203–1242.
[7] Mark Nadel, Infinitary intuitionistic logic from a classical point of view, Annals of Mathematical Logic 14 (1978) 159–191.
[8] Iwao Nishimura, On formulas in one variable in intuitionistic propositional calculus, Journal of Symbolic Logic 25 (1960) 327–331.
[9] Ladislav Rieger, On the lattice theory of Brouwerian propositional logic, Acta Facultatis Rerum Naturalium Universitatis Carolinae 189 (1949) 1–40.
[10] J.L.M. Vrancken, The algebra of communicating processes with empty process, Theoretical Computer Science 177 (1997) 287–328.