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A 3-Person Game on Polyhedral Sets

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Abstract—A problem of a Nash equilibrium point existence and calculating for a 3-person game on polyhedral sets with special payoff functions is considered. It is shown that its existence is equivalent to that of a saddle point in an auxiliary antagonistic game on polyhedral sets and that the saddle point calculating can be done using vector-solutions of a linear and quadratic programming problem formulated on the basis of the master 3-person game.

Keywords—3-person game, Antagonistic game, Equilibrium point, Saddle point, Polyhedral sets.

1. INTRODUCTION

An example of a 3-person game on polyhedral sets where existence of a Nash equilibrium point is equivalent to that of a saddle point in an auxiliary antagonistic game was presented by the author in [1]. A method for calculating such a saddle point for that type of antagonistic game on polyhedral sets was proposed by the author in [2]. The method enables us to calculate a saddle point for the solvable antagonistic game using vector-solutions of two auxiliary problems: linear and quadratic, so that the vector-solutions turn out to be components of the saddle point. In this article, we consider one more 3-person game for which a Nash point calculating can be also reduced to that of a saddle point in an auxiliary antagonistic game on polyhedral sets. Necessary and sufficient conditions of a Nash equilibrium point in this game are proposed. Using these conditions, it becomes possible to reduce calculating of the Nash point to that of a saddle point in that very type of antagonistic game for which the method proposed in [2] can be used.

2. THE PROBLEM STATEMENT AND MATHEMATICAL FORMULATION

Let us consider the following problem which can serve as a model for some practical situations. A company (evacuator) must evacuate from a territory a finite number of some waste products (for example, radioactive or hazardous) at certain volumes of each according to the existing environmental protection regulations. Such an evacuation implies the necessity of finding a recipient agreed to accept such products in certain amounts and kinds being paid for that along with a carrier agreed to transport such a cargo. The evacuator must pay both the recipient and carrier and is interested in minimizing its total expenditure. Accordingly the recipient is trying to get as much as possible from the evacuator nominating the prices for each evacuated product within some reasonable limits. However, the recipient in his turn is charged for acceptance of such waste products, for example, by the local government, and at the same time, may usually have

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX an additional revenue charging the carrier for certain services in points of the products' delivery (for example, ports, railroad stations, truck terminals, etc., located on the territory where the recipient agreed to accept the products). Some reasonable limits on tariffs are also applied, and the carrier can offer his services just in accordance with those limits trying in his turn to maximize his revenue.

In practical situations, the evacuator is limited in spending the money for the evacuation services so participants of the considered situation can be examined as players in a 3-person game having certain strategies and payoff functions. It is of importance and interest (from practical and scientific view points) to know if such a game has an equilibrium point and how it could be calculated, if so.

It was discussed and shown in [2] that, in many situations which are of interest for practice including the one described above, the sets of the players strategies are polyhedra determined by solvable systems of linear inequalities. Based on [2] and omitting secondary details, we shall further assume that the players allowable strategies are polyhedra M, Ω, H described by the following systems of inequalities

$$M = \left\{x \in \mathbb{R}_+^n : Ax \geq b\right\}, \Omega = \left\{y \in \mathbb{R}_+^n : By \geq d\right\}, H = \left\{u \in \mathbb{R}_+^n : Gu \geq g\right\},$$

where A, B, G and b, d, g are matrices and vectors of corresponding dimensions.We further assume that x is vector of prices paid by the evacuator for products to be evacuated, y and u are, respectively, vectors of volumes and transportation tariffs for products to be evacuated. We also assume that elements of all the considering matrices and vectors are real numbers.

So, we examine the 3-person game where the players are the evacuator, recipient and carrier with the following payoff functions:

the carrier: $f_1(x, y, u) = \max_{y \in \Omega}\langle y, u\rangle$, which is to be maximized on H;
the recipient: $f_2(x, y, u) = \max_{x \in M}\langle y, x\rangle + k \cdot \langle y, u\rangle - \langle q, y\rangle$, which is to be maximized on Ω;
the evacuator: $f_3(x, y, u) = \max_{y \in \Omega} \max_{u \in H} \{\langle y, u\rangle + \langle y, x\rangle\}$, which is to be minimized on M;

and where H, Ω, and M are sets of the players allowable strategies correspondingly, $0 < k < 1$.

We further consider a generalization of such a game where the players have the following payoff functions[1]:

$\tilde{f}_1(x, y, u) = \max_{y \in \Omega} \{\langle y, u\rangle + \langle y, Dx\rangle + \langle p, x\rangle\}$, which is to be maximized on H;
$\tilde{f}_2(x, y, u) = \max_{x \in M} \{\langle y, u\rangle + \langle y, Dx\rangle + \langle q, y\rangle + \langle p, x\rangle\}$, which is to be maximized on Ω;
$\tilde{f}_3(x, y, u) = \max_{y \in \Omega} \max_{u \in H} \{\langle y, u\rangle + \langle y, Dx\rangle + \langle p, x\rangle\}$, which is to be minimized on M;

and sets of their allowable strategies are H, Ω, M respectively. We shall assume that H, Ω, M are arbitrary (generally speaking) unbounded polyhedral sets given by joint systems of linear inequalities in nonnegative orthants of some spaces having finite dimensions and that $\langle y, u\rangle$ and $\langle y, x\rangle$ are bounded from above on H and M respectively $\forall y \in \Omega$ and p, q, D are, respectively, vectors and matrices of corresponding dimensions, the elements of which are real numbers.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR A NASH EQUILIBRIUM POINT IN THE GENERALIZED 3-PERSON GAME

Let

$$(x^*, y^*, u^*) \in M \times \Omega \times H \text{ and}$$

$$u^* \in \operatorname{Arg}\max_{u \in H}\langle y^*, u\rangle, y^* \in \operatorname{Arg}\max_{y \in \Omega}\left\{\max_{u \in H}\langle y, u\rangle + \langle q, y\rangle + \langle y, Dx^*\rangle + \langle p, x^*\rangle\right\}.$$

[1] One can easily show that the initial 3-person game is reducible to a game of such type.

THEOREM 1. *(x^*, y^*, u^*) is a Nash point in 3-person game on $M \times \Omega \times H$ with payoff functions $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$, if and only if (x^*, y^*) is a saddle point in an antagonistic game on $M \times \Omega$ with payoff function $\eta(x, y) = \max_{u \in H}\langle y, u\rangle + \langle q, y\rangle + \langle y, Dx\rangle + \langle p, x\rangle$.*

PROOF. *Necessity* follows from definitions of $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$, and obvious inequality

$$\max_{u\in H}\langle y, u\rangle + \langle q, y\rangle + \langle y, Dx^*\rangle + \langle p, x^*\rangle \le \max_{y\in\Omega}\left\{\max_{u\in H}\langle y, u\rangle + \langle q, y\rangle + \langle y, Dx^*\rangle + \langle p, x^*\rangle\right\} \forall y \in \Omega.$$

Sufficiency. Let (x^*, y^*) be a saddle point for $\eta(x, y)$ on $M \times \Omega$. Then $\max_{u\in H}\langle y, u\rangle + \langle q, y\rangle + \langle y, Dx^*\rangle + \langle p, x^*\rangle \le \langle y^*, u^*\rangle + \langle q, y^*\rangle + \langle y^*, Dx^*\rangle + \langle p, x^*\rangle$. From $\langle y, u^*\rangle + \langle q, y\rangle + \langle y, Dx^*\rangle + \langle p, x^*\rangle \le \max_{u\in H}\langle y, u\rangle + \langle q, y\rangle + \langle y, Dx^*\rangle + \langle p, x^*\rangle \forall y \in \Omega$ conclude that $\tilde{f}_2(x^*, y, u^*) \le \tilde{f}_2(x^*, y^*, u^*) \forall y \in \Omega$. Further, from $\langle y^*, u^*\rangle + \langle q, y^*\rangle + \langle y^*, Dx^*\rangle + \langle p, x^*\rangle \le \langle y^*, u^*\rangle + \langle q, y^*\rangle + \langle y^*, Dx\rangle + \langle p, x\rangle \forall x \in M$ conclude that $\tilde{f}_3(x, y^*, u^*) \ge \tilde{f}_3(x^*, y^*, u^*) \forall x \in M$.

Let's consider now $\tilde{f}_1(x^*, y^*, u) = \langle y^*, u\rangle + \langle y^*, Dx^*\rangle + \langle p, x^*\rangle$ and notice that from

$$\max_{u\in H}\langle y, u\rangle + \langle q, y\rangle + \langle y, Dx^*\rangle + \langle p, x^*\rangle \le \langle y^*, u^*\rangle + \langle q, y^*\rangle + \langle y^*, Dx^*\rangle + \langle p, x^*\rangle, \qquad \forall y \in \Omega,$$

it follows that

$$\max_{u\in H}\langle y^*, u\rangle + \langle q, y^*\rangle + \langle y^*, Dx^*\rangle + \langle p, x^*\rangle \le \langle y^*, u^*\rangle + \langle q, y^*\rangle + \langle y^*, Dx^*\rangle + \langle p, x^*\rangle,$$

which means, in particular, that

$$\langle y^*, u\rangle + \langle y^*, Dx^*\rangle + \langle p, x^*\rangle \le \langle y^*, u^*\rangle + \langle y^*, Dx^*\rangle + \langle p, x^*\rangle, \qquad \forall u \in H,$$

which is in its turn equal to $\tilde{f}_1(x^*, y^*, u) \le \tilde{f}_1(x^*, y^*, u^*) \forall u \in H$.

Theorem 1 is proved.

Then, there takes place the following obvious assertion.

ASSERTION 1. *(x^*, y^*) is a saddle point in an antagonistic game with payoff function*

$$\max_{u\in H}\langle y, u\rangle + \langle q, y\rangle + \langle y, Dx\rangle + \langle p, x\rangle,$$

if and only if $(\tilde{x}^*, y^*)$ is a saddle point in an antagonistic game with payoff function $\max_{u\in H}\langle y, u\rangle + \langle y, \tilde{D}\tilde{x}\rangle + \langle \tilde{p}, \tilde{x}\rangle$ on $\tilde{M} \times \Omega$, where $\tilde{M} = M \times \{1\}, \tilde{p} = (p, 0), \tilde{D} = [Dq], \tilde{x} = (x, 1)$.

Thus, it was shown that the master 3-person game is reducible to an antagonistic game on polyhedral sets for which the theory and solution method were proposed in [2]. We formulate the corresponding results below in notations of the present article. Proofs of the theorems formulated below were done in [2].

Let

$$\begin{aligned} \tilde{M} &= \left\{\tilde{x} \in \mathbb{R}_+^{m+1} : \tilde{A}\tilde{x} \ge b\right\}, \\ \tilde{A} &= [A0], \tilde{Q} = \left\{(\tilde{z}, y) \ge 0 : \tilde{z}\tilde{A} \le \tilde{p} + y\tilde{D}, By \ge d\right\}, \\ \tilde{P}(u) &= \left\{(t, \tilde{x}) \ge 0 : tB \le -u - \tilde{D}\tilde{x}, \tilde{A}\tilde{x} \ge b\right\}. \end{aligned}$$

THEOREM 2. [2] *In order for $(\tilde{x}^*, y^*)$ to be the saddle point of function $\tilde{\varphi}(\tilde{x}, y) = \max_{u\in H}\langle y, u\rangle + \langle y, \tilde{D}\tilde{x}\rangle + \langle \tilde{p}, \tilde{x}\rangle$ on $\tilde{M} \times \Omega$, it is necessary and sufficient that points $t^* \ge 0, u^* \ge 0, \tilde{z}^* \ge 0$, exist such that $(\tilde{z}^*, y^*, u^*)$ and $(t^*, \tilde{x}^*, u^*)$ are solutions, respectively to the problems*

$$\max_{(\tilde{z},y)\in\tilde{Q}} \{\langle b, \tilde{z}\rangle + \langle y, u\rangle\} \to \max_{u\in H}, \tag{1}$$

$$\min_{(t,\tilde{x})\in\tilde{P}(u)} \{\langle -d, t\rangle + \langle \tilde{p}, \tilde{x}\rangle\} \to \max_{u\in H}, \tag{2}$$

and the following equality is satisfied

$$\max_{y\in\Omega}\left\{\max_{u\in H}\langle y,u\rangle+\langle y,\tilde{D}\tilde{x}^*\rangle+\langle\tilde{p},\tilde{x}^*\rangle\right\}=\langle y^*,u^*\rangle+\langle y^*,\tilde{D}\tilde{x}^*\rangle+\langle\tilde{p},\tilde{x}^*\rangle. \tag{3}$$

Let

$$\tilde{b}=(0_n,0_n,b),\tilde{\mu}=(y,u,\tilde{z}),W=\begin{pmatrix}0&E&0\\0&0&0\\0&0&0\end{pmatrix},\quad \tilde{\Pi}=\begin{pmatrix}-\tilde{D}^\top&0&\tilde{A}^\top\\-B&0&0\\0&-G&0\end{pmatrix},$$

0_n is zero vector in $\mathbb{R}^n, \tilde{v}=(\tilde{p},-d,-g), \tilde{q}^*=(\tilde{D}\tilde{x}^*,0_n), \tilde{W}=\begin{pmatrix}0&E\\0&0\end{pmatrix}, S=\begin{pmatrix}-B&0\\0&-G\end{pmatrix}$,

E is the unit $(n\times n)$ matrix, $\xi=(y,u), w=(-d,-g)$.

ASSERTION 2. [2] *The solvability of problems* (1) *and* (3) *is equivalent to that of the respective quadratic programming problems*

$$\langle\tilde{\mu},W\tilde{\mu}\rangle+\langle\tilde{b},\tilde{\mu}\rangle\to\max_{\tilde{\mu}\geq 0:\tilde{\Pi}\tilde{\mu}\leq\tilde{v}}, \tag{4}$$

$$\langle\xi,\tilde{W},\xi\rangle+\langle\tilde{q}^*,\xi\rangle+\langle\tilde{p},\tilde{x}^*\rangle\to\max_{\xi\geq 0:S\xi\leq w}. \tag{5}$$

Furthermore, if $\mu^*=(y^*,u^*,\tilde{z}^*)$ *is the solution to* (4), $(t^*,\tilde{x}^*)$ *is the solution to linear programming problem*

$$\langle -d,t\rangle+\langle\tilde{p},\tilde{x}\rangle\to\min_{(t,\tilde{x})\in\tilde{P}(u^*)}, \tag{6}$$

and $\xi^*=(y^*,u^*)$ *is the solution to* (5), *then* $(\tilde{x}^*,y^*)$ *is the saddle point of function* $\tilde{\varphi}(\tilde{x},y)$ *on* $\tilde{M}\times\Omega$.

From Assertion 2, it follows that establishing the solvability of the 3-person game and searching for the Nash points in solvable games of this kind can be done using known, in particular, finite methods of solving corresponding mathematical programming problems [2].

ASSERTION 3. [2] *The solvability of the problem*

$$\max_{y\in\Omega}\left\{\max_{u\in H}\langle y,u\rangle+\langle y,\tilde{D}\tilde{x}\rangle+\langle\tilde{p},\tilde{x}\rangle\right\}\to\min_{\tilde{x}\in\tilde{M}}$$

is equivalent to that of the quadratic programming problem

$$\langle w,\lambda\rangle-\langle\xi,\tilde{W}\xi\rangle+\langle\tilde{p},\tilde{x}\rangle\to\min_{(\lambda,\xi,\tilde{x})\in\tilde{M}},$$

where

$$\tilde{M}=\left\{(\lambda,\xi,x):\lambda\geq 0,\xi\geq 0,\tilde{x}\geq 0,\tilde{A}\tilde{x}\geq b,\tilde{q}(\tilde{x})\leq -2\tilde{W}\xi+S^\top\lambda,\tilde{q}(\tilde{x})=(\tilde{D}\tilde{x},0_n)\right\}.$$

4. CONCLUSIVE REMARKS

1. The result presented in this article for the 3-person game should be considered as a step forward in widening of linear and quadratic programming possibilities to solve substantially nonlinear problems on polyhedral sets.

2. It should be played up that the proposed necessary and sufficient conditions for the Nash point are verifiable ones, and such a verification involves a software based on that for linear and quadratic programming. It enables us to solve large scale problems of the kind which are considered of importance for practice.

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