Moduli of twisted orbifold sheaves

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Abstract

We study stacks of slope-semistable twisted sheaves on orbisurfaces with projective coarse spaces and prove that in certain cases they have many of the asymptotic properties enjoyed by the moduli of slope-semistable sheaves on smooth projective surfaces.

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1. Introduction

The purpose of this paper is to streamline and broaden the moduli theory of twisted sheaves on surfaces. We also lay some foundations for the study of moduli of (twisted) sheaves on stacky surfaces. In particular, we will prove the following theorem. Let $k$ be a field and $r$ an integer which is invertible in $k$.

**Theorem.** Let $\mathcal{X} \to X$ be a $\mu_N$-gerbe on an $A_{N-1}$-orbisurface with projective coarse moduli space. Given an invertible sheaf $L \in \text{Pic}(X)$ and a positive integer $\Delta$, the stack $\text{Tw}^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ of totally regular slope-semistable $\mathcal{X}$-twisted sheaves of rank $r$, determinant $L$, and discriminant $\Delta$ is of finite type over $k$. If $\Delta$ is sufficiently large the stack $\text{Tw}^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ is also geometrically normal. In addition,

1. if there exists a locally free $\mathcal{X}$-twisted sheaf of rank $r$, then for any positive integer $B$, there is $\Delta > B$ such that $\text{Tw}^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ contains at least one geometrically integral connected component;
2. if $X$ is a smooth projective surface, then for all sufficiently large $\Delta$, the stack $\text{Tw}^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ is geometrically integral whenever it is non-empty, and it is non-empty for infinitely many values of $\Delta$. Moreover, if it is non-empty for $\Delta$, then it is non-empty for $\Delta + r \ell$ for any $\ell > 0$.

The condition that $X$ is an $A_{N-1}$-orbisurface means, roughly, that $X$ has isolated stacky points, each with stabilizer $\mu_N$. (We also permit the locus of stacky points to be empty, in which case $X$ is just a smooth projective surface.) The condition that a sheaf is totally regular means that the classes in $K$-theory of its (derived) representations at the stacky points of $X$ are multiples of the standard representation of $\mu_N$. These definitions, as well as the definition of (semi)stability and basic properties of the moduli problem, will be recalled in Sections 3.3.1 and 4.1, among other places. Suffice it to say that the theory described here is an extension of the classical theory of semistable sheaves on smooth projective surfaces, with some arithmetic ramifications (taken up elsewhere).

In previous work [12], we proved the second part of the above theorem under the additional assumption that the class of $\mathcal{X}$ in the Brauer group of $X \otimes_k \bar{k}$ had order $N$ and that $r = N$. The more general results proven here bring the structure theorems for moduli of twisted sheaves on smooth projective surfaces into alignment with those known for classical sheaves. The first part of the theorem uses a method shown to the author by Johan de Jong, which applies to a large range of moduli problems, notably in recent work of de Jong and Starr on rationally connected varieties. Here the asymptotic regularity properties of the moduli stacks lead to the existence of geometrically irreducible components, but we cannot yet conclude that the entire moduli problem becomes asymptotically irreducible.
The approach we take owes a debt to Adrian Langer, who, in a series of papers (see [7, 6, 8]) extended O’Grady’s beautiful methods [15] to all characteristics, clarifying the role of the Segre loci. It is O’Grady’s now standard outline, reproduced in [4] and [8], that we have followed. In several cases, Langer’s proofs carry over almost verbatim. When this is the case, we refer to his proofs and only indicate the changes that must be made in the present context.

There are three main ways in which our proof differs from the standard outline. The first is that the delicate numerical estimates needed for bounding the dimensions of the Segre loci are altered by the replacement of the classical Riemann–Roch formula by Toën’s orbifold Riemann–Roch formula. We include computations of the correction terms, which, in the case of totally regular sheaves, do not disturb the main properties of the relevant estimates.

The second difference between our proof and the standard outline is in the terminal steps of the proof of irreducibility, where Langer’s effective computations [8] do not work. This different proof is required by the nature of twisted sheaves and is very similar to the proof included in [12]. One amusing aspect of the theory of twisted sheaves that makes it hard to simply carry over classical results is the fact that there is no “trivial sheaf”; in fact, one does not even know a priori that there is a single semistable sheaf! This is proven below in Section 5.1.

The third difference lies in our independence from the methods of Geometric Invariant Theory (GIT). As we show, the standard O’Grady outline really leads to a description of the stack of \( \mu \)-semistable sheaves and its open substack of \( \mu \)-stable sheaves. No mention of Gieseker stability or the subtleties of GIT are necessary. We strongly believe that this approach clarifies the subject.

Finally, let us note that while it may be tempting to view the present work as a mere exercise in transferring classical results to the orbifold context, this is deeply misguided. The uninitiated reader is referred to [13] for examples of applications of the ideas discussed here and in [12] to division algebras over surfaces. Adding an orbifold structure to the underlying surface, giving rise to the spaces described in the present work, seems to have some relation to the cyclicity problem for division algebras of prime degree. While we have chosen to omit such arithmetic considerations from this paper, they remain a strong motivation for studying these moduli spaces.

2. Notation

We will say that a gerbe is non-trivial if its structure group is not the singleton group, and non-neutral if it does not have a section. (Similar conventions hold with the “non”s removed!) Thus, a residual gerbe on a stack is non-trivial when the corresponding “point” is a stacky point, but one can certainly have neutral non-trivial residual gerbes (e.g., if the field of moduli is algebraically closed!). We write \( \mathcal{B} \mu_N \) for the classifying stack of the group scheme \( \mu_N \); this is the stack-theoretic quotient \( \text{Spec} k/\mu_N \).

Let \( X \) be a smooth proper geometrically connected orbisurface over a field \( k \). By this we mean that \( X \) has trivial inertia in codimension 1, so that the stacky locus of \( X \) consists of isolated non-trivial residual gerbes. We will write \( X^{\text{sp}} \) for the open substack over which the inertia stack is trivial (the “spatial locus”); this is the largest open substack which is isomorphic to an algebraic space. Write \( \overline{X} \) for the coarse moduli space of \( X \) and \( \sigma : X \to \overline{X} \) for the map.

We will fix a \( \mu_N \)-gerbe \( \pi : \mathcal{X} \to X \) with \( N \) invertible in \( k \) for the entire paper. We will write \( D(\mathcal{X}) \) to denote the derived category of quasi-coherent \( \mathcal{X} \)-twisted sheaves; we will use a similar notation for any \( \mu_N \)-gerbe.

By definition, given a morphism of finite type \( X \to T \) of stacks, a “flat family of coherent sheaves over \( T \)” is a \( T \)-flat quasi-coherent \( \mathcal{O}_X \)-module of finite presentation. (The fibers become
coherent, but coherent is not the correct property for the family, as it is not preserved under base change on \( T \).)

Given a smooth surface \( Y \) over a field \( k \), we will let \( L_y \) be the invariant defined by Langer in Section 2.1 of [8]. This is constant that fixes the Bogomolov inequality in positive characteristic: for any slope-semistable sheaf \( E \) on \( Y \) of rank \( r \), we have that \( \Delta(D) \geq -(L_y^2/4H^2)r^2(r-1)^2 \).

### 3. Preliminaries on orbisurfaces and regular sheaves

#### 3.1. Intersection theory

**Theorem 3.1.1.** There exists a trace map \( H^2(X, \omega_X) \to k \) such that for all perfect complexes of quasi-coherent sheaves \( \mathcal{F} \) and \( \mathcal{G} \) on \( X \) and any integer \( i \), the pairing

\[
\text{Hom}_{D(X)}(\mathcal{F}, \mathcal{G}) \times \text{Hom}_{D(X)}(\mathcal{G}, \mathcal{F} \otimes \omega_X[2]) \to k
\]

induced by the cup product and the trace maps is perfect.

**Proof.** For the proof, we refer the reader to Proposition 1.9, Proposition 1.14, and Theorem 1.32 of [14]. \( \square \)

**Definition 3.1.2.** An invertible sheaf \( \mathcal{L} \) on \( X \) is **ample** if the non-vanishing loci of sections of powers of \( \mathcal{L} \) generate the topology on the underlying topological space \( |X| \). If \( X \) has an ample invertible sheaf, we will say that \( X \) is pseudo-projective.

Note that this definition is equivalent to the statement that some power of \( \mathcal{L} \) is the pullback of an ample invertible sheaf from \( \overline{X} \). There are more refined notions of ample sheaves (related to embeddings in weighted projective stacks), but they will have no place here (and seem to be overly specialized).

We use Vistoli’s intersection theory on \( X \) in what follows [21]. In particular, we have a theory of Chern classes and Todd classes, and a degree map \( \text{deg} : A^2(X) \to \mathbb{Q} \).

**Theorem 3.1.3.** Suppose \( H \) is an ample Cartier divisor on \( X \). If \( D \) is a divisor such that \( D^2 > 0 \) then \( D \cdot H \neq 0 \).

**Proof.** The proof is standard and familiar from the theory of surfaces. First, note that Toën’s Riemann–Roch formula [19] says that \( \chi(nD) \) is quadratic in \( n \) with leading coefficient \( D^2 \). Thus, using Theorem 3.1.1 and letting \( K \) denote a canonical divisor on \( X \), we have that either \( h^0(nD) \) or \( h^0(K-nD) \) goes to infinity with \( n \), and similarly for either \( h^0(-nD) \) or \( h^0(K+nD) \).

**Claim.** We cannot have both \( h^0(K-nD) \) and \( h^0(K+nD) \) going to infinity with \( n \).

Indeed, given \( \sigma \in \Gamma(X, K-nD) \), the map \( \tau \mapsto \tau \otimes \sigma : \Gamma(X, K+nD) \to \Gamma(X, 2K) \) is injective. But then \( h^0(K+nD) \) is bounded for all \( n \) by \( h^0(2K) \), so it cannot grow with \( n \).

We conclude that either \( h^0(nD) \) or \( h^0(-nD) \) is non-zero for some \( n > 0 \). Since \( H \) is ample, we have that \( H \cdot (nD) > 0 \) or \( H \cdot (-nD) > 0 \). In particular, \( H \cdot D \neq 0 \). \( \square \)
Corollary 3.1.4. If \( X \) is pseudo-projective then the intersection form on \( \text{NS}(X) \otimes \mathbb{R} \) has signature \((1, r - 1)\), the first basis vector coming from an ample divisor on \( X \).

Proof. Diagonalizing the form over \( \mathbb{R} \), the result follows from Theorem 3.1.3. \( \square \)

3.2. Uniformizations of \( \mu_N \)-gerbes on smooth pseudo-projective orbisurfaces

We work with \( \mu_N \)-gerbes throughout this paper. For basic properties of gerbes in the context of twisted sheaves, the reader is referred to [13] and [12]. For a proof that a gerbe on an orbisurface (or, more generally, a stack on a stack) is an (algebraic) stack, the reader is referred to Section 2.4 of [10].

Let \( k \) be an infinite field and \( X \) a proper smooth geometrically connected pseudo-projective orbisurface and \( X \to \mathcal{X} \) a \( \mu_N \)-gerbe, with \( N \) invertible in \( k \). By standard results [2,5,1], we know that \( X \) is a quotient stack. We recall the following result of Kresch and Vistoli [5].

Proposition 3.2.1. There is a finite flat generically separable morphism \( f : Y \to \mathcal{X} \) with \( Y \) a smooth geometrically connected projective surface over \( k \).

Idea of proof. We indicate the idea of the proof so that we can point out why the cover is generically separable (a statement which was left out of [5]). The proof proceeds by using the fact that \( \mathcal{X} \) is a quotient stack to note that there is a product \( P \to X \) of projective bundles which has a dense open representable substack whose complement has arbitrarily high codimension. The coarse space \( U \) of \( P \) is then shown to be projective, and the cover \( Y \) is found by taking general hyperplane sections of \( U \). Since a general such section will be generically separable over \( \overline{X} \), we will have that \( f \) is generically separable. \( \square \)

We can use the uniformization \( f \) to give a quick proof of algebraicity of certain stacks of sheaves. For proofs in more general contexts, the reader is referred to [12] and [11]. This more ad hoc approach suffices for the task at hand.

It is elementary that the fibered category \( \text{Sh}(\mathcal{X}) \) of flat families of coherent \( \mathcal{O}_{\mathcal{X}} \)-modules is a stack over \( k \). Because it is locally free, the uniformization \( f : Y \to \mathcal{X} \) induces a pullback morphism \( f^* : \text{Sh}(\mathcal{X}) \to \text{Sh}(Y) \).

Proposition 3.2.2. With the above notation, the morphism \( f^* \) is representable by separated schemes of finite presentation.

Proof. Let \( Y^{(2)} = Y \times \mathcal{X} \) and \( Y^{(3)} = Y \times Y \times \mathcal{X} \). The two projections \( p,q : Y^{(2)} \to Y \) are finite and flat, as are the three projections \( Y^{(3)} \to Y^{(2)} \) and (therefore) the three projections \( Y^{(3)} \to Y \). (It follows that \( Y^{(2)} \) and \( Y^{(3)} \) are projective, but there is no reason to believe that they are smooth or geometrically connected.)

Let \( \varphi : T \to \text{Sh}(Y) \) be a 1-morphism, corresponding to a flat family of coherent sheaves \( \mathcal{F} \) on \( Y \times T \). Define a functor \( \mathcal{D} \) on the category of \( T \)-schemes as follows: given a \( T \)-scheme \( S \to T \), the set \( \mathcal{D}(S) \) consists of isomorphisms \( \psi : p_S^* \mathcal{F}_S \sim q_S^* \mathcal{F}_S \) which satisfy the usual cocycle condition on \( Y^{(3)} \). It is clear that \( \mathcal{D} \) is a locally closed subfunctor of the functor \( H = \text{Hom}(p^* \mathcal{F}, q^* \mathcal{F}) \) parametrizing homomorphisms between the two pullbacks of \( \mathcal{F} \) to \( Y^{(2)} \). Since \( q^* \mathcal{F} \) is flat over \( T \), it is well known that \( H \) is a \( T \)-scheme of finite presentation. (That \( \mathcal{D} \) is locally of finite presentation is immediate.) \( \square \)
Corollary 3.2.3. The stack of coherent sheaves on $\mathcal{X}$ is an Artin stack locally of finite type over $k$.

Proof. By Proposition 3.2.2, this follows from the corresponding fact for $Sh(Y)$, which is Théorème 4.6.2.1 of [9].

Corollary 3.2.4. The stack of flat families of coherent $\mathcal{X}$-twisted sheaves is an Artin stack locally of finite type over $k$.

Proof. This follows immediately from the fact that this is an open (and closed) substack of the stack of coherent sheaves on $\mathcal{X}$.

We can also use $f$ to define numerical conditions which yield quasi-compact stacks of sheaves. The polarization $H$ on $\overline{X}$ induces a polarization $H|_Y$ on $Y$ which we will always use to define the projective structure on $Y$.

Definition 3.2.5. The $f$-Hilbert polynomial of a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ is the Hilbert polynomial of $f^*\mathcal{F}$, i.e., the numerical polynomial $P^f_{\mathcal{F}}(m) = \chi(Y, \mathcal{F}_Y(mHY))$.

Lemma 3.2.6. The $f$-Hilbert polynomial is constant in a flat family of coherent sheaves on $\mathcal{X}$ parametrized by a connected base $T$.

Proof. This follows immediately from the corresponding well-known fact on $Y$.

Corollary 3.2.7. Given a coherent $\mathcal{X}$-twisted sheaf $\mathcal{E}$ and a numerical polynomial $P$, the scheme $\text{Quot}(\mathcal{E}, P)$ parametrizing flat families of coherent quotients of $\mathcal{E}$ with $f$-Hilbert polynomial is proper.

Proof. Quasi-compactness (and separatedness) follows immediately from Proposition 3.2.2 and the corresponding fact for the Quot-scheme relative to $Y$. Properness follows by the usual proof of the valuative criterion (see e.g. the proof of Theorem 2.2.4 of [4]).

Further uses of the uniformization $f$ in studying semistable sheaves will be given in Section 4.3.

3.3. $A_{N-1}$-orbisurfaces and the Toën–Grothendieck–Hirzebruch–Riemann–Roch formula

In this section we discuss the orbisurfaces of interest to us and compute the correction terms to the classical Hirzebruch–Riemann–Roch formula for computing Euler characteristics of coherent sheaves.

A succinct description of Toën’s Riemann–Roch formula can be found in Appendix A of [20] or [19]. We refer the reader there or to [19] for details. The basic insight which Toën had is that the correct cohomology in which to work to produce a Riemann–Roch formula on an orbifold is that of the inertia stack. This yields correction terms to the standard Riemann–Roch formula arising from the additional components of the inertia stack. Our short-term goal is a calculation of these correction terms.
In brief, writing \( I \to X \) for the inertia stack (i.e., the relative group-stack over \( X \) representing the automorphism sheaves of the objects of \( X \)), for any locally free sheaf \( E \) on \( X \), Toën produces a weighted Chern character \( \tilde{\text{ch}}(E) \) and a weighted Todd class \( \tilde{Td}(E) \) (with the unadorned symbol \( \tilde{Td} \) denoting \( \tilde{Td}(TX) \)) lying in the cyclotomic Chow theory \( A(I) \otimes \mathbb{Q}(\mu_\infty) \). This is done by first promoting \( E \) to a class in \( K(I) \otimes \mathbb{Q}(\mu_\infty) \) and then taking the usual formula componentwise on \( I \). This promotion is described using the natural right action \( E \times I \to E \); the class supported on a section of \( I \) is the formal sum of roots of unity times eigensheaves in \( E \) for the action of that section. This is all very clearly explained in [20], building on the abstract foundations developed in [19].

3.3.1. \( A_{N-1} \)-orbisurfaces

We recall a definition from [10].

**Definition 3.3.1.1.** An orbisurface \( Z \) with coarse moduli space \( \overline{Z} \) is a Zariski \( A_{N-1} \)-orbisurface if

1. \( N \) is invertible on \( Z \);
2. it has isolated non-trivial closed residual gerbes \( \xi_1, \ldots, \xi_s \) with images \( p_1, \ldots, p_s \) in \( \overline{Z} \);
3. for each \( i = 1, \ldots, s \), the fiber product \( Z \times_{\overline{Z}} \text{Spec} \mathcal{O}_{\overline{Z},p_i} \) is isomorphic to the stack quotient \( [\text{Spec } R/\mu_N] \), where \( R \) is a regular local ring of dimension 2 with an action of \( \mu_N \) whose induced representation on the tangent space decomposes as \( \chi \oplus \chi^{-1} \), where \( \chi: \mu_N \to \mathbb{G}_m \) is the natural character.

Since we will only work with Zariski \( A_{N-1} \)-orbisurfaces in this paper, we will refer to these simply as \( A_{N-1} \)-orbisurfaces. The first condition ensures that \( Z \) is a (tame) Deligne–Mumford stack, rather than a (tame) Artin stack. Choosing a primitive root of unity defines an isomorphism between \( \mu_N \) and the constant group scheme \( \mathbb{Z}/N\mathbb{Z} \), showing that the stabilizer is unramified.

If \( \overline{Z} \) is a normal projective surface with isolated \( A_{N-1} \)-singularities (i.e., with local model \( k[[x, y, z]]/(z^n - xy) \)), then it is the coarse moduli space of a smooth pseudo-projective \( A_{N-1} \)-orbisurface, unique up to unique isomorphism over \( \overline{Z} \). In our case the surface \( \overline{Z} \) will arise by forming the cyclic cover of degree \( N \) of a smooth projective surface branched over a snc divisor. In this case the singularities have the form described above locally in the Zariski topology.

3.3.2. The correction terms

In this section we compute the correction to the naïve Riemann–Roch formula coming from the non-trivial residual gerbes of \( X \). The reader willing to accept the results of these calculations in the dimension estimates of Sections 4.4 and 4.5 is encouraged to skip this section on a first reading.

**Lemma 3.3.2.1.** Let \( \xi \) be a primitive \( N \)th root of unity and \( j < N \) a non-negative integer. We have

\[
\sum_{i=1}^{N-1} \frac{\xi^{ij}}{2 - \xi^i - \xi^{-i}} = \frac{j(j - N)}{2} + \frac{N^2 - 1}{12}.
\]
Proof. Write \( S = \mu_N(\overline{k}) \setminus \{1\} \) and \( 1 - S = \{y | 1 - y \in S\} \). The sum to be evaluated is

\[
\sum_{x \in S} \frac{x^j}{(1 - x) + (1 - x^{-1})}.
\]

Letting \( y = 1 - x \), we can rewrite the sum as

\[
- \sum_{y \in 1 - S} \frac{(1 - y)^{j+1}}{y^2}.
\]

Moreover, we know that the set over which the sum is taken is precisely the set of roots of the polynomial

\[
\sum_{t=0}^{N-1} \binom{N}{t+1} z^t = 0. \quad (3.3.2.1.1)
\]

Expanding the sum to be evaluated yields the double sum

\[
- \sum_{y \in 1 - S} \sum_{s=0}^r (-1)^s \binom{j + 1}{s} y^{s-2}.
\]

It thus suffices to evaluate the sums \( \tau(r) := \sum_{y \in S-1} y^r \) for integers \( r \) between \(-2\) and \( N - 1\). First suppose \( r > 0 \); the sum becomes

\[
\tau(r) = \sum_{x \in S} \sum_{q=0}^r (-1)^{r-q} \binom{r}{q} x^q = N
\]

by an elementary computation. It is also clear that \( \tau(0) = N - 1\). To compute \( \tau(-1) \) and \( \tau(-2) \), we argue as follows: letting \( y_1, \ldots, y_{N-1} \) be the elements of \( S - 1 \), we see that

\[
\tau(-1) = \sigma_{N-2}(y_1, \ldots, y_{N-1}) / \sigma_{N-1}(y_1, \ldots, y_{N-1}),
\]

where \( \sigma_i \) denotes the \( i \)th elementary symmetric function in \( N - 1 \) variables. Similarly,

\[
\tau(-2) = \tau(-1)^2 - 2\sigma_{N-3}(y_1, \ldots, y_{N-1}) / \sigma_{N-1}(y_1, \ldots, y_{N-1}).
\]

Using (3.3.2.1.1), we see that \( \sigma_i(y_1, \ldots, y_{N-1}) = \binom{N}{N-i} \). Putting this together, we find

\[
\tau(-1) = \frac{N - 1}{2}
\]

and

\[
\tau(-2) = \frac{(N - 1)(5 - N)}{12}.
\]
Basic algebraic manipulations now yield
\[- \sum_{s=0}^{j+1} (-1)^s \left( \frac{j+1}{s} \right) \tau(s-2) = \frac{j(j-N)}{2} + \frac{N^2-1}{12},\]
as desired. \(\square\)

Define a function \(f: \mathbb{Z}/N\mathbb{Z} \to \mathbb{Q}\) by the formula
\[f(x) = \frac{x(x-N)}{2} + \frac{N^2-1}{12},\]
where \(0 \leq x < N\) is one of the canonical representatives for \(\mathbb{Z}/N\mathbb{Z}\).

Now suppose \(F\) is a perfect complex on \(X\). Using the Toën–Riemann–Roch formula, we can write
\[\chi(F) = \deg(\text{ch}(F) \cdot \text{Td}_X) + \sum_{i=1}^{n} \delta_i(F),\]
where \(\delta_i(F)\) is a correction term (a priori lying in \(\mathbb{Q}(\mu_N)\), but actually lying in \(\mathbb{Q}\)) coming from contributions at the residual gerbe \(\xi_i\). Write \([L_{i}^* F] = \sum_{j=0}^{N-1} e_j^{(i)} \chi^j\) for the class in \(K\)-theory of the derived fiber of \(F\) over \(\xi_i\).

**Lemma 3.3.2.2.** We have \(\delta_i(F) = \frac{1}{N} \sum_{j=0}^{N-1} f(j) e_j^{(i)}\).

**Proof.** It suffices to prove this when \(k\) is algebraically closed (as the Euler characteristic is invariant under field extension, as is the intersection theory). In this case, the inertia stack \(\mathcal{I}(X) \to X\) has the form
\[\mathcal{I}(X) = X \sqcup \bigsqcup_{i=1}^{n} \bigsqcup_{\ell=1}^{N-1} \mathcal{B}_{\mu_N},\]
with one copy of \(\bigsqcup_{\ell=1}^{N-1} \mathcal{B}_{\mu_N}\) mapping to each \(\xi_i\) in the natural way (via the identification provided by \(\iota_i\)).

Consider the copy of \(\bigsqcup_{i=1}^{N-1} \mathcal{B}_{\mu_N}\) in the inertia stack lying over \(\xi_i\). On the \(\ell\)th component, a section acts on \(L_{i}^* F\) via the \(\ell\)th power of the natural action. Thus, decomposing \(\sum_{j=1}^{N-1} e_j^{(i)} \chi^j\) into weighted eigenbundles on the \(\ell\)th component yields \(\sum_{j=0}^{N-1} \xi^{j\ell} e_j^{(i)} \chi^{j\ell}\), so that the weighted Chern character \(\widetilde{\text{ch}}(F|_{\mu_N})\) equals \(\sum_{j=0}^{N-1} \xi^{j\ell} e_j^{(i)}\).

On the other hand, the Todd class which intervenes in the Toën–Riemann–Roch formula has the following form. Let \(E\) be the locally free sheaf \(\mathcal{F}_X|_{\mu_N}\) on \(\mathcal{B}_{\mu_N}\), viewed as the \(\ell\)th copy over \(\xi_i\). We know that \(E = \chi^{\ell} \oplus \chi^{-\ell}\), since \(X\) is an \(A_{N-1}\)-orbisurface. Using the standard formula (see e.g. A.0.5 of [20]), we find that the contribution to the Toën–Todd class coming
from this component is then

\[ \widetilde{Td}(E) = \frac{1}{2} - \zeta^\ell - \zeta^{-\ell}. \]

Thus, taking into account the fact that \( B\mu_n \rightarrow \text{Spec} \kappa \) has degree \( 1/N \), we find that the total correction term for the components of \( I(X) \) lying over \( \xi_i \) is

\[ \frac{1}{N} \sum_{\ell=1}^{N-1} \sum_{j=0}^{N-1} \frac{\zeta^{j\ell} e^{(i)}_j}{2 - \zeta^\ell - \zeta^{-\ell}} = \frac{1}{N} \sum_{j=0}^{N-1} e^{(i)}_j \sum_{\ell=1}^{N-1} \zeta^{j\ell} \left( \frac{2}{2 - \zeta^\ell - \zeta^{-\ell}} \right), \]

which proves the lemma. \( \square \)

The formula provides an immediate corollary.

**Corollary 3.3.2.3.** The Euler characteristic of \( \mathcal{O}_X \) satisfies

\[ \chi(X, \mathcal{O}_X) = \deg Td_X + n \frac{N^2 - 1}{12N}. \]

### 3.4. Twisted sheaves on \( AN_{-1} \)-orbisurfaces

In this section we fix an \( AN_{-1} \)-orbisurface \( X \) with coarse moduli space \( \bar{X} \) and a \( \mu_N \)-gerbe \( \mathcal{X} \rightarrow X \). We assume that the base field \( k \) is algebraically closed and denote the non-trivial residual gerbes of \( X \) by \( \xi_1, \ldots, \xi_n \), each of which is isomorphic to \( B\mu_N \) via a map \( \iota_i : B\mu_N \rightarrow X \). Given \( i = 1, \ldots, n \), we write \( \mathcal{X}_i = \mathcal{X} \times_X \xi_i \). Fix \((-1)\)-fold invertible \( \mathcal{X}_i \)-twisted sheaves \( \mathcal{L}_i \).

Twisting by \( \mathcal{L}_i \) and pulling back by \( \iota_i \) defines an equivalence of categories between \( D(\mathcal{X}_i) \) and the category of graded representations of \( \mu_N \).

**Notation 3.4.1.** Given a perfect complex \( F \) of \( \mathcal{X} \)-twisted sheaves, we thus get a class in \( K(\mu_N) \) by taking the class in \( K \)-theory associated to the perfect complex \( \mathcal{L}_i \otimes \mathcal{L}_i^* F \) of \( \mu_N \)-representations. We will always write \( F_i \) for this class.

We establish some of the basic properties of torsion free \( \mathcal{X} \)-twisted sheaves which are relevant to the problems at hand; in particular, we will focus on the role played by the non-trivial residual gerbes of \( X \). The reader unfamiliar with twisted sheaves is referred to [12] and [13] (and the references therein); we will not redevelop the theory from scratch in the present paper.

While we write in the language of twisted sheaves, when \( \mathcal{X} \) is the trivial \( \mu_N \)-gerbe, our results also apply to ordinary sheaves (and perfect complexes thereof) on \( X \). Removing the twisting class is straightforward in this case, but we will occasionally point out explicitly how a certain definition or property would look in the untwisted case.

Before proceeding, we record an orphan lemma, which is a twisted form of Theorem 3.1.1.1.

**Lemma 3.4.2.** Let \( \mathcal{E} \) and \( \mathcal{H} \) be perfect complexes of quasi-coherent \( \mathcal{X} \)-twisted sheaves. The cup product and trace maps induce a perfect pairing

\[ \text{Hom}(\mathcal{E}, \mathcal{H}) \times \text{Hom}(\mathcal{H}, \mathcal{E} \otimes \omega_X[2]) \rightarrow k. \]
Proof. This follows from Theorem 3.1.1 with $\mathcal{F} = \mathcal{O}_X$ and $\mathcal{I} = R\mathcal{H}\text{om}(\mathcal{O}, \mathcal{H})$, using the standard properties of perfect complexes. □

We start by noting a convention which will prove helpful.

**Definition 3.4.3.** Given a perfect complex $F$ of $\mathcal{X}$-twisted sheaves, the *normalized Chern classes*, denoted $\tilde{c}_i(F)$, are $Nc_i(F)$, where $c_i(F)$ denotes the usual $i$th Chern class in $A^i(\mathcal{X})$.

The purpose of normalizing the Chern classes is simply to correct the fact that $\pi_\ast \pi^\ast$ acts as multiplication by $1/N$ on $c_i$. Thus, if $G$ is a perfect complex on $X$, we have that $\pi_\ast \tilde{c}_i(\pi^\ast G) = c_i(G)$.

**Definition 3.4.4.** Given an $\mathcal{X}$-twisted sheaf $\mathcal{F}$, write $\text{Sing}(\mathcal{F})$ for the reduced closed substack structure on the complement of the locus over which $\mathcal{F}$ is locally free.

Since $\mathcal{X}$ and $X$ have the same topology, we can consider $\text{Sing}(\mathcal{F})$ as a subset of $X$. (This serves primarily to simplify notation and we will freely avail ourselves of this fact in the future.) Since $k$ is algebraically closed, we can define the length of a 0-dimensional twisted sheaf as follows.

**Definition 3.4.5.** Given an $\mathcal{X}$-twisted sheaf $\mathcal{Q}$ of dimension 0, the *length* of $\mathcal{Q}$ is the maximal length of a filtration $\mathcal{Q} = \mathcal{Q}_\ell \supset \mathcal{Q}_{\ell-1} \supset \cdots \supset \mathcal{Q}_0 = 0$ with non-trivial subquotients.

Using the convention of Definition 3.4.3, one can see that $\ell(\mathcal{Q}) = -\tilde{c}_2(\mathcal{Q})$.

**Definition 3.4.6.** A perfect complex $\mathcal{F} \in D(\mathcal{X})$ is totally regular if $\mathcal{F}_i$ is in the ideal $\mathbb{Z}_\rho \subset K(\mu_N)$ for each $i = 1, \ldots, n$.

A weaker notion which will often be sufficient for certain statements (but which will be relatively unimportant for this work) is the following.

**Definition 3.4.7.** A perfect complex $\mathcal{F} \in D(\mathcal{X})$ is totally positive if $\mathcal{F}_i$ is the class in $K(\mu_N)$ associated to a (non-virtual) representation of $\mu_N$ for each $i = 1, \ldots, n$.

The reader can readily check that these two properties are independent of the choices of the invertible sheaves $\mathcal{L}_i$ on $\mathcal{X}_i$ made at the beginning of the section.

Since any coherent sheaf on $X$ is perfect as an object of the derived category ($X$ being regular), we can ascribe the same properties to sheaves. We will do this freely.

**Lemma 3.4.8.** If $\mathcal{F}$ is a totally regular perfect complex of $\mathcal{X}$-twisted sheaves and $\mathcal{I}$ is an arbitrary perfect complex then

1. $\mathcal{F} \boxtimes \mathcal{I}$ is totally regular;
2. $R\mathcal{H}\text{om}(\mathcal{F}, \mathcal{I})$ is totally regular.

**Proof.** This immediately reduces to the analogous statement for objects in the derived category of $\mu_N$-representations, where this follows from the fact that $\mathbb{Z}_\rho$ is an ideal of $K(\mu_N)$. 

(We can use the tensor powers of the $\mathcal{L}_i$ on the various powers of $\mathcal{R}$ to get complexes of $\mu_N$-representations out of $R\mathcal{H}\text{om}(\mathcal{F}, \mathcal{G})$ or $\mathcal{F} \otimes \mathcal{G}$; since total regularity is independent of this local trivialization, we lose nothing by leaving such minor details unremarked upon.) □

Thus, if $\mathcal{F}$ is a totally regular perfect complex of $\mathcal{R}$-twisted sheaves, we can identify (via derived pushforward) the complex $R\mathcal{H}\text{om}(\mathcal{F}, \mathcal{F})$ with a totally perfect complex on $X$. We will implicitly do this below.

One thing which the reader should note is that the reflexive hull of a totally regular torsion free $\mathcal{R}$-twisted sheaf on $X$ is not necessarily totally regular. It is a simple matter to make local examples around any singular point, an exercise which we leave to the reader.

**Proposition 3.4.9.** If $\mathcal{F}$ is a totally regular perfect complex on $X$ then $\chi(X, \mathcal{F}) = \deg(\text{ch}(\mathcal{F}) \cdot Td_X)$.

**Proof.** This follows immediately from the calculations of Section 3.3.2, together with the fact that $\sum_{j=0}^{N-1} f(j) = 0$. □

In what follows we use the fact that for any field $L$ there is a canonical ring isomorphism $K(\mu_N) = \mathbb{Z}[x]/(x^N - 1)$ (which is functorial in $L$). We will write $K(\mu_N)$ for this constant ring.

**Proposition 3.4.10.** Let $T$ be a connected scheme and $\mathcal{F}$ a perfect complex on $T \times B\mu_N$. There exists a class $c \in K(\mu_N)$ such that for every geometric point $t : \text{Spec} \kappa \to T$, the class of the preimage $L_t^*\mathcal{F}$ in $K(\mu_N)$ is $c$.

**Proof.** We may assume that $T = \text{Spec} \ A$ is affine, so that there is a global resolution of $\mathcal{F}$ by locally free sheaves. This reduces us to the case in which $\mathcal{F}$ is itself locally free. The category of coherent sheaves on $B\mu_N \times T$ is naturally equivalent to the category of Hopf comodules $M$ over $A[x]/(x^N - 1)$ (with the Hopf $A$-algebra structure induced by the affine group scheme structure on $\mu_{N,T}$). Given such a comodule action $\alpha : M \to A[x]/(x^N - 1) \otimes M$, let $M_n \subset M$ be the $A$-submodule consisting of elements $m \in M$ such that $\alpha(m) = m \otimes x^n$. It is a standard calculation that $M_n$ is a sub-comodule and the natural map $\bigoplus_{n=0}^{N-1} M_n \to M$ is an isomorphism of comodules. When $M$ is a locally free $A$-module, each $M_n$ is also locally free, and the pullback by $t$ gives the decomposition of $t^*M$ into eigenspaces. If in addition $T$ is connected, each $M_n$ has constant rank, which shows that the class in $K$-theory of $t^*M$ is independent of $t$, as desired.

Alternatively, one can prove this using the upper semicontinuity of global sections for flat coherent sheaves on proper morphisms of Artin stacks: look at the morphism $\varphi : B\mu_{N,T} \to T$. Given a locally free sheaf $\mathcal{F}$ on $B\mu_{N,T}$ and a character $\chi$ of $\mu_N$, upper semicontinuity shows that the function $\Gamma(B\mu_{N,T}, t^*\mathcal{F} \otimes \chi^\vee)$ is upper semicontinuous, which shows that the eigenspace with character $\chi$ has upper semicontinuous dimension as a function on $T$. Since the fibers of $\mathcal{F}$ have constant dimension, the only way this can happen is if each representation has the same eigendecomposition and thus the same class in $K$-theory, as desired. □

The deformation theory of $\mathcal{R}$-twisted sheaves is governed by deformations of sheaves of modules in the étale topos of $\mathcal{R}$ and is thus described by the general theory of Illusie. In particular, we have the following proposition. Given a perfect complex $\mathcal{F}$ and an ideal $I$, there is a
trace map $\text{Ext}^i(\mathcal{F}, \mathcal{F} \otimes I) \to H^i(\mathcal{X}, I)$ arising from the trace map $\text{R} \mathcal{H}\text{om}(\mathcal{F}, \mathcal{F}) \to \mathcal{O}$ and the identification $\text{Ext}^i(\mathcal{F}, \mathcal{F} \otimes I) = \text{Ext}^i(\mathcal{O}, \text{R} \mathcal{H}\text{om}(\mathcal{F}, \mathcal{F}) \otimes I)$. We let $\text{Ext}^i(\mathcal{F}, \mathcal{F} \otimes I)_0$ denote the kernel of the trace map. As usual, when the rank of $\mathcal{F}$ is invertible in $k$, Serre duality induces an isomorphism $\text{Ext}^2(\mathcal{F}, \mathcal{F}_0) \sim \text{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_X)^\vee$.

**Proposition 3.4.11.** Let $A \to A_0$ be a surjection of $k$-algebras with kernel $I$ of square 0. Let $\mathcal{F}$ be a flat family of coherent $\mathcal{X}$-twisted sheaves parametrized by $A$ and let $\mathcal{L}_0$ be the determinant of $\mathcal{F}$. Fix a flat family $\mathcal{L}$ of invertible sheaves on $\mathcal{X}$ parametrized by $A$ along with an isomorphism $\mathcal{L} \otimes_A A_0 \to \mathcal{L}_0$.

(1) There is an element $\sigma$ of $\text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes I)_0$ such that $\sigma = 0$ if and only if there is a flat family of coherent $\mathcal{X}$-twisted sheaves $\mathcal{F}$ parametrized by $A$ and isomorphisms $\mathcal{F} \otimes_A A_0 \sim \mathcal{F}$ and $\text{det}\mathcal{F} \sim \mathcal{L}$ compatible with the given isomorphism $\mathcal{L} \otimes_A A_0 \sim \mathcal{L}_0$ and the identification of $\mathcal{L}_0$ with the determinant of $\mathcal{F}$.

(2) The set of isomorphism classes of such extensions is a pseudo-torsor under $\text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes I)_0$.

(3) The set of infinitesimal automorphisms of one such extension is trivial.

**Definition 3.4.12.** A coherent $\mathcal{X}$-twisted sheaf $\mathcal{F}$ is unobstructed if $\text{Ext}^2(\mathcal{F}, \mathcal{F})_0 = 0$.

**Definition 3.4.13.** Let $T$ be a $k$-scheme. A $T$-flat family of torsion free coherent sheaves on $X$ is a quasi-coherent sheaf $\mathcal{F}$ on $X \times T$ which is locally of finite presentation, flat over $T$, and such that for every geometric point $t \to T$, the fiber $\mathcal{F}_t$ is a torsion free coherent sheaf on $X \otimes_k k(t)$.

**Definition 3.4.14.** Given a torsion free sheaf $\mathcal{F}$ on $\mathcal{X}$,

(1) the soft singular locus of $\mathcal{F}$ is $\text{Sing}^{\text{soft}}(\mathcal{F}) := \text{Sing}(\mathcal{F}) \cap X^{\text{sp}}$;

(2) the soft hull of $\mathcal{F}$ is the sheaf $\mathcal{F}^{\text{soft}}$ with inclusion $\mathcal{F} \hookrightarrow \mathcal{F}^{\text{soft}}$ such that $(\mathcal{F} \hookrightarrow \mathcal{F}^{\text{soft}})|_{X^{\text{sp}}} = \mathcal{F}|_{X^{\text{sp}}} \hookrightarrow \mathcal{F}^{\text{vert}}|_{X^{\text{sp}}}$ and $\mathcal{F} \hookrightarrow \mathcal{F}^{\text{soft}}$ is isomorphic to the identity map in a neighborhood of $\text{Sing}(\mathcal{F}) \setminus \text{Sing}^{\text{soft}}(\mathcal{F})$;

(3) the soft colength of $\mathcal{F}$ is $\ell^{\text{soft}}(\mathcal{F}) := \ell(\mathcal{F}^{\text{soft}}/\mathcal{F})$.

**Definition 3.4.15.** A torsion free sheaf $\mathcal{F}$ on $X$ is softly reflexive if the map $\mathcal{F} \to \mathcal{F}^{\text{soft}}$ is an isomorphism.

Given a field $L \subset k$ and a sheaf $\mathcal{G}$ on $X \otimes L$, the length of $\mathcal{G}$ will always mean the length of $\mathcal{G} \otimes \bar{L}$ for any chosen algebraic closure of $L$. Thus, when we speak of the colength of the fiber of a family of torsion free sheaves, we mean the colength of the geometric fiber.

**Proposition 3.4.16.** Suppose $\mathcal{F}$ is a $T$-flat family of torsion free coherent sheaves on $\mathcal{X} \times T$. The function $t \mapsto \ell^{\text{soft}}(\mathcal{F}_t)$ is constructible.

**Proof.** It suffices to prove the following: assuming that $T$ is integral, we have that $\ell^{\text{soft}}(\mathcal{F}_0) = n$ if and only if there is a dense open subset $W \subset T$ such that $\ell^{\text{soft}}(\mathcal{F}_w) = n$ for all $w \in W$. 

The inclusion $\mathcal{F}_\eta \hookrightarrow \mathcal{F}_\eta^{\text{soft}}$ extends to a short exact sequence

$$0 \to \mathcal{F}_W \to \mathcal{F}_W' \to \mathcal{Q} \to 0$$

over some open subset $W$ of $T$, where $\mathcal{F}_W'$ has softly reflexive fibers. Moreover, shrinking $W$ if necessary, we may assume that $\mathcal{Q}$ is flat and quasi-finite over $W$. Finally, since the support of $\mathcal{Q}_\eta$ is contained in $X^{\text{sp}}_\eta$, we see that further shrinking $W$ we may suppose that $\text{Supp} \, \mathcal{Q} \subseteq X^{\text{sp}} \times W$. By Zariski’s Main Theorem, after shrinking $W$ one more time we may assume that $\mathcal{Q}$ is finite over $W$. It follows that each fiber of the exact sequence computes the soft hull of the fibers of $\mathcal{F}$, and the degree of $\mathcal{Q}$ over $W$ computes the (constant) colength of the (geometric) fibers.

**Definition 3.4.17.** Given a $T$-flat family $\mathcal{F}$ of torsion free coherent $\mathcal{X}$-twisted sheaves parametrized by $T$, the soft boundary of the family is

$$\partial^{\text{soft}} T := \{ t \in T \mid \ell^{\text{soft}}(\mathcal{F}_t) > 0 \}.$$

By Proposition 3.4.16, we know that $\partial^{\text{soft}} T$ is a locally constructible subset of $T$. Given a quasi-compact open subscheme $T' \subset T$, we can thus stratify $\partial^{\text{soft}} T'$ by locally closed subschemes. In particular, there will be a largest locally closed stratum, $(\partial^{\text{soft}} T')^{\text{dom}}$.

**Lemma 3.4.18.** Suppose $T$ is locally Noetherian. With the preceding notation, if $\partial^{\text{soft}} T \neq \emptyset$, then for all quasi-compact open subschemes $T' \subset T$ we have $\text{codim}(\partial^{\text{soft}} T')^{\text{dom}}, T') \leq r - 1$. Equivalently, if $t \in \partial^{\text{soft}} T$ is a minimal point then $\dim \partial T, t \leq r - 1$.

**Proof.** A standard argument (e.g., Lemma 9.2.1 of [4]) shows that

$$D := \{(t, u) \mid \mathcal{F}(t, u) \text{ is not locally free}\}$$

has codimension at most $r + 1$ in $X^{\text{sp}} \times T$. Moreover, $D \to T$ is quasi-finite and of finite type. It follows that the minimal points of the image all have codimension at most $r + 1 - 2 = r - 1$, as desired.

### 3.5. Soft Quot schemes

Let $\mathcal{F}$ be a softly reflexive torsion free $\mathcal{X}$-twisted sheaf.

**Definition 3.5.1.** The soft Quot space of length $\ell$ quotients of $\mathcal{F}$, denoted $\text{Quot}^{\text{soft}}(\mathcal{F}, \ell)$, is the algebraic space parametrizing quotients $\mathcal{F} \to \mathcal{Q}$ with $\mathcal{Q}$ a coherent twisted sheaf of length $\ell$ with support contained in $X^{\text{sp}}$.

It is a standard application of Artin’s theorem that this functor is an algebraic space. (The reader can see, e.g., [16] for a proof.) Since we constrain the support of the quotient to the spatial locus of $X$, we cannot hope that $\text{Quot}^{\text{soft}}$ is proper. However, we do have the following irreducibility result.

**Proposition 3.5.2.** The space $\text{Quot}^{\text{soft}}(\mathcal{F}, \ell)$ is irreducible of dimension $\ell(rk \mathcal{F} + 1)$.
**Proof.** The proof is exactly analogous to the proof of Lemma 2.2.7.28 of [12], and reduces to the proof of the classical case by Ellingsrud and Lehn [3]. □

4. Stability and boundedness for twisted sheaves on $X$

4.1. Slopes, $\mu$-stability, and the Segre invariant

Let $H$ be a fixed ample divisor on the coarse space $\overline{X}$. (We remind the reader that we denote the natural map $X \to \overline{X}$ by $\sigma$.)

**Definition 4.1.1.** Let $F$ be a perfect complex on $\mathcal{X}$ with positive rank $r$. The slope of $F$ (with respect to $H$) is

$$\mu(F) = \frac{H \cdot \overline{c}_1(F)}{r}.$$  

It is easy to see that the slope of $F$ is a rational number with denominator dividing $rN$. In particular, any bounded set of differences of slopes of perfect complexes on $\mathcal{X}$ has a minimal element.

**Lemma 4.1.2.** If $F$ is a perfect complex on $\mathcal{X} \times T$ with $T$ a connected scheme then for all pairs of geometric points $s, t \to T$, the slopes $\mu(F_t)$ and $\mu(F_s)$ are equal.

**Proof.** The proof is straightforward and comes down to showing that intersections of invertible sheaves are constant in a connected family. We refer the reader to Proposition 2.2.7.22 of [12] for an analogous proof with details. □

**Lemma 4.1.3.** If $\mathcal{E}$ is a coherent sheaf on $X$ with Hilbert polynomial

$$P(m) = \chi(X, \mathcal{E}(m)) = \alpha_2(\mathcal{E}) \frac{m^2}{2} + \alpha_1(\mathcal{E}) m + \alpha_0(\mathcal{E})$$

then

$$\deg \mathcal{E} = \alpha_1(\mathcal{E}) - \alpha_1(\mathcal{O}_X) \text{rk } \mathcal{E}.$$  

In particular, the slope of $\sigma_* \mathcal{E}$ equals the slope of $\mathcal{E}$.

**Proof.** We remind the reader that the slope of a coherent sheaf on $\overline{X}$ is computed as in Definition 1.2.11 of [4], using the coefficients of the Hilbert polynomial. (The formula is the one indicated as the conclusion of the lemma.)  

This follows immediately from the Toën–Grothendieck–Riemann–Roch formula. □

**Definition 4.1.4.** A torsion free sheaf $E$ on $\mathcal{X}$ is $\mu$-stable if $\mu(F) < \mu(E)$ for all subsheaves $F \subset E$ such that $0 < \text{rk } F < \text{rk } E$. The sheaf $E$ is $\mu$-semistable if $\mu(F) \leq \mu(E)$ for all non-zero subsheaves $F \subset E$.  


When $E$ is an $\mathcal{X}$-twisted sheaf, every subsheaf will also be $\mathcal{X}$-twisted, so the notion of stability for a twisted sheaf depends purely on the category of $\mathcal{X}$-twisted sheaves. In addition, it is standard that to verify semistability, it suffices to check that the slope does not increase among the same set of subsheaves as is used for checking stability.

One can measure the degree of stability of a sheaf using the following invariant.

**Definition 4.1.5.** Given a torsion free sheaf $E$ of rank at least 2 on $\mathcal{X}$, the Segre invariant of $E$, denoted $s(E)$, is the minimal value of $\mu(E_2) - \mu(E_1)$, where $E_1$ and $E_2$ are torsion free sheaves fitting into an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0.$$ 

The existence and uniqueness of the maximal destabilizing subsheaf (by a proof formally identical to that in [4]) imply that the set of such differences is in fact bounded and therefore there is a minimal such value (as implicitly asserted in the definition).

The Segre invariant, originally defined and studied by Segre and Nagata, was systematically studied by Langer in the context of moduli of sheaves [8]. He observed that the Segre invariant provides a satisfactory replacement for the notion of $\varepsilon$-stability used by O'Grady [15] when one wishes to study moduli in a characteristic-agnostic manner. Very positive Segre invariants indicate a high degree of stability, while very negative Segre invariants indicate a high degree of instability.

The following results will be useful to us; their proofs are formally identical to those in Corollary 2.3 and Lemma 2.4 of [6], so we omit them here. (The fact that we work with sheaves on stacks and Langer works with sheaves on varieties should not cause alarm – only the formal properties of the abelian categories of sheaves and the intersection theory play a role in the proofs.)

**Lemma 4.1.6.** Let $E$ be a torsion free sheaf on $\mathcal{X}$ of rank at least 2, and assume that $s(E)$ is realized by an exact sequence $0 \to E_1 \to E \to E_2 \to 0$.

1. If $E$ is $\mu$-semistable then both $E_1$ and $E_2$ are $\mu$-semistable.
2. If $E$ is $\mu$-stable then both $E_1$ and $E_2$ are $\mu$-stable.

The following is proven more generally for higher-dimensional varieties in Lemma 2.4 of [6], but we will only need it for surfaces.

**Lemma 4.1.7.** Suppose $E$ is a torsion free sheaf on $\mathcal{X}$ of rank at least 2. Let $D \subset X$ be a normal irreducible curve such that $E|_D$ is locally free. If $F$ is an elementary transformation of $E$ along a quotient of $E|_D$ then $s(E) \leq s(F) + D \cdot H$.

The following is referred to by Langer without proof. We give a proof here.

**Lemma 4.1.8.** Let $s \in \mathbb{Q}$ and let $F$ be a flat family of torsion free sheaves on $\mathcal{X} \times T$. There is a closed subset $T(s) \subset T$ parametrizing the geometric points $t$ such that $s(F_t) \leq s$.

**Proof.** First, we argue that the set is locally constructible; we may assume that $T$ is Noetherian. Theorem 2.3.2 of [4] shows that the Harder–Narasimhan filtrations on the $F_t$ form a flat family
over some dense open subscheme $U \subset T$. It follows (by stratifying $T$) that there is a maximal value $m$ for the slope of a subsheaf of a (geometric) fiber of $F$, i.e., for all exact sequences

$$0 \to E_1 \to F_t \to E_2 \to 0$$

we have that $\mu(E_1) \leq m$. In this case, $\mu(E_2) - \mu(E_1) \geq \mu(E_2) - m$, so that when $s(E_t) \leq s$ we have $\mu(E_2) \leq s + m$. On the other hand, Grothendieck’s boundedness theorem (Theorem 1.7.9 of [4]) shows that the scheme $Q$ of flat families of quotients of $F$ whose fibers have slopes bounded above by $s + m$ is of finite type. On the scheme $Q$, we can look at the closed subscheme $Q' \subset Q$ parametrizing exact sequences as above with $\mu(E_2) - \mu(E_1) \leq s$ (noting that the difference being computed is a locally constant function on $Q$ with finite range). By Chevalley’s theorem, $T(s)$ is constructible.

On the other hand, it is clear that $T(s)$ is closed under specialization, as we can spread out (over a base dvr) any sequence giving an upper bound for the Segre invariant (and the resulting slopes are constant in the fibers over the base dvr). Thus, $T(s)$ is closed, as desired. □

There is one final basic lemma concerning endomorphisms of semistable sheaves which we will need in the sequel.

**Lemma 4.1.9.** If $\mathcal{F}$ is a $\mu$-semistable $\mathcal{X}$-twisted sheaf of rank $r$ then $\dim \text{Hom}(\mathcal{F}, \mathcal{F}) \leq r^2$.

**Proof.** Any endomorphism of $\mathcal{F}$ must preserve the socle $\text{Soc}(\mathcal{F}) \subset \mathcal{F}$ (see Lemma 1.5.5ff of [4]); moreover, the quotient $\mathcal{F}/\text{Soc}(\mathcal{F})$ is also semistable. The result follows by induction from the polystable case, which itself follows immediately from the fact that stable sheaves are simple. □

**4.2. Discriminants and the Bogomolov inequality**

In this section we fix a uniformization $f : Y \to \mathcal{X}$ as in Section 3.2. Let $d$ be the degree of $Y$ over $X$ (not over $\mathcal{X}$—so $[Y : \mathcal{X}] = Nd$), and define $L_X = L_Y/d$, where $L_Y$ is the invariant defined by Langer in Section 2.1 of [8].

**Definition 4.2.1.** Given a coherent $\mathcal{X}$-twisted sheaf $\mathcal{E}$, the discriminant of $\mathcal{E}$, denoted $\Delta(\mathcal{E})$, is defined by

$$\Delta(\mathcal{E}) := 2 \text{rk}(\mathcal{E})c_2(\mathcal{E}) - (\text{rk} \mathcal{E} - 1)c_1^2(\mathcal{E}).$$

It is elementary that $\Delta(\mathcal{E}) = c_2(R\pi_* R\mathcal{H}om(\mathcal{E}, \mathcal{E}))$, and that the discriminant is invariant in a flat family of $\mathcal{X}$-twisted sheaves over a connected base.

**Notation 4.2.2.** Given $L \in \text{Pic}(\mathcal{X})$, we will write $\text{Tw}_{\mathcal{X}}^L(r, L, \Delta)$ for the stack of pairs $(\mathcal{F}, \psi)$, where $\mathcal{F}$ is a totally regular $\mu$-semistable $\mathcal{X}$-twisted sheaf of rank $r$ with discriminant $\Delta$ and $\psi : \text{det} \mathcal{F} \sim L$ is an isomorphism.
Lemma 4.2.3. If $E$ is totally regular of rank $r$ then

$$
\chi(E, E) := \chi(\mathcal{RHom}(E, E)) = -\Delta + r^2 \left( \chi(\mathcal{O}_X) - n \frac{N^2 - 1}{12N} \right).
$$

Proof. Since $\mathcal{RHom}(E, E)$ is totally regular with trivial determinant and rank $r^2$, Proposition 3.4.9 shows that

$$
\chi(E, E) = -\Delta + r^2 \deg \text{Td}_X.
$$

The formula now follows from Corollary 3.3.2.3. $\square$

As we will discuss below in Section 4.4, the dimension of $\mathcal{T}_X^\mu(L, \Delta)$ depends linearly upon $\Delta$.

Given a torsion free coherent $\mathcal{X}$-twisted sheaf $F$, the normalizations we have established show that $\mu(f^*F) = d\mu(F)$ and $\Delta(f^*F) = d\Delta(F)$.

Proposition 4.2.4 (Orbifold twisted Langer–Bogomolov inequality). If $E$ is a semistable $\mathcal{X}$-twisted sheaf then

$$
\Delta(E) \geq -\frac{L_X^2}{4H^2} \text{rk}(E)^2(\text{rk}(E) - 1).
$$

Proof. This follows immediately from the statement of Theorem 2.2 of [8] applied to $f^*E$, using the fact that $f^*F$ is semistable (see Lemma 4.3.1 below). $\square$

4.3. Moduli of $\mu$-semistable twisted sheaves

We retain the choice of uniformization $f : Y \to \mathcal{X}$ of Section 4.2.

In this section, we remark on the stack of semistable $\mathcal{X}$-twisted sheaves. Since we are not concerned with GIT quotients, we can consider the stack of all $\mu$-semistable sheaves and do not need to restrict our attention to an open sublocus of “Gieseker semistable sheaves.” The reader interested in such a thing can see Section 2.2.7.5 of [12] for one possible version of Gieseker stability in this context. While GIT does not apply to that definition of semistability, it will yield an open substack of the stack of semistable sheaves to which the present methods seem likely to extend, producing analogous results for that smaller moduli problem. The author has not carried out these calculations, so these statements should be taken with a (small) grain of salt.

The great advantage provided by considering only $\mu$-stability is that it affords an especially efficient development of the theory in the case at hand (i.e., a surface over a field). The reason for this is the following well-known lemma.

Lemma 4.3.1. A torsion free $\mathcal{X}$-twisted sheaf $\mathcal{F}$ is $\mu$-semistable if and only if $f^*\mathcal{F}$ is a $\mu$-semistable sheaf on $Y$.

Proof. The proof is identical to the proof of Lemma 3.2.2 of [4]. $\square$

The reader should note that the same is not true for $\mu$-stability.
This covering is useful for proving a basic algebraicity and boundedness statement. More general versions of this statement are true (cf. Section 2.3.2 of [12]), but this is all we will need in this paper.

**Lemma 4.3.2.** The stack of $\mu$-semistable $\mathcal{X}$-twisted sheaves with fixed $f$-Hilbert polynomial is of finite type over $k$.

**Proof.** This follows immediately from Lemma 3.2.2 and the corresponding fact for smooth projective surfaces (Theorem 4.2 of [7]).

**Proposition 4.3.3.** Given $L \in \text{Pic}(\mathcal{X})$, $r \geq 0$, and $\Delta \geq 0$, the stack $\text{Tw}^{ss}_{\mathcal{X}}(r, L, \Delta)$ an Artin stack of finite type over $k$.

**Proof.** Given $\mathcal{F}$, we know that $\Delta(f^*\mathcal{F}) = d\Delta(\mathcal{F})$. On the other hand, fixing the rank, determinant, and discriminant of a sheaf on $Y$ fixes its Hilbert polynomial (by appeal to the Riemann–Roch formula). Thus, the sheaves parametrized by $\text{Tw}^{ss}_{\mathcal{X}}(r, L, \Delta)$ have constant $f$-Hilbert polynomial. The result follows from Lemma 4.3.2.

An alternative approach to proving the results of Proposition 4.3.3 in the relative case (where a finite flat uniformization $Y \rightarrow \mathcal{X}$ may not exist) is given in Section 2.3 of [12]. The algebraicity is easily proven using Artin's theorem, while boundedness is a bit more complicated but not very difficult.

### 4.3.1. Pushing forward: openness of stability

We now turn to the relationship between stable and semistable sheaves. Again, in the case at hand there is a simple way to show that $\mu$-(semi)stability is open in a flat family. (For a proof in the relative case which uses different techniques and which is easily adaptable to the orbifold situation, the reader is again referred to Corollary 2.3.2.12 of [12].) We let $\text{Tw}_{\mathcal{X}}$ denote the stack of coherent $\mathcal{X}$-twisted sheaves.

Let $\mathcal{V}$ be a locally free $\mathcal{X}$-twisted sheaf of positive rank (for example, the reflexive hull of any coherent $\mathcal{X}$-twisted sheaf of positive rank, which will be locally free because $X$ is regular and 2-dimensional). Let $\mathcal{B} = \pi_*\sigma_*\mathcal{E}nd(\mathcal{V})$; this is a coherent sheaf of algebras on $\overline{X}$. An $\mathcal{X}$-twisted sheaf $\mathcal{F}$ gives rise to a right $\mathcal{B}$-module $M(\mathcal{F}) := \pi_*\sigma_*\mathcal{H}om(\mathcal{V}, \mathcal{F})$; this defines a map from $\text{Tw}_{\mathcal{X}}$ to the stack $\mathcal{M}_\mathcal{B}$ of $\mathcal{B}$-modules (since $\pi_*$ takes flat families to flat families, $X$ being tame). We will write this morphism as $M : \text{Tw}_{\mathcal{X}} \rightarrow \mathcal{M}_\mathcal{B}$.

Given a coherent sheaf $\mathcal{G}$ of $\mathcal{O}$-algebras (on either $X$ or $\overline{X}$), we will let $\text{Coh}_{2,1}(\mathcal{G})$ denote the Serre quotient of the category of coherent $\mathcal{G}$-modules by the subcategory consisting of modules supported in codimension 2. (The peculiar notation is meant to make this align with Section 1.6 of [4]. The reader looking for more details about this type of category is referred there.)

**Proposition 4.3.1.1.** A torsion free coherent $\mathcal{X}$-twisted sheaf $\mathcal{F}$ is $\mu$-(semi)stable if and only if $M(\mathcal{F})$ is a $\mu$-(semi)stable $\mathcal{B}$-module on $\overline{X}$.

**Proof.** First, it is easy to see that $\mathcal{F}$ is (semi)stable if and only if $\sigma_*\mathcal{H}om(\mathcal{V}, \mathcal{F})$ is a (semi)stable $\sigma_*\mathcal{E}nd(\mathcal{V})$-module. Thus, the content of the lemma is in the statement that $\pi_*$ reflects (semi)stability.
Let $\mathcal{A}$ be a locally free sheaf of $\mathcal{O}_X$-algebras with (reflexive) pushforward $\mathcal{B}$ on $\overline{X}$. Note that $X$ and $\overline{X}$ are isomorphic in codimension 1, which implies that the functor between Serre quotient categories

$$\text{Coh}_{2,1}(\mathcal{A}) \to \text{Coh}_{2,1}(\mathcal{B})$$

is an equivalence. Since $\sigma_*$ has vanishing higher direct images on the category of quasi-coherent sheaves, we know that for any $\mathcal{A}$-module $\mathcal{E}$ and any integer $m$, $\chi(X, \mathcal{E}(m)) = \chi(\overline{X}, \mathcal{E}(m))$. By Lemma 4.1.3, we see that the slope of $\mathcal{E}$ equals the slope of $\sigma_* \mathcal{E}$. We conclude that the notions of semistability coincide under pushforward, as they are defined by the same function on the same $K$-group.

**Proposition 4.3.1.2.** The substack $\text{Tw}_{\mathcal{X}}^s (r, L, \Delta) \subset \text{Tw}_{\mathcal{X}}^{ss} (r, L, \Delta)$ parametrizing $\mu$-stable twisted sheaves is open. Moreover, $\text{Tw}_{\mathcal{X}}^{ss} (r, L, \Delta)$ is an open substack of the stack of $\mathcal{X}$-twisted sheaves.

**Proof.** Lemma 3.7 of [18] shows that the substack $\mathcal{M}_{\mathcal{B}}^{ss}$ (respectively, $\mathcal{M}_{\mathcal{B}}^s$) of $\mu$-semistable (respectively $\mu$-stable) coherent $\mathcal{B}$-modules is open in $\mathcal{M}_{\mathcal{B}}$. Applying Proposition 4.3.1.1, we see that the substack of $\mathcal{X}$-parametrizing semistable $\mathcal{X}$-twisted sheaves is $\mathcal{M}^{-1} (\mathcal{M}_{\mathcal{B}}^{ss})$, and similarly for the stable loci.

In Sections 4.5 and 5, we will take up (among other things) the question of the density of $\text{Tw}_{\mathcal{X}}^s (r, L, \Delta)$ in $\text{Tw}_{\mathcal{X}}^{ss} (r, L, \Delta)$.

**4.3.2. Pushing forward, II: comparison of Harder–Narasimhan filtrations**

Let $\mathcal{E}$ be a torsion free sheaf on $X$. In the spirit of the last section, we compare properties of the Harder–Narasimhan filtration of $\mathcal{E}$ with those of the Harder–Narasimhan filtration of $\sigma_* \mathcal{E}$.

Recall that the Hilbert polynomial gives rise to a definition of stability for sheaves on the normal variety $\overline{X}$ (the coarse moduli space of $X$), and a similar notion of slope stability (by paying attention to only the first two coefficients of the reduced Hilbert polynomial).

**Lemma 4.3.2.1.** We have $\mu(\mathcal{E}) = \mu(\sigma_* \mathcal{E})$.

**Proof.** This follows from Lemma 4.1.3.

**Lemma 4.3.2.2.** A sheaf $\mathcal{E}$ is slope stable if and only if $\sigma_* \mathcal{E}$ is slope stable.

**Proof.** This follows immediately from Lemma 4.3.2.1 and the fact that slope-stability is determined in codimension 1, where $X$ and $\overline{X}$ are isomorphic.

**Proposition 4.3.2.3.** If $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$ is the Harder–Narasimhan filtration of $\mathcal{E}$ (with respect to slope stability), then $(\sigma_* \mathcal{E})_1 := \sigma_* (\mathcal{E}_1)$ defines the Harder–Narasimhan filtration on $\sigma_* \mathcal{E}$ (with respect to slope stability). Moreover, the slopes of the subquotients are the same.

**Proof.** This follows from the fact that $\sigma_*$ sends torsion free sheaves to torsion free sheaves condition along with Lemmas 4.3.2.2 and 4.3.2.1.
4.4. Dimension estimates at points of $\text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta)$

Let $\mathcal{E}$ be a semistable $\mathcal{X}$-twisted sheaf of rank $r$, determinant $L$, and discriminant $\Delta$. Letting $\text{Ext}^i(\mathcal{E}, \mathcal{E})_0$ denote the kernel of the trace map $\text{Ext}^i(\mathcal{E}, \mathcal{E}) \to H^i(X, \mathcal{O})$, it is standard that the miniversal deformation of $\mathcal{E}$ has dimension $d$ satisfying

$$\dim \text{Ext}^1(\mathcal{E}, \mathcal{E})_0 - \dim \text{Ext}^2(\mathcal{E}, \mathcal{E})_0 \leq d \leq \dim \text{Ext}^1(\mathcal{E}, \mathcal{E})_0$$

(see, e.g., Proposition 2.A.11 of [4] or the classic [17]). Thus, a lower bound for $d$ is provided by $\chi(X, \mathcal{O}) - \chi(\mathcal{E}, \mathcal{E}) + \dim \text{Hom}(\mathcal{E}, \mathcal{E})_0 \geq \chi(X, \mathcal{O}) - \chi(\mathcal{E}, \mathcal{E})$ (with equality holding when $\mathcal{E}$ is simple). Now, we know that the dimension of $\text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta)$ at $\mathcal{E}$ will be at least $d - \dim \text{Aut}(\mathcal{E}) + 1 \geq d - r^2 + 1$ (by Lemma 4.1.9), so that we conclude that

$$\dim_{[\mathcal{E}]} \text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta) \geq \chi(X, \mathcal{O}_X) - \chi(\mathcal{E}, \mathcal{E}) - r^2 + 1.$$  \hspace{1em} (4.4.0.1)

Using the Riemann–Roch formula, we find the following.

**Lemma 4.4.1.** Let $\mathcal{E}$ be a totally regular semistable $\mathcal{X}$-twisted sheaf of rank $r$, determinant $L$, and discriminant $\Delta$. There is a lower bound

$$\dim_{[\mathcal{E}]} \text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta) \geq \Delta - (r^2 - 1)\chi(X, \mathcal{O}_X) + r^2 n \frac{N^2 - 1}{12N} - r^2 + 1.$$

**Proof.** This results from combining Lemma 4.2.3 and Eq. (4.4.0.1) above. \hfill \Box

A similar argument, using the fact that stable sheaves are simple, shows that for any stable sheaf $\mathcal{E}$ there is an inequality

$$\Delta - (r^2 - 1)\chi(X, \mathcal{O}_X) + r^2 n \frac{N^2 - 1}{12N} \leq \dim_{[\mathcal{E}]} \text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta) \leq \Delta - (r^2 - 1)\chi(X, \mathcal{O}_X) + r^2 n \frac{N^2 - 1}{12N} + \dim \text{Ext}^2(\mathcal{E}, \mathcal{E})_0.$$

**Definition 4.4.2.** Given $r$, $L$, and $\Delta$, the expected dimension of $\text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta)$ is

$$\exp \dim \text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta) := \Delta - (r^2 - 1)\chi(X, \mathcal{O}_X) + r^2 n \frac{N^2 - 1}{12N}.$$  \hspace{1em} (4.4.2.1)

We see that for a stable $\mathcal{X}$-twisted sheaf $\mathcal{E}$ we have

$$\exp \dim \text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta) \leq \dim_{[\mathcal{E}]} \text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta) \leq \exp \dim \text{Tw}_{\mathcal{X}}^{s}(r, L, \Delta) + \dim \text{Ext}^2(\mathcal{E}, \mathcal{E})_0.$$  \hspace{1em} (4.4.2.1)

4.5. Dimensions of Segre loci

In this section we follow ideas established by O’Grady and Langer and produce bounds on the set of sheaves whose Segre invariant has a fixed upper bound.
4.5.1. An estimate

We use the function \( f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Q} \) from Section 3.3.2. The sequence \( f(0), f(1), \ldots, f(N - 1) \) gives rise to a (symmetric) circulant matrix \( (c_{ij}) = (\frac{1}{N} f(j - i)) \). It is well known that \( (c_{ij}) \) is diagonalizable (over \( \mathbb{R} \)) with eigenvalues \( \epsilon_m = \sum_{i=0}^{N-1} \frac{1}{N} f(i) \zeta^m i \).

**Lemma 4.5.1.1.** The eigenvalues \( \epsilon_m \) described above are all non-negative.

**Proof.** First, a preliminary reduction. If \( m = 0 \), this is a calculation which is left to the reader. For \( m > 0 \), the sum is taken over powers of a root of unity \( z \) such that \( z^d = 1 \) for some \( d \) dividing \( N \).

The sum simplifies to

\[
\epsilon_m = \sum_{i=0}^{N-1} \frac{i(i - N)}{2N} z^i = \frac{1}{2N} \left( \sum i^2 z^i - N \sum iz^i \right).
\]

It thus suffices to evaluate the exponential sums \( \sum i^r z^i \) for \( r = 1 \) and \( r = 2 \).

We recall the standard power sums

\[
\sum_{j=1}^{N-1} j x^j = \frac{(N - 1)x^{N+1} - Nx^N + x}{(x - 1)^2}
\]

and

\[
\sum_{j=1}^{N-1} j^2 x^j = \frac{(N - 1)^2 x^{N+2} + (-2N^2 + 2N + 1)x^{N+1} + N^2 x^N - x^2 - x}{(x - 1)^3}
\]

which can be derived by differentiating the standard geometric sum \( \sum_{j=1}^{N-1} x^j = (x^N - 1)/(x - 1) \). Using the fact that \( z^d = 1 \), we find that

\[
\sum_i iz^i = \frac{N}{z - 1}
\]

and

\[
\sum_i i^2 z^i = \frac{N(N - 2)z^2 + 2N(1 - N)z}{(z - 1)^3}.
\]

A simple calculation yields

\[
\epsilon_m = \frac{-2z}{(z - 1)^2},
\]

which we wish to show is positive. (The dependence on \( m \) is hidden in the choice of \( z \).) It is elementary to see that the argument of \( z/(z - 1)^2 \) is \( \pi \), which establishes the result. (E.g., one can easily show that it suffices to calculate the argument for any primitive such root of unity, as
the corresponding sums for different choices of \( z \) differ by positive real scalars; for \( z = e^{2\pi i/d} \), the calculation is an exercise in plane geometry, at least when \( d > 4 \). \( \square \)

In other words, the matrix \((c_{ij})\) is positive semidefinite.

Now suppose \( F \) is a perfect complex on \( X \). Using the Toën–Riemann–Roch formula, we can write

\[
\chi(F) = \deg(ch(F) \cdot Td_X) + \sum_{i=1}^{n} \delta_i(F),
\]

where \( \delta_i(F) \) is a correction term (a priori lying in \( \mathbb{Q}(\mu_N) \), but actually lying in \( \mathbb{Q} \)) coming from contributions at the residual gerbe \( \xi_i \). Write \( [L_i^*F] = \sum_{j=0}^{N-1} e_j^{(i)} \chi_j \) for the class in \( K \)-theory of the derived fiber of \( F \) over \( \xi_i \).

In this section, we will be interested in computing the correction terms to the Riemann–Roch formula for particular perfect complexes \( F \) arising as follows: let \( E \) be a totally regular twisted sheaf of rank \( r \) and

\[
0 \to E_1 \to E \to E_2 \to 0
\]

a non-trivial exact sequence of torsion free sheaves. Let \( F = R\mathcal{H}om(E_1, E_2) \).

**Proposition 4.5.1.2.** Using the above notation, for any exact sequence (4.5.1.1.1) and any \( i = 1, \ldots, n \) we have that \( \delta_i(F) \leq 0 \).

**Proof.** We work at a fixed residual gerbe \( \xi_i \) and omit \( i \) from the notation. Using the notation of Section 3.4, for a sheaf \( G \) we will write \( \overline{G} \) for \( \overline{G_i} \). Let \( \overline{E_1} = \sum d_i \chi_i =: P \), and let \( d \) be the column vector \((d_i)\). Let \( s = \sum d_i < N \) be the rank of \( E_1 \). We claim that

\[
\delta(F) = -d^T(c_{ij})d + \frac{s}{N} \sum_{j=0}^{N-1} f(j).
\]

Then, since \((c_{ij})\) is positive semidefinite, it follows that \( \delta(F) \) is bounded by \((s/N) \sum f(j) = 0 \).

To prove the claim, note that there is an equality \( \overline{E_2} = \rho - P \) in \( K(\mu_N) \), so that \( F = P^\vee \otimes (\rho - P) = r\rho - P^\vee \otimes P \). We can write \( P^\vee \otimes P = \sum_{i,j} d_i d_{i+j} \chi_j \). Thus, since \( \chi(F) \) depends only on the class of \( F \) in \( K \)-theory, the correction term becomes

\[
\delta(F) = \frac{s}{N} \sum f(j) - \sum_{j=0}^{N-1} \left( \frac{1}{N} f(j) \sum_{i=0}^{N-1} d_i d_{i+j} \right).
\]

The second sum in this expression is a quadratic form in the \( d_i \) in which the coefficient of \( d_a d_b \) is given by \(-\frac{1}{N} f(b-a) - \frac{1}{N} f(a-b) = -\frac{2}{N} f(b-a) \) when \( a \neq b \) and by \(-\frac{N^2 - 1}{12N} = -f(0) \) when \( a = b \). The matrix corresponding to this form is precisely \(- (c_{ij})\), as desired. \( \square \)
4.5.2. Flag spaces of semistable twisted sheaves

We recall a few basic facts on dimensions of substacks of the stack of semistable twisted sheaves which admit a specified flag structure. We have followed §3.7 of [8] and Appendix 2.A of [4] in our presentation, with a few modifications appropriate to the situation at hand.

Given rational polynomials $P_1, \ldots, P_k$, let $\text{Filt}(P_1, \ldots, P_k)$ denote the stack of filtered $\mathcal{X}$-twisted sheaves $F_0 \mathcal{E} \subset F_1 \mathcal{E} \subset \cdots \subset F_k \mathcal{E} = \mathcal{E}$ such that the $f$-Hilbert polynomial of $\mathcal{E}_i := F_i \mathcal{E} / F_{i-1} \mathcal{E}$ is $P_i$ for $i = 1, \ldots, k$. The map $F_i \mathcal{E} \mapsto \mathcal{E} / F_0 \mathcal{E}$ defines a morphism of Artin stacks $q : \text{Filt}(P_1, \ldots, P_k) \to \text{Tw}_{\mathcal{X}}$.

Lemma 4.5.2.1. The reduced structure on the image of $q$ is a closed substack of $\text{Tw}_{\mathcal{X}}$.

Proof. It is enough to check this after pulling back to a smooth cover of $\text{Tw}_{\mathcal{X}}$, so this reduces to the following: if $\mathcal{E}$ is a $S$-flat coherent $\mathcal{X} \times S$-twisted sheaf with torsion free fibers, then the locus parametrizing sheaves admitting a flag of the prescribed type is locally closed in $S$. We proceed by induction on $k$, the case when $k = 0$ following from the fact that the geometric Hilbert polynomial is constant in a flat family. We know that $Q := \text{Quot}(\mathcal{E}, P_k)$ is proper over $S$ by Corollary 3.2.7. Over $Q$, there is a two-step filtration $F_k \mathcal{E}_Q \subset \mathcal{E}_Q$ with quotient having geometric Hilbert polynomial $P_k$. By induction on $k$, the subset of $Q$ containing the rest of the flag is closed; since $Q$ is proper over $S$, the image is closed, as desired.

We will (temporarily) denote the image of $q$ by $\text{Tw}_{\mathcal{X}}(P_1, \ldots, P_k)$ (following Langer).

Proposition 4.5.2.2. The dimension of $\text{Tw}_{\mathcal{X}}(P_1, \ldots, P_k)$ at a point $\mathcal{E}$ admitting a filtration $0 = F_0 \mathcal{E} \subset F_1 \mathcal{E} \subset \cdots \subset F_k \mathcal{E} = \mathcal{E}$ is bounded above by

$$\sum_{i \geq j} \dim \text{Ext}^1(\mathcal{E}_i, \mathcal{E}_j) + \dim \text{Aut}(\mathcal{E}) - 1,$$

where $\mathcal{E}_i := F_i \mathcal{E} / F_{i-1} \mathcal{E}$. If $\mathcal{E}$ is semistable of rank $r$ then $\dim \text{Aut}(\mathcal{E}) \leq r^2$.

Proof. Fix a locally free $\mathcal{X}$-twisted sheaf $\mathcal{V}$. We can also realize the stack $\text{Tw}_{\mathcal{X}}$ as the stack $\text{Mod}_{\mathcal{A}}$ of coherent right $\mathcal{A}$-modules (by sending $\mathcal{E}$ to $\pi_* \mathcal{H}om(\mathcal{V}, \mathcal{E})$), where $\mathcal{A} = \pi_* \mathcal{E} \text{nd}(\mathcal{V})$ is an Azumaya algebra on $X$. Given an $\mathcal{A}$-module $\mathcal{F}$, we also have the usual Hilbert polynomial $P_{\mathcal{F}}(m) = \chi(X, F(m))$, where we twist by a fixed polarization of the coarse moduli space of $X$; as usual, for large values of $m$ we have $P_{\mathcal{F}}(m) = H^0(X, F(m))$ (as Serre’s theorem applies).

The stack $\text{Tw}_{\mathcal{X}}$ is a union of open substacks $\text{Tw}_{\mathcal{X}}^{(m)}$ consisting of twisted sheaves $\mathcal{E}$ such that $\mathcal{H}om(\mathcal{V}, \mathcal{E})$ is $m$-regular in the sense of Mumford. We can realize any such $\mathcal{E}$ as a quotient of

$$\text{Hom}(\mathcal{V}, \mathcal{E}(m)) \otimes_k \mathcal{V}(-m).$$

Thus, if we fix the Hilbert polynomial $P_{\mathcal{H}om(\mathcal{V}, \mathcal{E})}$ we get a quasi-compact stack. It follows that the stack $\text{Filt}(P_1, \ldots, P_k)^{(m)}$ of filtrations on $m$-regular twisted sheaves breaks up as a finite disjoint union of corresponding stacks of filtered $\mathcal{A}$-modules (over a finite collection of sequences of Hilbert polynomials for the subquotient $\mathcal{A}$-modules), each of which is open and closed in $\text{Filt}(P_1, \ldots, P_k)$. 


Thus, to estimate the dimension of $\text{Tw}_{\mathcal{X}}(P_1, \ldots, P_k)$, we can also assume that the filtrations in question have fixed $\mathcal{A}$-Hilbert polynomials $Q_1, \ldots, Q_k$, and that the ambient sheaves have Castelnuovo–Mumford regularity at most $m$. Let $\mathcal{H} = \mathcal{A}(-m)\mathcal{Q}_{\text{com}(\mathcal{X}, \mathcal{E})(m)}$. Standard methods (just as in Proposition 3.8 of [8]) show that the flag space $Y$ parametrizing $\mathcal{A}$-module filtrations $G_0 \mathcal{H} \subset \cdots \subset G_k \mathcal{H} = \mathcal{H}$ such that the Hilbert polynomials of the subquotients are $P_i$ and the $\mathcal{A}$-Hilbert polynomials are $Q_i$ has dimension at most

$$\sum_{i \geq j} \dim \text{Ext}^1(\mathcal{E}_i, \mathcal{E}_j) + \dim \text{End}(\mathcal{H}) - 1.$$ 

On the other hand, there is a map $\alpha : Y \to \text{Tw}_{\mathcal{X}}$ sending a filtration to $\mathcal{H}/G_0 \mathcal{H}$. This morphism obviously factors through the map $\beta : Y \to \mathcal{Q} := \text{Quot}(\mathcal{V}(-m)\mathcal{Q}_{\text{com}(\mathcal{X}, \mathcal{E})(m)}, \mathcal{P})$; moreover, the image of $\beta$ is the preimage of the image of $\alpha$. Thus, to get an upper bound on the dimension of $\text{Tw}_{\mathcal{X}}(P_1, \ldots, P_k)$ at $\mathcal{E}$, it suffices to subtract from $\dim Y$ a lower bound for the dimension of the fibers of $Q \to \text{Tw}_{\mathcal{X}}$ near $\mathcal{E}$. Elementary considerations show that the fiber dimension at $\mathcal{E}$ is given by $\dim \text{End}_{\mathcal{A}}(\mathcal{H}) - \dim \text{Aut}(\mathcal{E})$, which gives the desired result.

If $\mathcal{E}$ is semistable, then $\dim \text{Hom}(\mathcal{E}, \mathcal{E}) \leq r^2$ by 4.1.9, from which it follows that $\dim \text{Aut}(\mathcal{E}) \leq r^2$. \hfill \Box

4.5.3. The dimension estimate

This section is again very close to the approach taken by Langer in Section 7 of [8]. We recall the principal results and give proofs only when they are different from those of Langer. Following Langer, define $f(r) = -1 + \sum_{i=1}^{r} \frac{1}{i}$. We retain the notation from Section 4.2. Given a torsion free coherent $\mathcal{X}$-twisted sheaf $F$, it follows from Lemma 4.3.1 that the Harder–Narasimhan filtration on $F$ pulls back to the Harder–Narasimhan filtration on $f^* F$. The calculations of Section 4.2 show that $\mu_{\text{max}}(f^* F) = d \mu_{\text{max}}(F)$.

**Lemma 4.5.3.1.** Let $F_1$ and $F_2$ be torsion free slope-semistable coherent $\mathcal{X}$-twisted sheaves such that the rank and slope of $F_i$ are $r_i$ and $\mu_i$, respectively. There is an inequality

$$\mu_{\text{max}}(\mathcal{H}\text{om}(F_1, F_2)) \leq \mu_2 - \mu_1 + (r_1 + r_2 - 2)L_X.$$ 

**Proof.** It is clear that we can replace $F_1$ and $F_2$ by their reflexive hulls and thus assume that they are locally free. Thus, we may assume that the formation of $\mathcal{H}\text{om}(F_1, F_2)$ commutes with pullback to $Y$. Taking into account the fact that $L_Y = dL_X$ and similarly for the slopes in question, the fact to be proved follows immediately from the corresponding fact for sheaves on smooth projective surfaces. This is contained in the proof of Corollary 4.2 of [8] (and is a short deduction from Eq. (2.1.1) of [8]). \hfill \Box

**Lemma 4.5.3.2.** Let $E_1$ and $E_2$ be torsion free slope-semistable $\mathcal{X}$-twisted sheaves. Let $r_i$ and $\mu_i$ denote the rank and slope of $E_i$. If $\mu_1 > \mu_2$ then $\text{Hom}(E_1, E_2) = 0$. Otherwise, we have

$$\dim \text{Hom}(E_1, E_2) \leq \frac{r_1 r_2}{2H^2} \left( \mu_2 - \mu_1 + \frac{3}{2}H^2 + (r_1 + r_2 - 2)L_X \right)^2 + \frac{r_1 r_2}{8} \left( 2f(r_1 r_2) - 9 \right)H^2.$$
Proof. The proof is identical to the proof of Corollary 4.2 in [8], using Lemma 4.5.3.1 to bound

\[ \mu_{\text{max}}(\mathcal{H}om(E_1, E_2)) = \mu_{\text{max}}(\sigma_\ast \mathcal{H}om(E_1, E_2)) \]

and Theorem 4.1 of [8] to bound \( H^0(\overline{X}, \sigma_\ast \pi_\ast \mathcal{H}om(E_1, E_2)) \). \( \square \)

**Corollary 4.5.3.3.** There exists a constant \( \beta_\infty \) depending only on \( X \) and \( r \) such that for any semistable \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) of rank \( r \) invertible in \( k \) we have

\[ \dim \text{Ext}^2(\mathcal{F}, \mathcal{F})_0 \leq \beta_\infty. \]

**Proof.** This follows immediately from Lemma 4.5.3.2 with \( E_1 = \mathcal{F} \) and \( E_2 = \mathcal{F} \otimes \omega_X \). \( \square \)

**Corollary 4.5.3.4.** There exists a constant \( \beta(r) \) depending only on \( X \) and \( r \) such that for any semistable \( \mathcal{X} \)-twisted sheaf \( \mathcal{E} \) of rank \( r \) and determinant \( L \), we have

\[ \dim_\mathcal{E} \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta) \leq \Delta + \beta(r). \]

**Proof.** The proof is identical to the proof of Lemma 8.5 of [8]. \( \square \)

**Corollary 4.5.3.5.** We have \( \dim \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta) \leq \exp \dim \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta) + \beta_\infty. \)

**Proof.** This follows from Corollary 4.5.3.3 and Eq. (4.4.2.1). \( \square \)

Using 4.5.3.2, we can now give an estimate of the dimension of the locus \( \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta)(s) \) parametrizing semistable \( \mathcal{X} \)-twisted sheaves with Segre invariant at most \( s \).

**Lemma 4.5.3.6.** Let \( E \) be a slope semistable \( \mathcal{X} \)-twisted sheaf of rank \( r \). Let the Segre invariant \( s(E) \) be realized by an exact sequence

\[ 0 \to E_1 \to E \to E_2 \to 0, \]

and let \( r_i := \text{rk } E_i \). There exists a constant \( \alpha(r) \) depending only on \( X, \mathcal{X}, Y, H \) and \( r \) such that

\[ \sum_{i \geq j} \dim \text{Ext}^1(E_i, E_j) \leq \frac{\max(r_1, r_2) + r}{2r} \left( \Delta(E) + \frac{r_1 r_2}{H^2} s^2 \right) + \frac{r_1 r_2}{2H^2} (3H^2 - KH + 2(r - 2)L_X) s + \alpha(r). \]

**Proof.** The reader should compare this with Proposition 7.1 of [8]. The proof is identical, with the following modifications: First, in place of Langer’s Bogomolov inequality (Theorem 2.2 of [8]) we use Proposition 4.2.4 (and the orbifold Hodge Index Theorem 3.1.3 above). In place of Langer’s Corollary 4.2 of [8], we use Lemma 4.5.3.2. The final modification requires that we explain the main outline of the proof.
We start with the equality
\[ \sum_{i \geq j} \dim \text{Ext}^1(E_i, E_j) = \sum_{i \geq j} \left( \dim \text{Hom}(E_i, E_j) + \dim (E_j \otimes \omega_X) \right) + \chi(E_1, E_2) - \chi(E, E). \]

Lemma 4.5.3.2 yields bounds for the terms in the sum on the right-hand side, so it remains to bound the difference \( \chi(E_1, E_2) - \chi(E, E) \), and this is the point where one must modify the proof. The following lemma plays a central role.

**Lemma 4.5.3.7.** Given a perfect complex \( F \) of coherent \( \mathcal{O}_X \)-modules of positive rank \( r \), we have
\[ \chi(F) = -\frac{\Delta(F)}{2r} + \frac{r}{2} (\xi - K_X) + \chi(\mathcal{O}_X) + c(F), \]
where \( \xi = c_1(F)/r \) and \( c(F) \) is a correction term depending only upon the classes \([L_i^* F]\) in \( K(B_{\mu N})\). If \( F = R\sigma_* \mathcal{H}om(E_1, E_2) \) with \( 0 \to E_1 \to E \to E_2 \to 0 \) an exact sequence of twisted sheaves as in (4.5.1.1) above, then \( c(F) \) is bounded in terms of \( X, N, n, \) and \( r \).

**Proof.** The Toën–Riemann–Roch formula shows that \( \chi(F) = \deg(\text{ch}(F) \cdot \text{Td}_X) + c'(F) \), where \( c'(F) \) is a correction term depending only upon the \([L_i^* F]\). Expanding out the intersection product in Chern classes and using Corollary 3.3.2.3 yields the first part of the lemma. The second follows from Proposition 4.5.1.2. \( \square \)

Let \( \xi = \tilde{c}_1(E_1)/r_1 - \tilde{c}_1(E_2)/r_2 \). The preceding lemma shows that
\[ \chi(E_1, E_2) - \chi(E, E) = \Delta(E) - \left( \frac{\Delta_1}{2r_1} + \frac{\Delta_2}{2r_2} \right) + \frac{1}{2} r_1 r_2 (\xi - K_X) \xi + (r_1 r_2 - r^2) \chi(\mathcal{O}_X) + c, \]
where \( c \) is a correction term which is bounded above in terms of \( X, N, n, \) and \( r \). The proof of Proposition 7.1 of [8] yields an upper bound on \( \chi(E_1, E_2) - \chi(E, E) - c \) (by following the arguments verbatim); since \( c \) is bounded, this yields a bound on \( \chi(E_1, E_2) - \chi(E, E) \) (and changes Langer’s constant \( a_2(r) \) by a constant). The rest is precisely as in the proof of [8]. \( \square \)

**Proposition 4.5.3.8.** There exists a constant \( B \) such that for any \( s \geq 0 \),
\[ \dim \text{Tw}^s_{\mathcal{X}}(r, L, \Delta)(s) \leq \left( 1 - \frac{1}{2r} \right) \Delta + \frac{2r^2}{9H^2} s^2 + \frac{r^2}{8H^2} \left[ 3H^2 - KH + 2(r - 2)L_X \right] s + B. \]

**Proof.** The proof is identical to the proof of Theorem 7.2 in [8], using Proposition 4.5.2.2 in place of Langer’s Proposition 3.8 and Lemma 4.5.3.6 in place of Langer’s Proposition 7.1 \( \square \)

5. **Asymptotic properties of moduli**

In this section we study what happens to the stack \( \text{Tw}^s_{\mathcal{X}}(r, L, \Delta) \) as \( \Delta \) goes to infinity.
5.1. Non-emptiness

Definition 5.1.1. Given a torsion free $\mathcal{E}$-twisted sheaf $\mathcal{V}$, the discrepancy of $\mathcal{V}$ is

$$\delta(\mathcal{V}) := \max_{\mathcal{V}' \subseteq \mathcal{V}} \{ \mu(\mathcal{V}') - \mu(\mathcal{V}) \},$$

where $\mathcal{V}' \subset \mathcal{V}$ ranges over proper subsheaves satisfying $0 < \text{rk} \mathcal{V}' < \text{rk} \mathcal{V}$.

It is clear that $\delta(\mathcal{V})$ has a denominator bounded by $N \text{rk}(\mathcal{V})!$. If $\mathcal{V}$ is unstable then $\delta(\mathcal{V}) = \mu_{\text{max}}(\mathcal{V}) - \mu(\mathcal{V})$, while $\mathcal{V}$ is stable if and only if $\delta(\mathcal{V}) < 0$. It is elementary that the discrepancy is invariant under twisting by an invertible sheaf.

Proposition 5.1.2. Suppose $\mathcal{V}$ is a torsion free totally regular $\mathcal{E}$-twisted sheaf of rank $r$ with determinant $L$. Given any $c \in \mathbb{Q}$, there exists $c > c_0$ and a subsheaf $\mathcal{W} \subset \mathcal{V}$ with rank $r$, determinant $L$ and $\deg c_2(\mathcal{V}) > c$ such that $\delta(\mathcal{W}) < \delta(\mathcal{V})$. In particular, there is such a subsheaf $\mathcal{W}$ which is $\mu$-stable.

Proof. This is similar to Theorem 5.2.5 of [4]. The reader will note that the proof of [4] uses the fact that the trivial bundle is semistable, a fact which is not available to us. The proof here shows that even if we start with an unstable ambient sheaf $\mathcal{V}$, we can at least ensure that the subsheaf will have smaller discrepancy. Since the discrepancies have bounded denominator, iterating this process will ultimately produce a stable sheaf.

A preliminary reduction: replacing $\mathcal{V}$ by $\mathcal{V}(-tH)$ for large $t$, we may assume that

$$\mu(\mathcal{V}) < \min\{0, (r - 1)\delta(\mathcal{V})\}.$$ 

Since the discrepancy of a sheaf is invariant under twisting, we can twist the resulting subsheaf of $\mathcal{V}(-tH)$ back up by $tH$ to achieve the desired result.

Let $C$ be a smooth curve belonging to the linear system $|rmH|$ for large $m$ which lies in $X^{\text{op}}$ and avoids the singular points of $\mathcal{V}$; write $\mathcal{E} := \mathcal{E} \times_X C$. A result of Grothendieck (Lemma 1.7.9 of [4]) shows that the space $\mathcal{Q}$ of locally free quotients $F$ of $\mathcal{V}|_{\mathcal{E}}$ with $\text{rk}(F) > r$ and $\mu(F) \leq \frac{1}{r}C \cdot H$ is quasi-compact. Let $\mathcal{M}$ be an invertible $\mathcal{E}$-twisted sheaf (which exists by Tsen’s theorem). Let $q : \operatorname{pr}_1^* \mathcal{V}|_{\mathcal{E}} \to \mathcal{F}$ on $\mathcal{E} \times Q$ be the universal quotient of $\mathcal{V}|_{\mathcal{E}}$ satisfying those two conditions. For large $s$ we know that $(\operatorname{pr}_2)_*(\mathcal{F}^\vee \otimes \operatorname{pr}_1^* \mathcal{M}(sH))$ is a locally free sheaf on $Q$; it is thus the sheaf of sections of some geometric vector bundle $\mathcal{V} \to Q$ whose fiber over a quotient $\mathcal{V}|_{\mathcal{E}} \to F$ is $\text{Hom}(F, \mathcal{M}(sH))$. The quotient map $q$ induces a map $\tilde{q} : \mathcal{V} \to \text{Hom}(\mathcal{V}|_{\mathcal{E}}, \mathcal{M}(sH))$ which is a linear embedding when restricted to each fiber over $Q$. Since $\text{rk}(\mathcal{F}) < r$, we see that for sufficiently large $s$ the map $\tilde{q}$ cannot be dominant. Thus, there exists some surjective map $\varphi : \mathcal{V} \to \mathcal{M}(sH)$ which does not factor through any quotient parametrized by $Q$.

We claim that the kernel $\mathcal{W}$ of one such map $\varphi : \mathcal{V} \to \mathcal{M}(sH)$ satisfies $\delta(\mathcal{W}) < \delta(\mathcal{V})$. Let us grant this for a moment. Standard computations show that $\text{det}(\mathcal{W}) = \text{det}(\mathcal{V})(-C)$, so that the sheaf $\mathcal{W}(-mH)$ will then have determinant $L$ and $\delta(\mathcal{W}) = \delta(\mathcal{W}(-mH)).$ Taking the kernel of a general quotient $\mathcal{W} \to \mathcal{Q}$ with $\mathcal{Q}$ of finite length will then make the second Chern class arbitrarily large.
Thus, it remains to establish the assertion about the discrepancy. Given a saturated subsheaf \( \mathcal{W}' \subset \mathcal{W} \), let \( \mathcal{V}' \subset \mathcal{V} \) be its saturation in \( \mathcal{V} \). The sheaf \( \mathcal{R} := \mathcal{V}' / \mathcal{W}' \) is a subsheaf of \( M(sH) \).

If \( \mathcal{R} \neq 0 \) then we have that
\[
\det (\mathcal{W}') \sim \det (\mathcal{V}') (-C),
\]
so that
\[
\mu (\mathcal{W}') = \mu (\mathcal{V}') - \frac{C \cdot H}{\text{rk}(\mathcal{W}')} < \delta (\mathcal{V}) + \mu (\mathcal{V}) - \frac{C \cdot H}{r} = \delta (\mathcal{W}) + \mu (\mathcal{W}),
\]
so that \( \mu (\mathcal{W}') - \mu (\mathcal{W}) < \delta (\mathcal{V}) \), as desired.

### Corollary 5.1.3
Given \( r, L \), and \( c_0 \), if there exists a totally regular torsion free \( \mathcal{X} \)-twisted sheaf \( \mathcal{V} \) of rank \( r \) and determinant \( L \) then \( \text{Tw}_{\mathcal{X}/k}^s (r, L, c) \neq \emptyset \) for some \( c > c_0 \). If in addition \( \mathcal{V} \) is locally free at the non-trivial residual gerbes of \( X \) then there is a point of \( \text{Tw}_{\mathcal{X}/k}^s (r, L, c) \) for some \( c > c_0 \) parametrizing a sheaf which is locally free at the non-trivial residual gerbes of \( X \).

**Proof.** The first statement follows directly from Proposition 5.1.2. The second follows from the proof: modifications are made along general curves and points, and thus do not change the sheaf on the (discrete) stacky locus of \( X \).

5.2. Shrinking of the boundary

Given a locally closed substack \( Z \subset \text{Tw}_{\mathcal{X}/k}^s (r, L, \Delta) \), we will write \( Z^s \) for the open substack \( Z \cap \text{Tw}_{\mathcal{X}/k}^s (r, L, \Delta) \).

**Proposition 5.2.1.** There exists a constant \( D \) such that for \( \Delta \geq D \), the open substack \( \text{Tw}_{\mathcal{X}/k}^s (r, L, \Delta) \subset \text{Tw}_{\mathcal{X}/k}^s (r, L, \Delta) \) is everywhere dense.

**Proof.** By definition, the complement \( \text{Tw}_{\mathcal{X}/k}^s (r, L, \Delta) \setminus \text{Tw}_{\mathcal{X}/k}^s (r, L, \Delta) (0) \) (the locus parametrizing semistable twisted sheaves with non-positive Segre invariant). By Proposition 4.5.3.8 we have that
\[
\dim \text{Tw}_{\mathcal{X}/k}^s (r, L, \Delta) \leq \left( 1 - \frac{1}{2r} \right) \Delta + B.
\]
On the other hand, we know from Lemma 4.4.1 that \( \dim \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta) \geq \Delta - (r^2 - 1) \chi(X, \mathcal{O}_X) + r^2 n^2 \frac{N^2 - 1}{12N} - r^2 + 1 \). When \( \Delta \) is sufficiently large the latter is strictly larger than the former, which implies the result.

**Hypothesis 5.2.2.** For the rest of this section, we will assume that \( \Delta \geq D \), so that every irreducible component of \( \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta) \) contains geometrically \( \mu \)-stable twisted sheaves.

Fix a positive integer \( n \) and let \( C \subset X_{\text{sp}} \) be a general curve in \( |\mathcal{O}_X(nH)| \). Write \( \mathcal{C} = \mathcal{X} \times_X C \).

**Proposition 5.2.3.** If \( Z \) is a closed irreducible substack of \( \text{Tw}_{\mathcal{X}}^{ss}/k(r, L, \Delta) \) which meets the stable locus such that \( \partial_{\text{soft}} Z = \emptyset \) and \( \dim Z > \dim \text{Tw}_{\mathcal{X}}^{ss}(r, L_C) \) then there is a point of \( Z \) whose restriction to \( \mathcal{C} \) is not stable.

**Proof.** The proof is exactly same as the proof of Lemma 3.2.4.13 of [12]. Let us recapitulate the main points.

Suppose that every sheaf in \( Z \) has (geometrically) stable restriction to \( C \), so that there is an induced map \( Z \to \text{Tw}_{\mathcal{X}}^{ss}(r, L_C) \). Let \( \bar{Z} \) denote the sheafification of \( Z^s \) (so that \( Z^s \to \bar{Z} \) is a \( \mathbb{G}_m \)-gerbe) and let \( \text{Tw} \) denote the coarse moduli space of \( \text{Tw}_{\mathcal{X}}^{ss}(r, L_C) \) (so that, again, we have a gerbe). There is a diagram

\[
\begin{array}{ccc}
Z & \longrightarrow & \text{Tw}_{\mathcal{X}}^{ss}(r, L_C) \\
\downarrow & & \downarrow \\
Z^s & \longrightarrow & \text{Tw}.
\end{array}
\]

By assumption, \( \dim \bar{Z} > \dim \text{Tw} \), so that a general geometric fiber \( F \) of \( \bar{Z} \to \text{Tw} \) has dimension at least 1. Let \( D^s \) denote the (possibly affine) normalization of a connected component of \( F \).

By Tsen’s theorem, there is a generically (on \( D^0 \)) separable map \( \varphi : D^0 \to Z^s \) such that the induced map \( D^0 \to \text{Tw}_{\mathcal{X}}^{ss}(r, L_C) \) is essentially constant. Using Langton’s theorem (Appendix 2.B of [4], which easily adapts to the situation at hand and magically does not require any extension of the fraction field to achieve a semistable extension), we can extend \( \varphi \) to a map \( \varphi' : D \to Z \), where \( D \) is the smooth projective completion of \( D^0 \). (Indeed, Langton’s theorem extends the family corresponding to \( \varphi \) to a family of semistable twisted sheaves over the base \( D \); the fact that \( Z \) is closed in \( \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta) \) then shows that the limits along \( D \) must lie in \( Z \). This is true even though the ambient stack \( \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta) \) is highly non-separated in general.) Since \( \text{Tw} \) is separated, we see that \( \varphi' \) collapses \( D \) to a point of \( \text{Tw} \); giving rise to an essentially constant family of restrictions.

Thus, we have a family \( \mathcal{F} \) on \( \mathcal{X} \times D \) such that the restriction to \( \mathcal{C} \times D \) has fibers which are all mutually isomorphic stable locally free \( \mathcal{C} \)-twisted sheaves. By an analysis identical to that of Lemma 3.1.4.6ff of [12], we deduce that for any positive integer \( n \) there is a snc divisor \( C^{(n)} \) in \( |nC| \) such that the restriction of \( \mathcal{F} \) to \( (C^{(n)} \times_X \mathcal{X}) \times D \) is an essentially constant family of...
simple locally free sheaves. If \( G \) is one of the fibers of \( \mathcal{F} \), we thus see that the induced Kodaira–Spencer map \( \tau : H^1(X, \delta nd(\mathcal{G})) \to H^1(C^{(n)}, \delta nd(\mathcal{G}|_{C^{(n)}})) \) sends the element \( \varepsilon \) corresponding to the family parametrized by \( \varphi' \) to 0. On the other hand, since \( C^{(n)} \) is arbitrarily ample, we have that \( \tau \) is injective. It follows that \( \varepsilon = 0 \); applying this argument at the points of \( D \) shows that \( \varphi' \) is essentially constant (as it is generically separable and has trivial tangent map), which is a contradiction. \( \square \)

Keeping the notation of Proposition 5.2.3, let \([\mathcal{F}] \in Z\) be a sheaf with non-stable restriction to \( C \), so that there is an exact sequence

\[
0 \to \mathcal{F}' \to \mathcal{F}|_C \to \mathcal{F}'' \to 0
\]

of locally free twisted sheaves on \( C \) with \( \mu(\mathcal{F}') \geq \mu(\mathcal{F}'') \). Write \( r' = \text{rk} \mathcal{F}' \) and \( r'' = \text{rk} \mathcal{F}'' \).

**Proposition 5.2.4.** With the above notation, if

\[
\dim Z > \dim \text{Tw}_{\mathcal{X}/k}^{ss}(r, L, c) + r' r'' (1 + C(K - C))
\]

then \( s(\mathcal{F}) \leq 2CH \).

**Proof.** The proof is identical to the proof of Proposition 8.4 of [8], using Lemma 4.1.7 in place of Langer’s Lemma 1.4. The reader nervous about stacky issues need only note that \( C \) lies entirely in \( X^{\text{sp}} \), and Corollary 2.2.9 of [4] applies equally to twisted sheaves on curves. \( \square \)

Let \( \theta(r) = \frac{73}{36} r^2 + r - 1 \).

**Theorem 5.2.5.** There exist constants \( A_1, C_1, C_2 \in \mathbb{Q} \) depending only on \( X \) and \( r \) such that if \( \Delta \geq A_1 \) and \( Z \subset \text{Tw}_{\mathcal{X}/k}^{ss}(r, L, \Delta) \) is an irreducible closed substack with

\[
\dim Z \geq \left( 1 - \frac{r - 1}{2 \theta(r)} \right) \Delta + C_1 \sqrt{\Delta} + C_2
\]

then \( \partial_{\text{soft}} Z^s \neq \emptyset \).

**Proof.** The proof is identical to the proof of Theorem 8.1 of Langer, using Propositions 5.2.3 and 5.2.4 in place of Langer’s Propositions 8.2 and 8.4, and using Corollary 4.5.3.4 in place of Langer’s Lemma 8.5.

There is one point which deserves a bit of elaboration. Langer (in the proof of Theorem 8.1) and Huybrechts–Lehn (in the proof of their analogous Theorem 9.2.2) reduce to proving that \( \partial_{\text{soft}} Z \neq \emptyset \) by invoking the fact that \( \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta) \) is a quotient stack. Let us rapidly sketch the intrinsic form of this argument: we know that the codimension of the dominant component of \( \delta_{\text{soft}} Z \) (when it is non-empty, as the proof shows) has codimension at most \( r - 1 \) by Lemma 3.4.18. On the other hand, we know that the dimension of \( \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta)(0) \) is asymptotically equal to \( (1 - \frac{1}{\theta}) \Delta \), which grows more slowly than \( (1 - \frac{1}{2 \theta(r)}) \Delta \). Thus, for \( \Delta \) sufficiently large we will have that the codimension of \( \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta)(0) \) is larger than \( \dim Z - r + 1 \), implying that the dominant component of \( \delta_{\text{soft}} Z \) must meet \( \text{Tw}_{\mathcal{X}}^{ss}(r, L, \Delta) \) and therefore \( \partial_{\text{soft}} Z^s \neq \emptyset \).
Lemma 5.2.8. Let $C_3 = \max\{C_2 + \beta_\infty \frac{A_1}{\Delta} + 2\beta_\infty - (r^2 - 1)\chi(X, \mathcal{O}_X) + r^2 n \frac{N^2 - 1}{12n}\}$. In particular, $C_3 \geq C_2$. Given a substack $Z \subset \text{Tw}^s_{\mathcal{X}}(r, L, \Delta)$, let $\beta(Z) = \min\{\dim \text{Hom}_0(E, E \otimes \omega_X): [E] \in Z\}$. We always have that $\beta(Z) \leq \beta_\infty$ defined in Corollary 4.5.3.3.

Following Friedman and [4], we make the following definition.

Definition 5.2.6. An $\mathcal{X}$-twisted sheaf $\mathcal{F}$ is good if it is $\mu$-stable and $\text{Ext}^2(\mathcal{F}, \mathcal{F})_0 = 0$.

Let $W \subset \text{Tw}^s_{\mathcal{X}}(r, L, \Delta)$ be the reduced closed substack parametrizing semistable $\mathcal{X}$-twisted sheaves $\mathcal{F}$ such that $s(\mathcal{F}) = 0$ or $\text{Ext}^2(\mathcal{F}, \mathcal{F})_0 \neq 0$. In other words, $W$ is the complement of the good locus.

Theorem 5.2.7. For all $\Delta \geq A_1$,
\[
\dim W \leq \left(1 - \frac{r - 1}{2\theta(r)}\right)\Delta + C_1 \sqrt{\Delta} + C_3.
\]

Proof. The proof again follows a standard outline, with some slight complications arising from the fact that we only consider the soft boundaries.

Suppose $Z$ is an irreducible component of $W$ of maximal dimension. Arguing by contradiction, we assume that $\dim Z > (1 - \frac{r - 1}{2\theta(r)})\Delta + C_1 \sqrt{\Delta} + C_3$. Applying Theorem 5.2.5, we know that $\beta_{\text{soft}}^Z \neq 0$. Let $D$ be the dominant component of $\beta_{\text{soft}}^Z$, and let $\ell$ be the colength of the sheaves parametrized by the points of $D$. Let $Y$ be an irreducible component of $D$; the formation of the soft hull yields a morphism $p : Y \to \text{Tw}^s_{\mathcal{X}}(r, L, \Delta - 2r \ell)$. Let $Z_1$ be the closure of $p(Y)$. As long as $\beta_{\text{soft}}^Z \neq 0$, we can iterate this procedure and produce $Z_2$, etc. Note that at each step, the discriminant $\Delta$ decreases. By Proposition 4.2.4, this process must terminate (as the discriminant is bounded below), say at $Z_t$.

We can also understand this process by “running it in reverse.” That is to say: a general point $\mathcal{E}$ of $Z_t$ is softly reflexive, and there is an obvious induced map $\text{Quot}_{\text{soft}}(\mathcal{E}, \ell) \to \text{Tw}^s_{\mathcal{X}}(r, L, \Delta)$ whose image contains $p^{-1}(\{\mathcal{E}\})$. Using Proposition 3.5.2, we conclude that $\dim p^{-1}(\{\mathcal{E}\}) \leq \ell(r + 1)$. If this is an equality, then we see that a general soft quotient of $\mathcal{E}$ appears in $Y$.

Lemma 5.2.8. The kernel $\mathcal{F}$ of a general quotient $\mathcal{E} \to \mathcal{Q}$ satisfies
\[
\dim \text{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_X)_0 < \dim \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_X)_0.
\]

Proof. First, we note that there is always a natural inclusion $\text{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_X)_0 \hookrightarrow \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_X)_0$ arising from the fact that $\mathcal{E}$ is the soft hull of $\mathcal{F}$.

Given $f \in \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_X)_0$, the fact that it is traceless shows that there is a collection of $\ell$ points $p_i$ of $X^{\text{op}}$ such that the fiber of $f$ at each $p_i$ is not a multiple of the identity (with respect to any local trivialization of $\omega_X$). Thus, there is a quotient $\mathcal{E}(p_i) \to \mathcal{Q}_i$ of length $1$ whose kernel is not preserved by $f$. The point $\mathcal{E} \to \oplus \mathcal{Q}_i$ thus has a kernel $\mathcal{F}$ such that $\text{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_X)_0$ does not contain $f$, as desired. \qed
Using the preceding lemma and Lemma 3.4.18, we thus find inequalities

$$\dim Z_1 \geq \dim Y - \ell(r + 1) \geq \dim Z - (r - 1) - \ell(r + 1) \geq \dim Z - (2r - 1)\ell - 1,$$

and, moreover, if equality holds we know that $\beta(Z) < \beta(Z_1)$. (In general, since $\beta$ is upper semicontinuous, we know that $\beta(Z) \leq \beta(Z_1)$.)

Following Langer, write $\epsilon_i = \dim Z_i - (\dim Z_i - 1 - (2r - 1)\ell - 1)$, so that $\beta(Z_{i-1}) \leq \beta(Z_i) \leq \beta_\infty$ for all $i$, and whenever $\epsilon_i = 0$ the inequality is strict. We conclude that $\sum \epsilon_i \geq t - \beta_\infty$ and that

$$\dim Z_t = \dim Z - (2r - 1)\sum \ell_i - t + \sum \epsilon_i.$$  

Moreover, since we are taking soft hulls, it is easy to see that $\Delta = \Delta_t + 2r\sum \ell_i$, from which we deduce that

$$\dim Z_t \geq \dim Z - \left(1 - \frac{1}{2r}\right)(\Delta - \Delta_t) - \beta_\infty.$$  

Using the assumption about $\dim Z$, we thus conclude that

$$\dim Z_t - \left(1 - \frac{1}{2r}\right)\Delta_t \geq \left(1 - \frac{1}{2r}\right)\Delta + C_1\sqrt{\Delta} + C_3 - \beta_\infty.$$  

Since $\partial^{\text{soft}}Z_t^\infty = \emptyset$, we have by Theorem 5.2.5 that

$$\dim Z_t - \left(1 - \frac{1}{2r}\right)\Delta_t < \left(1 - \frac{1}{2r}\right)\Delta + C_1\sqrt{\Delta} + C_2.$$  

Using the fact that $\Delta > \Delta_t$ and the description of $C_3$, we immediately get a contradiction by comparing the two inequalities. □

**Corollary 5.2.9.** There exists a constant $A_2 \geq A_1$ such that whenever $\Delta \geq A_2$,

1. every irreducible component of $\mathcal{Tw}^{ss}_{12}(r, L, \Delta)$ is generically good, hence contains $\mathcal{Tw}^{ss}_{12}(r, L, \Delta)$ as a dense open substack which is generically smooth of the expected dimension;
2. the stack $\mathcal{Tw}^{ss}_{12}(r, L, \Delta)$ is normal is a normal local complete intersection.

By “local complete intersection” we mean that for any smooth cover by a scheme $U \to \mathcal{Tw}^{ss}_{12}(r, L, \Delta)$, we have that $U$ is a local complete intersection over the base. (It is fruitful to compare this with the classical statement – as in Theorem 9.3.3 of [4] – where the use of GIT means that only the stable locus is understood to be lci, as the local structure at boundary points deteriorates in the passage to the GIT quotient.)

**Proof.** The first part is a direct result of Theorem 5.2.7. To prove the second part, one could argue as in the proof of Theorem 9.3.3 (and thus Theorem 2.2.8) of [4] and invoke a smooth cover by a Quot scheme. Instead, let us give an intrinsic proof.
Let $\mathcal{F}$ be a semistable $\mathcal{X}$-twisted sheaf of determinant $L$ and discriminant $\Delta$, and let $(V, v) \to T^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ be a smooth map from a pointed scheme whose completion at $v$ gives a miniversal deformation of $\mathcal{F}$ over $k$. Since the good locus is dense, there will be points $w \in V(k)$ parametrizing good (hence stable) $\mathcal{X}$-twisted sheaves $\mathcal{G}$. The expected dimension of $T^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ at $[\mathcal{G}]$ is $\chi(\mathcal{O}) - \chi(\mathcal{G}, \mathcal{F})$, while the expected dimension at $\mathcal{F}$ is $\chi(\mathcal{O}) - \chi(\mathcal{F}, \mathcal{F}) + \dim\text{Hom}_0(\mathcal{F}, \mathcal{F})$; the latter term is recognizable as the dimension of the tangent space to $\text{Aut}_0(\mathcal{F})$ (where this means automorphisms in the moduli problem, i.e., automorphisms which preserve the determinant).

By the assumption that $\mathcal{G}$ is good, we have that the dimension of $T^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ at $[\mathcal{G}]$ is equal to $\chi(\mathcal{O}) - \chi(\mathcal{G}, \mathcal{F})$, so that $\dim_0 V = \chi(\mathcal{O}) - \chi(\mathcal{G}, \mathcal{F}) + \dim(V/T^{ss}_{\mathcal{X}/k}(r, L, \Delta))$, where the last term is the (constant) relative dimension of the smooth morphism $V \to T^{ss}_{\mathcal{X}/k}(r, L, \Delta)$. Since $\mathcal{F}$ and $\mathcal{G}$ belong the same flat family, we see that $\chi(\mathcal{F}, \mathcal{F}) = \chi(\mathcal{G}, \mathcal{F})$, from which we conclude that $\dim(V/T^{ss}_{\mathcal{X}/k}(r, L, \Delta)) = \dim\text{Hom}_0(\mathcal{F}, \mathcal{F})$. Since $\dim_0 V = \dim_0 V$, we see that $V$ has the expected dimension at $v$. Schlessinger’s criterion (e.g., Proposition 2.1.A.11 of [4]) implies that $V$ is lci at $v$, which implies (by definition) that $T^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ is lci at $\mathcal{F}$, as desired.

Now that we know $T^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ is lci with singular locus of arbitrarily large codimension (for large enough $\Delta$), we see that it is R1 and S2, and is thus normal by Serre’s criterion. □

### 5.3. Irreducibility

We conclude with two irreducibility results. The first one is significantly stronger, but only applies to the classical situation in which $X \to X^\text{sp}$ is an isomorphism. The second uses asymptotic normality to ensure that there is a geometrically irreducible component of the stack for sufficiently large $\Delta$ and works without any hypothesis on the stackiness of $X$; it uses a trick due to de Jong and Starr and shown to the author by de Jong.

**Theorem 5.3.1.** There exists $A_3 \geq A_2$ such that for any $\Delta \geq A_3$, any irreducible component of $T^{ss}_{\mathcal{X}/k}(r, L, \Delta)$ contains:

1. a point $[\mathcal{F}]$ which is good and locally free, and
2. a point $[\mathcal{F}]$ which is good such that $\ell^{\text{soft}}(\mathcal{F}) = 1$.

**Proof.** The standard proof of this fact (as in Section 9.6 of [4]) works. We recall the numerical proof of the second, and leave the similar proof of the first to the reference (where it is spelled out in greater detail).

To prove the second, let $A_3 = A_2 + 2r(\frac{D_{SS}}{r-1} + 1)$. If $\Delta \geq A_3$ then $\Delta \geq A_2$, and thus $\partial^{\text{soft}} Z^s \neq \emptyset$. Let $Y$ be an irreducible component of the dominant component of $\partial^{\text{soft}} Z^s$; suppose the soft colength of the sheaves parametrized by $Y$ is $\ell$, and let $\rho : Y \to T^{ss}_{\mathcal{X}/k}(r, L, \Delta - 2r\ell)$ be the morphism coming from formation of the soft hull. Let $Z'$ be the closure of $\rho(Y)$. Proposition 3.5.2 shows that the fibers of $\rho$ have dimension at most $\ell(r+1)$, so that $

\dim Z' \geq \dim Y - \ell(r + 1)$. In addition, Lemma 3.4.18 shows that $\dim Y \geq \dim Z - (r - 1)$. Putting these together yields

$$\dim Z' \geq \exp\dim T^{ss}_{\mathcal{X}/k}(r, L, \Delta - 2r\ell) + (\ell - 1)(r - 1),$$
since \( \exp \dim \mathbf{Tw}^s_{\mathcal{E}}(r, L, \Delta - 2r\ell) = \dim \mathbf{Tw}^s_{\mathcal{E}}(r, L, \Delta) - 2r\ell \) (since the latter has the expected dimension). Now, if \( \Delta - 2r\ell \geq A_2 \) then \( Z' \) is generically good by Theorem 5.2.7, in which case \( \dim Z' \leq \exp \dim \mathbf{Tw}^s_{\mathcal{E}}(r, L, \Delta - 2r\ell) \), which shows that \((r - 1)(\ell - 1) = 0\), so that \( \ell = 1 \). On the other hand, if \( \Delta - 2r\ell < A_2 \) then \( 2r\ell > \Delta - A_2 \geq A_3 - A_2 = 2r(\beta + 1) \), so that \((r - 1)(\ell - 1) > \beta_\infty \) and \( \dim Z' \geq \exp \dim \mathbf{Tw}^s_{\mathcal{E}}(r, L, \Delta - 2r\ell) + \beta_\infty \), which contradicts Corollary 4.5.3.5. \(\square\)

**Theorem 5.3.2.** The following asymptotic irreducibility statements hold:

1. If \( X \to X^{sp} \) is an isomorphism, then for all sufficiently large values of \( \Delta \) the stack \( \mathbf{Tw}^s_{\mathcal{E}}(r, L, \Delta) \) is geometrically integral whenever it is non-empty, and if it is non-empty then \( \dim \mathbf{Tw}^s_{\mathcal{E}}(r, L, \Delta + 1) \).
2. In general, if there exists a totally regular locally free \( \mathcal{E} \)-twisted sheaf of rank \( r \), then for any positive integer \( E \), there is \( \Delta > E \) such that \( \mathbf{Tw}^s_{\mathcal{E}}(r, L, \Delta) \) contains a geometrically integral connected component when it is non-empty. Moreover, if it is non-empty for one value of \( \Delta \) then it is non-empty for infinitely many.

The proof of Theorem 5.3.2 works precisely as in the proof of Theorem 3.2.4.11 of [12], using the fact that \( X \) has no stacky points (in part (1)) or that the sheaves are totally regular (in part (2)). We sketch the main points of the proof using a sequence of propositions.

**Proposition 5.3.3.** Let \( \mathcal{E} \) and \( \mathcal{F} \) be two locally free totally regular \( \mathcal{E} \)-twisted sheaves of rank \( r \) with determinant \( L \). For sufficiently large \( m \), the cokernel of a general map \( \mathcal{E} \to \mathcal{F}(m) \) is an invertible twisted sheaf supported on a smooth member of \( |\mathcal{O}(rm)| \) in \( X^{sp} \).

**Proof.** We may choose \( m \) sufficiently large that for any closed substack \( Y \subset X \) of length at most 3 the restriction map \( \text{Hom}(\mathcal{E}, \mathcal{F}(m)) \to \text{Hom}_Y(\mathcal{E}_Y, \mathcal{F}(m)_Y) \) is surjective. Let \( A \) be the affine space whose \( k \)-points are \( \text{Hom}(\mathcal{E}, \mathcal{F}(m)) \), and let \( \Phi : \mathcal{E}_A \to \mathcal{F}(m)_A \) be the universal map over \( \mathcal{E} \times A \). For a point \( x \in X^{sp}(k) \) write \( Y_x = \text{Spec} \mathcal{O}_{X, x}/m^2_x \). There is a (non-canonical) isomorphism \( \text{Hom}_{Y_x}(\mathcal{E}_{Y_x}, \mathcal{F}(m)_{Y_x}) \cong \text{M}_r(\mathcal{O}_{X, x}/m^2_x) \), where \( r = \text{rk} \mathcal{E} = \text{rk} \mathcal{F} \). We can write an element of the latter as \( A = A_0 + xA_1 + yA_2 \) with the \( A_i \) elements of \( \text{M}_r(k) \). We claim that the locus of \( A \) such that \( \det A = 0 \in \mathcal{O}_{X, x}/m^2_x \) is a cone of codimension at least 3. To see this, we recall the Jacobi formula:

\[
\det(A) = \det A_0 + \text{Tr}(\text{adj}(A_0)(A_1x + A_2y)).
\]

The condition that \( \det A_0 = 0 \) has codimension 1. It is clear from the formula that the vanishing to first order in \( x \) and \( y \) will also be conditions of codimension at least 1.

Thus, there is a cone \( U_x \subset A \) of codimension at least 3 such that of \( A \) is not in \( U_x \), then the map \( \mathcal{E} \to \mathcal{F}(m) \) corresponding to \( A \) does not vanish in the second jet bundle at \( x \). Since the \( U_x \) vary algebraically with \( x \), we see that there is a locally closed subset \( U \subset A \) of codimension at least 1 whose complement \( V \) parametrizes maps whose determinants are smooth at each \( x \in X^{sp}(k) \). The condition that the map be an isomorphism in the fibers over \( X \setminus X^{sp} \) is open (and non-empty, since both \( \mathcal{E} \) and \( \mathcal{F} \) are totally regular). Finally, a similar argument shows that the condition that the cokernel of \( \mathcal{E} \to \mathcal{F}(m) \) have rank 1 at each point of its support is also a non-empty open subset. Putting these together yields the proposition. \(\square\)
Now let $\mathcal{Q}$ be the cokernel of a general map as in Proposition 5.3.3. Write $\mathcal{Q} = i_\ast \mathcal{M}$, where $i : C \to X^{sp}$ is the inclusion of the (smooth) support of $\mathcal{Q}$ (which lies in $|rmH|$) and $\mathcal{M}$ is an invertible twisted sheaf on $C$. Since $\mathcal{E}$ and $\mathcal{F}$ have the same determinant and discriminant, we know the degree of $\mathcal{M}$. (To see this, one could for example pull back to a finite flat cover $Y \to X$ to reduce to the case where $X$ is a smooth projective surface. Now knowing the determinant and discriminant of $\mathcal{E}$ and $\mathcal{F}$ tells us their Hilbert polynomials, whence we know the Hilbert polynomial of $\mathcal{M}$, which tells us the degree.)

This leads us to the second major step of the proof of Theorem 5.3.2, which is collected in the following pair of propositions.

**Proposition 5.3.4.** Let $\mathcal{C} \to C \to S$ be a $\mu_N$-gerbe on a proper smooth family of geometrically connected curves over a connected locally Noetherian base scheme $S$. Given a rational number $d$, the relative twisted Picard space $\text{Pic}_{\mathcal{C}/S}(d)$ of invertible twisted sheaves on the fibers of $\mathcal{C}$ with degree $d$ is a proper smooth algebraic $S$-space with geometrically integral fibers. Thus, if $S$ is integral then so is $\text{Pic}_{\mathcal{E}/S}$.

**Proof.** The properness of $\text{Pic}_{\mathcal{C}/S}(d)$ is a standard result, and works just as in the classical case (see Section 3.1 of [12] for details). Thus, it remains to see that the geometric fibers are integral. If $S = \text{Spec} \kappa$ then there is an invertible $C$-twisted sheaf, say $L$. Twisting down by $L$ gives an isomorphism between $\text{Pic}_{\mathcal{C}/S}(d)$ and $\text{Pic}_{\mathcal{C}/S}(d')$ for a suitable integer $d'$. Since the latter is a torsor under the Jacobian of $C$, we immediately see that it is integral, as desired.

**Proposition 5.3.5.** Let $\mathcal{V}$ and $\mathcal{W}$ be two locally free totally regular $X$-twisted sheaves with the same rank, determinant, and discriminant. There exists finite colength twisted subsheaves $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{W}' \subset \mathcal{W}$ with the same colength and an irreducible family $\mathcal{F}$ of $X$-twisted sheaves containing both $\mathcal{V}'$ and $\mathcal{W}'$. Moreover, if $\mathcal{V}$ and $\mathcal{W}$ are good, then we can assume that the members of $\mathcal{F}$ are also good.

**Proof.** The proof is identical to the proof of Proposition 3.2.4.22 of [12]. The idea is as follows: choose $m$ so that there are exact sequences

$$0 \to \mathcal{V}(-m) \to \mathcal{W} \to \mathcal{P} \to 0$$

and

$$0 \to \mathcal{V}(-m) \to \mathcal{V} \to \mathcal{Q} \to 0,$$

where $\mathcal{P}$ and $\mathcal{Q}$ are invertible sheaves supported on smooth members of $|rmH|$ lying in $X^{sp}$. Using Proposition 5.3.4, we can connect $\mathcal{P}$ and $\mathcal{Q}$ in an irreducible family, say $\mathcal{D}$ parametrized by $T$.

In general, given an extension $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ with $\mathcal{C}$ an invertible sheaf supported on a smooth curve, taking the preimage of $\mathcal{C}(-\ell)$ under an inclusion $\mathcal{C}(-\ell) \to \mathcal{C}$ induced by a section of $\mathcal{O}(\ell)$ yields a finite colength subsheaf $\mathcal{B}' \subset \mathcal{B}$. Moreover, taking large values of $\ell$ makes the space $\text{Ext}^1(\mathcal{C}(-\ell), \mathcal{A})$ behave better in a family (e.g., as $\mathcal{C}$ varies the $\text{Ext}^1$ spaces form a vector bundle). Taking $\ell$ sufficiently large that $\text{Ext}^1(\mathcal{V}(-m), \mathcal{D}_t(-\ell))$ behaves well as $t$ varies, we get a family of sheaves interpolating between $\mathcal{V}'$ and $\mathcal{W}'$, as desired. (One goal of increasing $\ell$ is to ensure that general such extensions are torsion free in general fibers. Once this
is achieved, the good locus will be further open subset, preserving irreducibility of the base of the family.) We refer the reader to the proof of [12] for details. □

Proof of Theorem 5.3.2(1). The crux of the proof lies in Theorem 5.3.1. (The mechanism is familiar from [4,8,12].) Let \( \mathcal{E}(r, L, \Delta) \) denote the set of irreducible components of \( \text{Tw}^{ss}_{X'}(r, L, \Delta) \). Suppose \( \Delta \geq A_3 \). Given an element \([Z] \in \mathcal{E}(r, L, \Delta)\), we then have that there is a point \( z \in Z \) parametrizing a good locally free \( \mathcal{E} \)-twisted sheaf \( \mathcal{V} \). Choose a closed point \( x \in X \) and let \( \mathcal{V}' \subseteq \mathcal{V} \) be the kernel of a quotient \( \mathcal{V} \twoheadrightarrow \mathcal{D} \), where \( \mathcal{D} \) has length 1 and is supported at \( x \). The sheaf \( \mathcal{V}' \) lies in a well-defined component of \( \text{Tw}^{ss}_{X'}(r, L, \Delta + 2r) \) because it is good (and thus cannot lie in an intersection of components), yielding a map \( \phi(\Delta) : \mathcal{E}(r, L, \Delta) \twoheadrightarrow \mathcal{E}(r, L, \Delta + 2r) \). Taking a further quotient of \( \mathcal{F} \) at such a skyscraper twisted sheaf produces an element \( q \in \mathcal{E}(r, L, \Delta + 4r) \) which is easily seen to be equal to \( \phi(\Delta + 2r) \circ \phi(\Delta)([Z]) \) (and similarly for higher compositions and further quotients).

On the other hand, any element \([W] \in \mathcal{E}(r, L, \Delta + 2r)\) contains a point \( w \) parametrizing a good torsion free \( \mathcal{E} \)-twisted sheaf \( \mathcal{F} \) with colength 1. Letting \( \mathcal{V}' := \mathcal{F}/\mathcal{V} \), we see that \([W]\) is the image of the component containing \( \mathcal{V} \) under the map \( \phi(\Delta) \). This shows that \( \phi(\Delta) \) is surjective for all \( \Delta \geq A_3 \). This yields a sequence of surjections \( \mathcal{E}(r, L, \Delta) \twoheadrightarrow \mathcal{E}(r, L, \Delta + 2r) \twoheadrightarrow \cdots \) for each \( \Delta \). Since \( \Delta \) has bounded denominator, there are finitely many such sequences containing all possible values of \( \Delta \geq A_3 \).

By Proposition 5.3.5, starting with \( \mathcal{V}' \) and \( \mathcal{W}' \), locally free good sheaves corresponding to components \( A \) and \( B \) described by \( \mathcal{E}(r, L, \Delta) \), there exists some \( \ell \) and good colength \( \ell \) subsheaves \( \mathcal{V}' \subseteq \mathcal{V} \) and \( \mathcal{W}' \subseteq \mathcal{W} \) which lie in the same irreducible component of \( \text{Tw}^{ss}_{X'}(r, L, \Delta + 2r) \). This shows that the composition \( \phi(\Delta + 2r) \circ \phi(\Delta + 2r(\ell - 1)) \circ \cdots \circ \phi(\Delta) \) collapses the two points \( A \) and \( B \) to the same point. Since each \( \mathcal{E}(r, L, \Delta) \) is finite, we see that the sequence of surjections eventually results in singletons. Since there are only finitely many such sequences to consider, we see that for \( \Delta \) sufficiently large, \( \mathcal{E}(r, L, \Delta) \) is a singleton, i.e., that \( \text{Tw}^{ss}_{X'}(r, L, \Delta) \) is irreducible. □

Proof of Theorem 5.3.2(2). We sketch the proof, leaving a few details to the reader. A more complete treatment of a similar statement will appear in a forthcoming paper on the period-index problem for Brauer groups of surfaces over finite fields.

Since extension to the perfect closure of \( k \) is radicial, in order to prove the statement we may assume that the base field \( k \) is perfect. The absolute Galois group of \( k \) acts on the components of \( \text{Tw}^{ss}_{X'/k}(r, L, \Delta) \otimes_k \overline{k} \) (for any chosen algebraic closure of \( k \)), and a component which is Galois-fixed corresponds to a geometrically integral component over \( k \).

For sufficiently large \( \Delta \), the irreducible components of \( \text{Tw}^{ss}_{X'/k}(r, L, \Delta) \otimes_k \overline{k} \) are normal. Moreover, the map \( \phi(\Delta) : \mathcal{E}(r, L, \Delta) \twoheadrightarrow \mathcal{E}(r, L, \Delta + 2r) \) defined above in the proof of Theorem 5.3.2(1) is Galois equivariant. Since \( \mathcal{E}(r, L, \Delta) \) is finite, if we can show that iteration of the maps \( \phi(\Delta) \) eventually brings any two points together, we will thus find a point in \( \mathcal{E}(r, L, \Delta + 2r) \) (for some \( \ell \)) which is Galois fixed, yielding the desired component. Since we no longer know that \( \phi(\Delta) \) is surjective, we cannot conclude that this is the only component.

The proof that the \( \phi(\Delta) \) are contracting is similar to Proposition 5.3.3. Given good torsion free \( \mathcal{E} \)-twisted sheaves \( \mathcal{E} \) and \( \mathcal{F} \), there are cofinite length subsheaves \( \mathcal{E}' \) and \( \mathcal{F}' \) which have isomorphic fibers over every Henselization of \( X' \). Now the analogue of Proposition 5.3.3 for torsion free \( \mathcal{E} \)-twisted sheaves yields further cofinite length subsheaves \( \mathcal{E}'' \subseteq \mathcal{E}' \) and \( \mathcal{F}'' \subseteq \mathcal{F}' \) which lie in a connected family. Since \( \mathcal{E}'' \) and \( \mathcal{F}'' \) are good, this shows that the components containing \( \mathcal{E}' \) and \( \mathcal{F}' \) are contracted by a suitable iterate of the maps \( \phi(\Delta) \). □
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