Twisted cocycles of Lie algebras and corresponding invariant functions

Jiří Hrivnák, Petr Novotný

Department of Physics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Břehová 7, 115 19 Prague 1, Czech Republic

ARTICLE INFO

Article history:
Received 1 October 2008
Accepted 2 November 2008
Available online 18 December 2008
Submitted by H. Schneider

Keywords:
Lie algebra
Invariant
Contraction
Nilpotent

ABSTRACT

We consider finite-dimensional complex Lie algebras. Using certain complex parameters we generalize the concept of cohomology cocycles of Lie algebras. A special case is generalization of 1-cocycles with respect to the adjoint representation – so called \((\alpha, \beta, \gamma)\)-derivations. Parametric sets of spaces of cocycles allow us to define complex functions which are invariant under Lie isomorphisms. Such complex functions thus represent useful invariants – we show how they classify three and four-dimensional Lie algebras as well as how they apply to some eight-dimensional 1-parametric nilpotent continua of Lie algebras. These functions also provide necessary criteria for existence of 1-parametric continuous contraction.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

The search for a new concept of invariant characteristics of Lie algebras led to the definition of \((\alpha, \beta, \gamma)\)-derivations in [1]. Consider a complex Lie algebra \(\mathcal{L}\) and its derivation \(D \in \text{der} \mathcal{L}^0\), i.e. a linear operator \(D \in \text{End} \mathcal{L}^0\) satisfying for all \(x, y \in \mathcal{L}\) the equation \(D[x, y] = [Dx, y] + [x, Dy]\). For fixed \(\alpha, \beta, \gamma \in \mathbb{C}\) is a linear operator \(A \in \text{End} \mathcal{L}^0\) called an \((\alpha, \beta, \gamma)\)-derivation if for all \(x, y \in \mathcal{L}\) the equation

\[
\alpha A[x, y] = \beta [Ax, y] + \gamma [x, Ay]
\]

holds [1]. In this article we denote the set of all operators satisfying (1) by \(\text{der}_{(\alpha, \beta, \gamma)} \mathcal{L}\). Investigating the spaces \(\text{der}_{(\alpha, \beta, \gamma)} \mathcal{L}\), various Jordan and Lie operator algebras were obtained. In [1] was also shown that the dimensions of these operator algebras and, in fact, the dimensions of all spaces \(\text{der}_{(\alpha, \beta, \gamma)} \mathcal{L}\)
as well as their intersections form invariant characteristics of Lie algebras. The invariance of these dimensions enabled us to define so called invariant function \( \psi \mathcal{L} \) by the relation
\[
\psi \mathcal{L}(\alpha) = \dim \text{der}_{(\alpha,1,1)} \mathcal{L}.
\]
This complex function \( \psi \mathcal{L} \) turned out to be very valuable: it was used to classify all complex three-dimensional indecomposable Lie algebras. More significantly, it resolved some parametric continua of nilpotent Lie algebras.

Prior to invention of the invariant function \( \psi \mathcal{L} \), the isomorphism problem for algebras inside nilpotent parametric continuum had to be solved explicitly. If algebras inside a nilpotent parametric continuum are considered then all well known characteristics such as dimensions of derived, lower central and upper central sequences, Lie algebra of derivations, radical, nilradical, ideals, subalgebras [2,3] and megaideals [4] naturally coincide. Moreover, due to Engel's theorem none of the 'trace' invariants based on the adjoint representation such as \( C_{pq} \) [5] or \( \chi_i \) [6] exists in this case. The nilpotent parametric continua appeared for example in [7] where all graded contractions [8,9] of the Pauli graded \( \mathfrak{sl}_3(C) \) were found. The invariant function \( \psi \mathcal{L} \) was successfully applied in [1] to some of the nilpotent parametric continua from [7]. This invariant function did not, however, resolve all nilpotent continua which were discussed in [7].

Often, the search for new invariant characteristics of Lie algebras is motivated by the classification of degenerations [5,11,12] or 1-parametric continuous contractions [13]. The applicability of the invariant function \( \psi \mathcal{L} \) (or its possible generalizations) in this field remained an open problem.

Precisely these facts motivated the work undertaken in this article. The idea of finding new invariant characteristics of Lie algebras via generalizing standard Chevalley cohomology cocycles is followed. New invariant functions are defined and applied not only to some cases from [7] but to all four-dimensional complex Lie algebras. Application of these generalized cocycles to 1-parametric continuous contractions is discussed. It is shown that these new invariants of Lie algebras constitute an essential tool for resolving parametric continua of nilpotent Lie algebras and provide powerful necessary contraction criteria.

In Appendix A, the tables of the invariant functions \( \psi, \varphi, \varphi^0 \) for two, three and four-dimensional Lie algebras are listed.

In Appendix B, proof of the classification Theorem 3.6 is located.

2. Twisted cocycles of Lie algebras

Let \( \mathcal{L} \) be an arbitrary complex Lie algebra and \((V,f)\) its representation. We denote by \( C^q(\mathcal{L}, V) \) the vector space of all \( V \)-cochains of dimension \( q \) for \( q \in \mathbb{N} \) and \( C^0(\mathcal{L}, V) = V \). We generalize the notions of cocycles analogously to \((\alpha,\beta,\gamma)\)-derivations. Let \( \kappa = (\kappa_{ij}) \) be a \((q+1) \times (q+1)\) complex symmetric matrix. We call \( c \in C^q(\mathcal{L}, V), q \in \mathbb{N} \) for which
\[
0 = \sum_{i=1}^{q+1} (-1)^{i+1} \kappa_{ii} f(x_i) c(x_1, \ldots, \hat{x_i}, \ldots, x_{q+1}) + \sum_{i,j=1}^{q+1} (-1)^{i+j} \kappa_{ij} c([x_i, x_j], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_{q+1})
\]
the twisted \( q+1 \)-cochains of \( \mathcal{L} \) with respect to \( \kappa \).
Lemma 2.2.\nspaces equal

In this section we investigate in detail the space $1386$

J. Hrivnak, P. Novotny / Linear Algebra and its Applications 430 (2009) 1384–1403

Proof. Suppose $\alpha$ coc.

Six permutations of the variables $\alpha$ coc.

Lemma 2.1. The definition of $(\alpha, \beta, \gamma)$-derivations is now included in the definition of twisted cocycles. Considering the adjoint representation and its 1-dimensional twisted cocycles, we immediately have:

$$Z^1(\mathcal{L}, \text{ad}_\alpha) = \text{der}_{(\alpha, \beta, \gamma)} \mathcal{L}. \tag{5}$$

In this section we investigate in detail the space $Z^2(\mathcal{L}, \text{ad}_\alpha)$. For this purpose it may be more convenient to use different notation, analogous to that of derivations, defined by

$$\text{coc}(\alpha, \beta, \gamma) \mathcal{L} = Z^2(\mathcal{L}, \text{ad}_\alpha, (\alpha_1, \alpha_2, \alpha_3)) \tag{6}$$

i.e. in the space $\text{coc}(\alpha, \beta, \gamma) \mathcal{L}$ are such $B \in C^2(\mathcal{L}, \mathcal{L})$ which for all $x, y, z \in \mathcal{L}$ satisfy

$$0 = \alpha_1 B(x, [y, z]) + \alpha_2 B(z, [x, y]) + \alpha_3 B(y, [z, x])$$

+ $\beta_1 [x, B(y, z)] + \beta_2 [z, B(x, y)] + \beta_3 [y, B(z, x)]. \tag{7}$

Six permutations of the variables $x, y, z \in \mathcal{L}$ in the defining Eq. (7) give

Lemma 2.1. Let $\mathcal{L}$ be a complex Lie algebra. Then for any $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{C}$ are all the following six spaces equal:

1. $\text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}$;
2. $\text{coc}(\alpha_2, \alpha_1, \alpha_3, \beta_2, \beta_1, \beta_3) \mathcal{L}$;
3. $\text{coc}(\alpha_3, \alpha_2, \alpha_1, \beta_3, \beta_2, \beta_1) \mathcal{L}$;
4. $\text{coc}(\alpha_3, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_3) \mathcal{L}$;
5. $\text{coc}(\alpha_2, \alpha_3, \alpha_1, \beta_2, \beta_3, \beta_1) \mathcal{L}$;
6. $\text{coc}(\alpha_2, \alpha_3, \alpha_1, \beta_3, \beta_1, \beta_2) \mathcal{L}$.

Lemma 2.2. Let $\mathcal{L}$ be a complex Lie algebra. Then for any $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{C}$ is the space $\text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}$ equal to all of the following:

1. $\text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L} \cap \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}$;
2. $\text{coc}(\beta_1, \beta_2, \beta_3, \beta_1, \beta_2, \beta_3) \mathcal{L} \cap \text{coc}(\beta_1, \beta_2, \beta_3, \beta_1, \beta_2, \beta_3) \mathcal{L}$;
3. $\text{coc}(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3) \mathcal{L} \cap \text{coc}(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3) \mathcal{L}$;
4. $\text{coc}(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3) \mathcal{L} \cap \text{coc}(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3) \mathcal{L}$.

Proof. Suppose $\alpha$, $\alpha$, $\alpha$, $\beta$, $\beta$, $\beta$ $\in \mathbb{C}$ are given. We demonstrate the proof on the case $1$, the proof of the other cases is analogous. Let $B \in \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}$; then for arbitrary $x, y, z \in \mathcal{L}$ we have

$$0 = \alpha_1 B(x, [y, z]) + \alpha_2 B(z, [x, y]) + \alpha_3 B(y, [z, x])$$

+ $\beta_1 [x, B(y, z)] + \beta_2 [z, B(x, y)] + \beta_3 [y, B(z, x)] \tag{8}$

$$0 = \alpha_1 B(x, [y, z]) + \alpha_2 B(y, [z, x]) + \alpha_3 B(x, [y, z])$$

+ $\beta_1 [y, B(z, x)] + \beta_2 [y, B(z, x)] + \beta_3 [x, B(y, z)]. \tag{9}$
By adding and subtracting Eqs. (8) and (9) we obtain
\[
0 = (\alpha_1 + \alpha_3)B(x, [y, z]) + (\alpha_2 + \alpha_1)B(z, [x, y]) + (\alpha_3 + \alpha_2)B(y, [z, x]) \\
+ (\beta_1 + \beta_3)[x, B(y, z)] + (\beta_2 + \beta_1)[z, B(x, y)] + (\beta_3 + \beta_2)[y, B(z, x)] \\
0 = (\alpha_1 - \alpha_3)B(x, [y, z]) + (\alpha_2 - \alpha_1)B(z, [x, y]) + (\alpha_3 - \alpha_2)B(y, [z, x]) \\
+ (\beta_1 - \beta_3)[x, B(y, z)] + (\beta_2 - \beta_1)[z, B(x, y)] + (\beta_3 - \beta_2)[y, B(z, x)]
\]
(10)
and thus \(\text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\) is the subset of the intersection. Similarly, starting with Eqs. (10) and (11) we obtain Eqs. (8) and (9) and the remaining inclusion is proven. \(\square\)

We show in the following theorem that four parameters are sufficient for the description of all spaces of two-dimensional twisted cocycles.

**Theorem 2.3.** Let \(\mathcal{L}\) be a Lie algebra. Then for any \(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{C}\) there exist \(\alpha, \beta, \gamma, \delta \in \mathbb{C}\) such that the subspace \(\text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L} \subset \mathbb{C}^2(\mathcal{L}, \mathcal{L})\) is equal to some of the following 16 subspaces:

1. \(\text{coc}(\alpha_0, \beta_0, \gamma_0, \delta_0) \mathcal{L}\); \(\text{coc}(\alpha_1, \beta_1, \gamma_1, \delta_1) \mathcal{L}\); \(\text{coc}(\alpha_2, \beta_2, \gamma_2, \delta_2) \mathcal{L}\);
2. \(\text{coc}(\alpha_0, \beta_0, \gamma_1, \delta_1) \mathcal{L}\); \(\text{coc}(\alpha_1, \beta_1, \gamma_0, \delta_0) \mathcal{L}\); \(\text{coc}(\alpha_2, \beta_2, \gamma_2, \delta_2) \mathcal{L}\);
3. \(\text{coc}(\alpha_0, \beta_0, \gamma_0, \delta_1) \mathcal{L}\); \(\text{coc}(\alpha_1, \beta_1, \gamma_1, \delta_0) \mathcal{L}\); \(\text{coc}(\alpha_2, \beta_2, \gamma_2, \delta_2) \mathcal{L}\);
4. \(\text{coc}(\alpha_0, \beta_0, \gamma_1, \delta_0) \mathcal{L}\); \(\text{coc}(\alpha_1, \beta_1, \gamma_0, \delta_1) \mathcal{L}\); \(\text{coc}(\alpha_2, \beta_2, \gamma_2, \delta_2) \mathcal{L}\).

**Proof.** Suppose \(\alpha_2 + \alpha_3 = 0\) and \(\beta_2 + \beta_3 = 0\). Then the following four cases are possible:

1. \(\alpha_2 = -\alpha_3 = 0 \text{ and } \beta_2 = -\beta_3 = 0\). In this case we have
   \(\text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L} = \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\).
2. \(\alpha_2 = -\alpha_3 = 0 \text{ and } \beta_2 = -\beta_3 \neq 0\). In this case we have
   \(\text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L} = \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\)
   \(= \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\)
   \(= \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\).
3. \(\alpha_2 = -\alpha_3 \neq 0 \text{ and } \beta_2 = -\beta_3 = 0\). In this case we have
   \(\text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L} = \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\)
   \(= \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\)
   \(= \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\).
4. \(\alpha_2 = -\alpha_3 \neq 0 \text{ and } \beta_2 = -\beta_3 \neq 0\). In this case we have
   \(\text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L} = \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\)
   \(= \text{coc}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mathcal{L}\).

Discussion of the three remaining cases of the values of \(\alpha_2 + \alpha_3\) and \(\beta_2 + \beta_3\) is similar to the previous case and we omit it. \(\square\)
It may be more convenient, sometimes, to use different distribution of the cocycle spaces than in Theorem 2.3. Henceforth, we investigate mainly the cocycle space $\text{cocl}_{(1,1,1,\ldots,\lambda)}$ which for $\lambda \neq 0$ fits in the class $\text{cocl}_{(\alpha,\beta,\gamma,\beta,\gamma,1,1)}$. Let $g : \mathcal{L} \to \mathcal{L}'$ be an isomorphism of Lie algebras $\mathcal{L}$ and $\mathcal{L}'$. Then the mapping $\varphi : C^q(\mathcal{L}, \mathcal{L}') \to C^q(\mathcal{L}, \mathcal{L}')$, $q \in \mathbb{N}$ defined for all $c \in C^q(\mathcal{L}, \mathcal{L}')$ and all $x_1, \ldots, x_q \in \mathcal{L}$ by
\[
(\varphi c)(x_1, \ldots, x_q) = gc^{-1}x_1, \ldots, gc^{-1}x_q,
\]
is an isomorphism of vector spaces $C^q(\mathcal{L}, \mathcal{L}')$ and $C^q(\mathcal{L}, \mathcal{L}')$. For any complex symmetric $(q + 1)$-square matrix $\kappa$
\[
\varphi(Z^q(\mathcal{L}, \text{ad}_{\varphi}, \kappa)) = Z^q(\mathcal{L}', \text{ad}_{\varphi}, \kappa)
\]
holds.

**Proof.** Suppose we have $g : \mathcal{L} \to \mathcal{L}'$ such that for all $x, y \in \mathcal{L}$
\[
[x, y]_{\mathcal{L}'} = g[g^{-1}x, g^{-1}y]_{\varphi}
\]
holds. It is clear that the map $\varphi : C^q(\mathcal{L}, \mathcal{L}') \to C^q(\mathcal{L}, \mathcal{L}')$, $q \in \mathbb{N}$ is linear and bijective, i.e. it is an isomorphism of these vector spaces. By putting $f = \text{ad}_{\varphi}$ and rewriting definition (3) we have for $c \in Z^q(\mathcal{L}, \text{ad}_{\varphi}, \kappa)$ and all $x_1, \ldots, x_q \in \mathcal{L}$
\[
0 = \sum_{i=1}^{q+1} (-1)^{i+1} k_{ii} [g^{-1}x_i, c(g^{-1}x_1, \ldots, g^{-1}x_i, \ldots, g^{-1}x_{q+1})]_{\varphi}
+ \sum_{i,j} (-1)^{i+j} k_{ij} c([g^{-1}x_i, g^{-1}x_j]_{\varphi}, g^{-1}x_1, \ldots, g^{-1}x_i, \ldots, g^{-1}x_j, \ldots, g^{-1}x_{q+1}).
\]
Applying the mapping $g$ on this equation and taking into account that $k_{ij} \in \mathbb{C}$ one has
\[
0 = \sum_{i=1}^{q+1} (-1)^{i+1} k_{ii} [x_i, (\varphi c)(x_1, \ldots, x_i, \ldots, x_{q+1})]_{\varphi}
+ \sum_{i,j} (-1)^{i+j} k_{ij} (\varphi c)([x_i, x_j]_{\varphi}, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_{q+1}),
\]
i.e. $\varphi c \in Z^q(\mathcal{L}', \text{ad}_{\varphi}, \kappa)$. □

**Corollary 3.2.** For any $q \in \mathbb{N}$ and any complex symmetric $(q + 1)$-square matrix $\kappa$ is the dimension of the vector space $Z^q(\mathcal{L}, \text{ad}_{\varphi}, \kappa)$ an invariant characteristic of Lie algebras.

Sixteen parametric spaces in Theorem 2.3 allow us to define 16 invariant functions of up to four variables. However, a complete analysis of possible outcome is beyond the scope of this work. Rather empirically, following calculations in dimension four and eight, we pick up two 1-parametric sets of vector spaces to define two new invariant functions of a $n$-dimensional Lie algebra $\mathcal{L}$. We call functions $\varphi, \varphi^0 : \mathbb{C} \to \{0, 1, \ldots, n^2(n - 1)/2\}$ defined by the formulas
\[
(\varphi \mathcal{L})(\alpha) = \dim \text{cocl}_{(1,1,1,\alpha,\alpha,\alpha)} \mathcal{L},
(\varphi^0 \mathcal{L})(\alpha) = \dim \text{cocl}_{(0,1,\alpha,1,1)} \mathcal{L},
\]
the invariant functions corresponding to two-dimensional twisted cocycles of the adjoint representation of a Lie algebra $L$.

From Theorem 3.1 follows immediately:

**Corollary 3.3.** If two complex Lie algebras $L$, $\tilde{L}$ are isomorphic, $L \cong \tilde{L}$, then it holds:

1. $\psi_L = \psi_{\tilde{L}}$,
2. $\varphi_L = \varphi_{\tilde{L}}$,
3. $\varphi^0_L = \varphi^0_{\tilde{L}}$.

### 3.1 Invariant functions $\psi$, $\varphi$ and $\varphi^0$ of low-dimensional Lie algebras

We now investigate the behaviour of the functions $\varphi$, $\varphi^0$ in dimensions three and four. It was shown in [1] that the invariant function $\psi$ classifies indecomposable three-dimensional complex Lie algebras. Moreover, observing the tables in Appendix A, the following theorem holds.

**Theorem 3.4 (Classification of three-dimensional complex Lie algebras).** Two three-dimensional complex Lie algebras $L$, $\tilde{L}$ are isomorphic if and only if $\psi_L = \psi_{\tilde{L}}$.

Observing the tables of $\varphi^0$, we may also derive a quite interesting fact – the function $\varphi^0$ alone also classifies three-dimensional Lie algebras:

**Theorem 3.5 (Classification of three-dimensional complex Lie algebras II).** Two three-dimensional complex Lie algebras $L$ and $\tilde{L}$ are isomorphic if and only if $\varphi^0_L = \varphi^0_{\tilde{L}}$.

Theorem 3.4 (or 3.5) provided complete classification of three-dimensional complex Lie algebras. We show in this section that combined power of the functions $\psi$ and $\varphi$ distinguishes among all complex four-dimensional Lie algebras. We define the **number of occurrences** of $j \in \mathbb{C}$ in a complex function $f$. Let $j$ be in the range of values of $f$. If there exist only finitely many mutually distinct numbers $x_1, \ldots, x_m \in \mathbb{C}$ for which $f(x_1) = \cdots = f(x_m) = j$ holds then we write

$$f : j_m$$

and say that $j$ **occurs** in $f$ $m$-times; otherwise we write $f : j$.

**Theorem 3.6 (Classification of four-dimensional complex Lie algebras).** Two four-dimensional complex Lie algebras $L$ and $\tilde{L}$ are isomorphic if and only if $\psi_L = \psi_{\tilde{L}}$ and $\varphi_L = \varphi_{\tilde{L}}$.

**Proof.** See Appendix B. □

An efficient algorithm for the identification of four-dimensional Lie algebras was quite recently published in [6]. We may now formulate an alternative algorithm: take a four-dimensional complex Lie algebra $L$ and

1. Calculate $\psi_L$ and $\varphi_L$.
2. The range of values of the functions $\psi$ and $\varphi$ and the number of their occurrences determines the label $(g-k)$, $k = 1, \ldots, 34$ in Appendix A.
3. The algebra is now identified up to the exact value of parameter(s) of the parametric continuum. These parameters are determined in the following cases:

   (g-8) Pick any of the two values $z \in \mathbb{C}, z \neq 1$, which satisfy $\psi_L(z) = 6$, and put $a = z$. Then $L \cong g_{3,4}(a) \oplus g_1$ holds.
There are two different complex numbers \( z_1, z_2 \neq 0 \) which satisfy \( \varphi \mathcal{L}(z_1) = \varphi \mathcal{L}(z_2) = 13 \). If \( z_1 - 1 = 2/z_2 \) holds then put \( a = z_1 - 1 \), otherwise put \( a = z_2 - 1 \). Then \( \mathcal{L} \cong g_{4,2}(a) \) holds.

There are three mutually different complex numbers \( z_1, z_2, z_3 \neq 0, -1 \) which satisfy \( \varphi \mathcal{L}(z_1) = \varphi \mathcal{L}(z_2) = \varphi \mathcal{L}(z_3) = 13 \). Put \( a = \frac{z_1 + 1}{z_2 + 1} \), \( b = \frac{z_2 + 1}{z_3 + 1} \). Then \( \mathcal{L} \cong g_{5,5}(a, b) \) holds.

Pick any of the six values \( z \in \mathbb{C} \), which satisfy \( \varphi \mathcal{L}(z) = 5 \), and put \( a = z \). Then \( \mathcal{L} \cong g_{4,5}(a, -1 - a) \) holds.

Pick any of the two values \( z \in \mathbb{C}, z \neq 1 \), which satisfy \( \varphi \mathcal{L}(z) = 6 \), and put \( a = z \). Then \( \mathcal{L} \cong g_{4,5}(a, a^2) \) holds.

Take the value \( z \in \mathbb{C} \), which satisfies \( \varphi \mathcal{L}(z) = 15 \), and put \( a = z - 1 \). Then \( \mathcal{L} \cong g_{4,5}(a, 1) \) holds.

Pick any of the two values \( z \in \mathbb{C} \), which satisfy \( \varphi \mathcal{L}(z) = 13 \), and put \( a = z + 1 \). Then \( \mathcal{L} \cong g_{4,5}(a, 1) \) holds.

Pick any of the two values \( z \in \mathbb{C}, z \neq 2 \) which satisfy \( \varphi \mathcal{L}(z) = 4 \), and put \( a = z \). Then \( \mathcal{L} \cong g_{4,8}(a) \) holds.

We demonstrate the above algorithm of identification on the following example.

**Example 3.1.** In [6], a four-dimensional algebra \( \mathcal{L}_1 \) was introduced:

\[
\mathcal{L}_1: \quad [e_1, e_2] = -e_1 - e_2 + e_3, \quad [e_1, e_3] = -6e_2 + 4e_3, \quad [e_1, e_4] = 2e_1 - e_2 + e_4,
\]

\[
[e_2, e_3] = 3e_1 - 9e_2 + 5e_3, \quad [e_2, e_4] = 4e_1 - 2e_2 + 2e_4, \quad [e_3, e_4] = 6e_1 - 3e_2 + 3e_4.
\]

(1) Computing the functions \( \varphi \mathcal{L}_1 \) and \( \varphi \mathcal{L}_1 \) one obtains:

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \psi \mathcal{L}_1(\alpha) )</th>
<th>( \alpha )</th>
<th>( \varphi \mathcal{L}_1(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>12</td>
<td>2</td>
</tr>
</tbody>
</table>

(2) The combination of occurrences \( \varphi \mathcal{L}_1 : 61, 52, 4 \) and \( \varphi \mathcal{L}_1 : 13_2, 12 \) is unique for the case (g-11).

(3) Since for \( z_1 = 3, z_2 = 1 \) the equality \( z_1 - 1 = 2/z_2 \) holds, one has \( a = z_1 - 1 = 2 \) and \( \mathcal{L}_1 \cong g_{4,2}(2) \).

4. Contractions of Lie algebras

4.1. Continuous contractions of Lie algebras

Suppose we have an arbitrary Lie algebra \( \mathcal{L} = (V, [\cdot, \cdot]) \) and a continuous mapping \( U : (0, 1) \rightarrow GL(V) \), i.e. \( U(\varepsilon) \in GL(V), 0 < \varepsilon \leq 1 \). If the limit

\[
[x,y]_0 = \lim_{\varepsilon \to 0^+} U(\varepsilon)^{-1}[U(\varepsilon)x, U(\varepsilon)y]
\]

exists for all \( x, y \in V \) then we call the algebra \( \mathcal{L}_0 = (V, [\cdot, \cdot]_0) \) a 1-parametric continuous contraction (or simply a contraction) of the algebra \( \mathcal{L} \) and write \( \mathcal{L} \rightarrow \mathcal{L}_0 \). We call the contraction \( \mathcal{L} \rightarrow \mathcal{L}_0 \) proper if \( \mathcal{L} \neq \mathcal{L}_0 \). Contraction to the Abelian algebra is always possible via \( U(\varepsilon) = \varepsilon I \).

It is well known that if \( \mathcal{L} \rightarrow \mathcal{L}_0 \) is any 1-parametric continuous contraction of a Lie algebra \( \mathcal{L} \) then \( \mathcal{L}_0 \) is also a Lie algebra. Invariant characteristics of Lie algebras change after a contraction. The relation among these characteristics before and after a contraction form useful necessary contraction criteria. For example, such a set of these criteria, which provided the complete classification of contractions of three and four-dimensional Lie algebras, has been found in [13]. Our aim is to state new necessary contraction criteria using \( (\alpha, \beta, \gamma) \)-derivations and twisted cocycles.

**Theorem 4.1.** Let \( \mathcal{L} \) be a complex Lie algebra, \( \mathcal{L} \rightarrow \mathcal{L}_0 \) and \( q \in \mathbb{N} \). Then for any \( (q + 1) \times (q + 1) \) complex symmetric matrix \( \kappa \)

\[
\dim Z^q(\mathcal{L}, \text{ad}_{\mathcal{L}, \kappa}) \leq \dim Z^q(\mathcal{L}_0, \text{ad}_{\mathcal{L}_0, \kappa}) \quad (15)
\]

holds.
Let $\mathcal{L} \to \mathcal{L}_0$ be a contraction of a complex Lie algebra $\mathcal{L}$, where $\mathcal{L}_0$ is a proper contraction of a complex Lie algebra $\mathcal{L}$. Then it holds
\[
\dim C^q(\mathcal{L}, \mathcal{L}_0) = \dim C^q(\mathcal{L}, \mathcal{L}_0, \epsilon), \quad 0 < \epsilon \leq 1, \; q \in \mathbb{N}.
\]
\textbf{Proof.} Since $\psi_{\mathcal{L}}(\alpha) = \dim Z^1(\mathcal{L}, \text{ad}\mathcal{L}, \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix})$ the first inequality follows from (15) and the second from $\psi_{\mathcal{L}}(1) = \dim \text{der}\mathcal{L}$ and (22). □

\textbf{Corollary 4.4.} If $\mathcal{L}_0$ is a contraction of a complex Lie algebra $\mathcal{L}$ then it holds:

1. $\psi_{\mathcal{L}} \leq \psi_{\mathcal{L}_0}$
2. $\psi^0_{\mathcal{L}} \leq \psi^0_{\mathcal{L}_0}$.

\textbf{Proof.} Since $\psi_{\mathcal{L}}(\alpha) = \dim Z^2(\mathcal{L}, \text{ad}\mathcal{L}, \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \\ 1 & \alpha \end{pmatrix})$ the first inequality follows from (15); the proof of the second condition is analogous. □

\section*{4.2. Continuous contractions of low-dimensional Lie algebras}

We have used the invariant function $\psi$ to classify all three-dimensional Lie algebras in Theorem 3.4. We now employ the necessary contraction criterion of Corollary 4.3 to describe all possible contractions among these algebras. The behaviour of the function $\psi$ determines the classification and contractions of three-dimensional Lie algebras. Contractions of three-dimensional algebras were the most recently classified in [13]:

\textbf{Theorem 4.5.} Only the following proper contractions among three-dimensional Lie algebras exist:

1. $g_{3,4}(-1)$ is a contraction of $\text{sl}(2, \mathbb{C})$,
2. $g_{3,3}$ is a contraction of $g_{3,2}$,
3. $g_{3,1}$ is a contraction of $g_{3,2}, g_{3,4}(\alpha), g_{2,1} \oplus g_1$ and $\text{sl}(2, \mathbb{C})$,
4. all algebras contract to the Abelian algebra.

Analysis of all possible pairs of three-dimensional Lie algebras leads us to the following theorem.

\textbf{Theorem 4.6} (Contractions of three-dimensional complex Lie algebras). Let $\mathcal{L}, \mathcal{L}_0$ be two three-dimensional complex Lie algebras. Then there exists a proper 1-parametric continuous contraction $\mathcal{L} \to \mathcal{L}_0$ if and only if

$$\psi_{\mathcal{L}} \leq \psi_{\mathcal{L}_0} \quad \text{and} \quad \psi_{\mathcal{L}}(1) < \psi_{\mathcal{L}_0}(1).$$

\textbf{Proof.} $\Rightarrow$: This implication is, in fact, Corollary 4.3.

$\Leftarrow$: This implication follows from a direct comparison of the tables of the invariant functions $\psi$ of three-dimensional Lie algebras in Appendix A and Theorem 4.5.

We discuss the application of the criteria of the Corollaries 4.3 and 4.4 to the four-dimensional Lie algebras in the following examples.

\textbf{Example 4.1.} To demonstrate behaviour of the functions $\psi, \varphi$ and $\varphi^0$ in dimension four, we consider the following sequence of contractions [5,13]:

$$\text{sl}(2, \mathbb{C}) \oplus g_1 \to g_{4,8}(-1) \to g_{3,4}(-1) \oplus g_1 \to g_{4,1} \to g_{3,1} \oplus g_1 \to 4g_1.$$
Table 1
Invariant functions $\psi$, $\varphi$ and $\psi^0$ of the contraction sequence: $\text{sl}(2, \mathbb{C}) \oplus g_1 \to g_{4,8}(-1) \to g_{3,4}(-1) \oplus g_1 \to g_{4,1} \to g_{3,1} \oplus g_1 \to 4g_1$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\varphi(\alpha)$</th>
<th>$\psi^0(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-1$ 0 1 2</td>
<td>$-1$ 0 1 2</td>
<td>0 1 2</td>
</tr>
<tr>
<td>$[1\text{mm}] \text{sl}(2, \mathbb{C}) \oplus g_1$</td>
<td>6 4 4 2 1 14 12 12 10 9 0 0 1 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g_{4,8}(-1)$</td>
<td>6 4 5 4 4 14 12 13 12 12 0 0 1 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g_{3,4}(-1) \oplus g_1$</td>
<td>7 7 6 5 5 16 16 14 14 3 3 3 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g_{4,1}$</td>
<td>7 7 7 7 7 16 16 15 15 3 3 3 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g_{3,1} \oplus g_1$</td>
<td>10 11 10 10 10 19 20 19 19 8 11 8 8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4g_1$</td>
<td>16 16 16 16 16 24 24 24 24 24 24 24 24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2
Invariant functions $\psi$, $\varphi$ and $\psi^0$ of the contraction: $g_{4,2}(a) \to g_{4,5}(a, 1)$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\varphi(\alpha)$</th>
<th>$\psi^0(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a$ $\frac{a}{2}$</td>
<td>$1 + a$ $\frac{a}{4}$</td>
<td>$2 + 1 + a$ $\frac{a}{4}$</td>
</tr>
<tr>
<td>$g_{4,2}(a)$</td>
<td>6 5 5 4 13 13 12 3 1 1 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g_{4,5}(a, 1)$</td>
<td>8 6 6 4 15 13 13 7 2 2 0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 4.2. Consider the pair of parametric four-dimensional Lie algebras $g_{4,2}(a), a \neq 0, \pm 1, -2$ and $g_{4,5}(a'), a' \neq 0, \pm 1, -2$. There are two possibilities, how the corresponding tables of the invariant functions $\psi$, in Appendix A cases (g-11) and (g-20), can satisfy $\psi_{g_{4,2}}(a) \leq \psi_{g_{4,5}}(a', 1)$. The first possibility leads to conditions $a' + 1 = 2/a$ and $a + 1 = 2/a'$ — these have solutions $a = a' = 1, -2$ and we excluded them. The second possibility implies $a = a'$. The necessary condition 1. of Corollary 4.4 therefore admits only the contraction $g_{4,2}(a) \to g_{4,5}(a, 1)$. This contraction indeed exists [13]. In Table 2 we summarize the behaviour of the functions $\psi$, $\varphi$ and $\psi^0$. Note that the function $\psi$ grows only at the points $1, a, \frac{1}{a}$ and the function $\psi$ only at one(!) point $1 + a$.

Example 4.3. Consider the pair of four-dimensional Lie algebras $g_{4,7}$ and $g_{4,2}(1)$. The necessary conditions $\psi_{g_{4,7}} \leq \psi_{g_{4,2}}(1)$, $[\psi_{g_{4,7}}(1) < [\psi_{g_{4,2}}(1)](1)$ and $\psi_{g_{4,7}} \leq \psi_{g_{4,2}}(1)$ are satisfied. But since it holds

$$1 = [\psi^0_{g_{4,7}}] \left( \frac{3}{2} \right) > [\psi^0_{g_{4,2}}(1)] \left( \frac{3}{2} \right) = 0,$$

a contraction is not possible.

4.3. Graded contractions of Lie algebras

Consider a graded contraction of the Pauli graded $\text{sl}(3, \mathbb{C})$ which was in [7] denoted by $\epsilon^{17,7}(\alpha)$, $\alpha \neq 0$. We may determine this graded contration by listing its non-zero commutation relations in $\mathbb{Z}_3$-labeled basis $\{l_{01}, l_{02}, l_{10}, l_{20}, l_{21}, l_{11}, l_{22}, l_{12}, l_{21}\}$:

$$L^{17,7}(\alpha) : \quad [l_{01}, l_{10}] = -al_{11}, \quad [l_{01}, l_{20}] = l_{21}, \quad [l_{01}, l_{11}] = l_{12}, \quad [l_{01}, l_{22}] = l_{20},$$

$$[l_{02}, l_{10}] = l_{12}, \quad [l_{02}, l_{22}] = l_{21}, \quad [l_{10}, l_{11}] = l_{21}, \quad a \neq 0.$$ 

Lie algebras $L^{17,7}(\alpha)$ are all indecomposable, nilpotent and their derived series, lower central series, upper central series and the number of formal Casimir invariants [16] coincide. The invariant function $\psi$ has the following form:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_{L^{17,7}(\alpha)}(\alpha)$</td>
<td>20</td>
<td>19</td>
</tr>
</tbody>
</table>
In this case, the function $\psi$ completely fails – does not depend on $a \neq 0$. We are able, however, to advance by calculation of the function $\varphi$:

$$a = 1, a = -1$$

$$\begin{array}{c|cccc}
\alpha & 0 & 1 & -1 & -1 \\
\varphi(L_{17,7}(1)(\alpha)) & 112 & 83 & 81 & 80 \\
\end{array}$$

$$\begin{array}{c|cccc}
\alpha & 0 & 1 & -1 & -1 \\
\varphi(L_{17,7}(-1)(\alpha)) & 104 & 83 & 81 & 80 \\
\end{array}$$

$$a = \frac{1}{4} + \frac{\sqrt{7}}{4} i$$

$$\begin{array}{c|cccc}
\alpha & 0 & 1 & -1 & -1 \\
\varphi(L_{17,7}(\frac{1}{4} + \frac{\sqrt{7}}{4} i)(\alpha)) & 104 & 82 & 82 & 80 \\
\end{array}$$

$$a = \frac{1}{4} - \frac{\sqrt{7}}{4} i$$

$$\begin{array}{c|cccc}
\alpha & 0 & 1 & -1 & -1 \\
\varphi(L_{17,7}(\frac{1}{4} - \frac{\sqrt{7}}{4} i)(\alpha)) & 104 & 82 & 82 & 80 \\
\end{array}$$

$$a = \frac{1}{4} + \frac{\sqrt{7}}{4} i, a = \frac{1}{4} - \frac{\sqrt{7}}{4} i$$

$$\begin{array}{c|cccc}
\alpha & 0 & 1 & -1 & -1 \\
\varphi(L_{17,7}(\alpha)(\alpha)) & 104 & 83 & 81 & 80 \\
\end{array}$$

In order to verify that

$$\varphi(L_{17,7}(a) : 1041, 821, 812, 80, \ a \neq 0, \pm 1, \frac{1}{3}, \frac{1}{4} \pm \frac{\sqrt{7}}{4} i)$$

we have to check the equality

$$-a = -\frac{1}{2} + \frac{1}{2a},$$

which has the solutions $\frac{1}{4} \pm \frac{\sqrt{7}}{4} i$. Thus, (23) is verified.

We proceed to solve the relation

$$\varphi(L_{17,7}(a) = \varphi(L_{17,7}(a'), \ a, a' \neq 0, \pm 1, \frac{1}{3}, \frac{1}{4} \pm \frac{\sqrt{7}}{4} i$$

and we obtain

1. If $-a = -a', -\frac{1}{2} + \frac{1}{2a} = \frac{1}{2} + \frac{1}{2a'}$ then $a = a'$.
2. If $-a = \frac{1}{2} + \frac{1}{2a}, -a' = -\frac{1}{2} + \frac{1}{2a'}$ then $a = a' = \frac{1}{4} \pm \frac{\sqrt{7}}{4} i$.

The second case is not possible. Observing that all other tables of the function

$$\varphi(L_{17,7}(a), a = \pm 1, \frac{1}{3}, \frac{1}{4} \pm \frac{\sqrt{7}}{4} i)$$

are mutually different, we have: if $\varphi(L_{17,7}(a) = \varphi(L_{17,7}(a'), a, a' \neq 0$ then $a = a'$. We conclude that even though the function $\psi$ did not distinguish the algebras in nilpotent parametric continuum, the function $\varphi$ provided their complete description.

5. Concluding remarks

- The invariant functions $\psi$ and $\varphi$ are able to classify all four-dimensional Lie algebras and provide also necessary contraction criteria. In order to obtain stronger contraction criteria, we also
defined the function $\phi^0$ – a supplement to the functions $\psi$ and $\varphi$ (see Example 4.3). However, the combined forces of the Corollaries 4.3 and 4.4 do not provide us with a complete classification of contractions of four-dimensional Lie algebras.

- Existence of invariant functions, arising from the concept of two-dimensional twisted cocycles and allowing classification of continuous contractions of four-dimensional complex Lie algebras, remains an open problem. The complete description of the spaces of two-dimensional twisted cocycles for four-dimensional Lie algebras would solve the existence of such functions explicitly. However, such a complete description seems, at the moment, out of reach.

- The contraction criterion formulated in Theorem 4.1 is a natural generalization of contraction criteria which involve standard cohomology cocycles – see e.g. [11,14].

- The invariant function $\psi$ can be easily generalized and used as an invariant of an arbitrary anti-commutative or commutative algebra – it can, for example, describe all two-dimensional complex Jordan algebras and their contractions [17].

- In contrast to algebraic approach of the 'trace' invariants $C_{pq}$ and $x_i$ [5,6], the concept of invariant cocycles provides a new knowledge about the dimensions of highly non-trivial structures, interlaced with given Lie algebra. Considering parametric continua of nilpotent algebras, which firstly appear in dimension 7, the invariant functions $\psi, \varphi, \phi^0$ seem to be even more important. In these cases, the behaviour of these invariant functions is quite unique and irreplaceable.

### Acknowledgements

The authors are grateful to J. Tolar for numerous stimulating discussions. Partial support by the Ministry of Education of Czech Republic (projects MSM6840770039 and LC06002) is gratefully acknowledged.

### Appendix A

Appendix A contains the classification of complex Lie algebras up to dimension four and the invariant functions $\psi, \varphi, \phi^0$. We basically follow the notation of [13]. Instead of the symbols $\psi_L, \varphi_L, \phi^0_L$, abbreviated symbols $\psi, \varphi, \phi^0$ are used. Blank spaces in the tables of the functions $\psi, \varphi, \phi^0$ denote general complex numbers, different from all previously listed values, e.g. it holds:

$$\psi_{g_{3,4}}(-1)(\alpha) = 3, \quad \alpha \in \mathbb{C}, \alpha \neq \pm 1.$$ 

#### Two-dimensional complex Lie algebras

<table>
<thead>
<tr>
<th>$g_1$ : Abelian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\psi(\alpha)$</td>
</tr>
</tbody>
</table>

$g_{2,1} : [e_1, e_2] = e_1$

| $\alpha$ | $1$ | $0$ | $\psi(\alpha)$ | $2$ | $\psi^0(\alpha)$ |
| $\psi(\alpha)$ | $2$ | $3$ | $2$ |

#### Three-dimensional complex Lie algebras

<table>
<thead>
<tr>
<th>$g_1$ : Abelian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\psi(\alpha)$</td>
</tr>
</tbody>
</table>

$g_{2,1} \oplus g_1 : [e_1, e_2] = e_2$
Four-dimensional complex Lie algebras

<table>
<thead>
<tr>
<th>$g_{3,1}$</th>
<th>$g_{3,2}$</th>
<th>$g_{3,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\psi(\alpha)$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

$g_{3,1} : [e_2, e_3] = e_1$

<table>
<thead>
<tr>
<th>$g_{3,4}(-1)$</th>
<th>$g_{3,4}(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\psi(\alpha)$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>5</td>
</tr>
</tbody>
</table>

$g_{3,4}(-1) : [e_1, e_3] = e_1, [e_2, e_3] = -e_2$

<table>
<thead>
<tr>
<th>$sl(2, \mathbb{C})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>-1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

$sl(2, \mathbb{C}) : [e_1, e_2] = e_1, [e_2, e_3] = e_3, [e_1, e_3] = 2e_2$

### (g-1) $4g_1$ : Abelian

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>24</td>
<td>24</td>
</tr>
</tbody>
</table>

### (g-2) $g_{2,1} \oplus 2g_1 : [e_1, e_2] = e_1$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>0</td>
<td>11</td>
<td>14</td>
<td>8</td>
</tr>
</tbody>
</table>

### (g-3) $g_{2,1} \oplus g_{2,1} : [e_1, e_2] = e_1, [e_3, e_4] = e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
<td>12</td>
<td>10</td>
</tr>
</tbody>
</table>

### (g-4) $g_{3,1} \oplus g_1 : [e_2, e_3] = e_1$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>0</td>
<td>11</td>
<td>19</td>
<td>11</td>
</tr>
</tbody>
</table>

### (g-5) $g_{3,2} \oplus g_1 : [e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>1</td>
<td>13</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>12</td>
<td>3</td>
</tr>
</tbody>
</table>

### (g-6) $g_{3,3} \oplus g_1 : [e_1, e_3] = e_1, [e_2, e_3] = e_2$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>12</td>
<td>3</td>
</tr>
</tbody>
</table>

### (g-7) $g_{3,4}(a) : [e_1, e_3] = e_1, [e_2, e_3] = ae_2, a \neq 0, \pm 1$
(g-7) \( g_{3,4}(-1) \oplus g_1 : [e_1, e_3] = e_1, [e_2, e_3] = -e_2 \)

\[
\begin{array}{c|ccc}
\alpha & 1 & 0 & -1 \\
\hline
\psi(\alpha) & 6 & 7 & 7 \\
\phi(\alpha) & 15 & 16 & 16 \\
\end{array}
\]

(g-8) \( g_{3,4}(a) \oplus g_1 : [e_1, e_3] = e_1, [e_2, e_3] = ae_2, a \neq 0, \pm 1 \)

\[
\begin{array}{c|c|c|c}
\alpha & 1 & a & \frac{1}{a} \\
\hline
\psi(\alpha) & 6 & 7 & 6 \\
\phi(\alpha) & 13 & 13 & 13 \\
\end{array}
\]

(g-9) \( \text{sl}(2, \mathbb{C}) \oplus g_1 : [e_1, e_2] = e_1, [e_2, e_3] = e_3, [e_1, e_3] = 2e_2 \)

\[
\begin{array}{c|c|c|c}
\alpha & 1 & 0 & -1 \\
\hline
\psi(\alpha) & 4 & 4 & 6 \\
\phi(\alpha) & 12 & 12 & 14 \\
\end{array}
\]

(g-10) \( g_{4,1} : [e_2, e_4] = e_1, [e_3, e_4] = e_2 \)

\[
\begin{array}{c|c|c|c}
\alpha & 1 & 2 & 1 \\
\hline
\psi(\alpha) & 6 & 7 & 2 \\
\phi(\alpha) & 13 & 12 & 13 \\
\end{array}
\]

(g-11) \( g_{4,2}(a) : [e_1, e_4] = ae_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3, a \neq 0, \pm 1, -2 \)

\[
\begin{array}{c|c|c|c|c}
\alpha & 1 & a & \frac{1}{a} \\
\hline
\psi(\alpha) & 6 & 5 & 4 \\
\phi(\alpha) & 13 & 13 & 12 \\
\end{array}
\]

(g-12) \( g_{4,2}(1) : [e_1, e_4] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3 \)

\[
\begin{array}{c|c|c|c|c}
\alpha & 1 & 2 & 2 \\
\hline
\psi(\alpha) & 8 & 4 & 1 \\
\phi(\alpha) & 15 & 12 & 7 \\
\end{array}
\]

(g-13) \( g_{4,2}(-2) : [e_1, e_4] = -2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3 \)

\[
\begin{array}{c|c|c|c|c}
\alpha & 1 & -2 & -1 \\
\hline
\psi(\alpha) & 6 & 5 & 5 \\
\phi(\alpha) & 15 & 12 \\
\end{array}
\]

(g-14) \( g_{4,2}(-1) : [e_1, e_4] = -e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3 \)

\[
\begin{array}{c|c|c|c|c}
\alpha & 1 & -1 & 2 \\
\hline
\psi(\alpha) & 6 & 6 & 4 \\
\phi(\alpha) & 13 & 16 & 12 \\
\end{array}
\]

(g-15) \( g_{4,3} : [e_1, e_4] = e_1, [e_3, e_4] = e_2 \)

\[
\begin{array}{c|c|c|c|c}
\alpha & 1 & 0 & 1 \\
\hline
\psi(\alpha) & 6 & 7 & 6 \\
\phi(\alpha) & 16 & 13 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
\alpha & 1 & 2 & 3 \\
\hline
\psi(\alpha) & 6 & 7 & 6 \\
\phi(\alpha) & 16 & 13 \\
\end{array}
\]
\( g_{4,4} \): \([e_1, e_4] = e_1, [e_2, e_4] = e_1 + e_2, [e_3, e_4] = e_2 + e_3 \)

\[
\begin{array}{|c|}
\hline
\alpha \\quad 1 \\
\psi(\alpha) \quad 6 \\
\hline
\end{array}
\quad \begin{array}{|c|}
\alpha \quad 2 \\
\psi(\alpha) \quad 13 \\
\hline
\end{array}
\quad \begin{array}{|c|}
\alpha \quad 1 \\
\psi(\alpha) \quad 3 \\
\hline
\end{array}
\]

\( g_{4,5}(a, b) \): \([e_1, e_4] = ae_1, [e_2, e_4] = be_2, [e_3, e_4] = e_3, a \neq 0, \pm 1, \pm b, 1/b, b^2, -1 - b, b \neq 0, \pm 1, \pm a, 1/a, a^2, -1 - a \)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\alpha & 1 & a & \frac{1}{a} & b & \frac{1}{b} & \frac{a}{b} & \frac{b}{a} \\
\psi(\alpha) & 6 & 5 & 5 & 5 & 5 & 5 & 4 \\
\hline
\end{array}
\quad \begin{array}{|c|c|c|c|c|c|}
\alpha & a + b & \frac{1-a}{b} & \frac{1+b}{a} \\
\psi(\alpha) & 13 & 13 & 13 & 13 \\
\hline
\end{array}
\]

\( g_{4,5}(a, -1 - a) \): \([e_1, e_4] = ae_1, [e_2, e_4] = (-1 - a)e_2, [e_3, e_4] = e_3, a \neq 0, \pm 1, -2, -1/2, -1/2 \pm i\sqrt{3}/2 \)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\alpha & 1 & a & \frac{1}{a} & -1 - a & -\frac{1}{1+a} & -\frac{a+1}{a} \\
\psi(\alpha) & 6 & 5 & 5 & 5 & 5 & 4 \\
\hline
\end{array}
\quad \begin{array}{|c|c|c|c|}
\alpha & a + a^2 & \frac{a^2 + 1}{a^2} & \frac{a^2 + 1}{a} \\
\psi(\alpha) & 13 & 13 & 13 & 13 \\
\hline
\end{array}
\]

\( g_{4,5}(a, a^2) \): \([e_1, e_4] = ae_1, [e_2, e_4] = a^2e_2, [e_3, e_4] = e_3, a \neq 0, \pm 1, \pm i, -1/2 \pm i\sqrt{3}/2 \)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\alpha & 1 & a & \frac{1}{a} & a^2 & \frac{1}{a^2} & \frac{a}{a^2} \\
\psi(\alpha) & 6 & 6 & 6 & 5 & 5 & 4 \\
\hline
\end{array}
\quad \begin{array}{|c|c|c|c|}
\alpha & a + a^2 & \frac{a^2 + 1}{a^2} & \frac{a^2 + 1}{a} \\
\psi(\alpha) & 13 & 13 & 13 & 13 \\
\hline
\end{array}
\]

\( g_{4,5}(a, 1) \): \([e_1, e_4] = ae_1, [e_2, e_4] = e_2, [e_3, e_4] = e_3, a \neq 0, \pm 1, -2 \)

\[
\begin{array}{|c|c|c|}
\hline
\alpha & 1 & a & \frac{1}{a} \\
\psi(\alpha) & 8 & 6 & 6 \\
\hline
\end{array}
\quad \begin{array}{|c|c|}
\alpha & 1 + a & \frac{1}{a} \\
\psi(\alpha) & 15 & 13 \\
\hline
\end{array}
\]

\( g_{4,5}(a, -1) \): \([e_1, e_4] = ae_1, [e_2, e_4] = -e_2, [e_3, e_4] = e_3, a \neq 0, \pm 1, \pm i \)

\[
\begin{array}{|c|c|c|c|}
\hline
\alpha & 1 & a & \frac{1}{a} \\
\psi(\alpha) & 6 & 5 & 5 \\
\hline
\end{array}
\quad \begin{array}{|c|c|}
\alpha & -1 + a & -\frac{1}{a} \\
\psi(\alpha) & 13 & 13 \\
\hline
\end{array}
\]

\( g_{4,5}(1, 1) \): \([e_1, e_4] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_3 \)

\[
\begin{array}{|c|c|}
\hline
\alpha & 1 \\
\psi(\alpha) & 12 \\
\hline
\end{array}
\quad \begin{array}{|c|c|}
\alpha & 2 \\
\psi(\alpha) & 18 \\
\hline
\end{array}
\quad \begin{array}{|c|c|}
\alpha & 2 \\
\psi(\alpha) & 18 \\
\hline
\end{array}
\]
(g-23) $g_{4,5}(-1, 1) : [e_1, e_4] = -e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>20</td>
<td>13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>2</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

(g-24) $g_{4,5}(-2, 1) : [e_1, e_4] = -2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>-2</th>
<th>-1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>8</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>2</th>
<th>-1</th>
<th>1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>7</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

(g-25) $g_{4,5} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) : [e_1, e_4] = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) e_1, [e_2, e_4] = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) i e_2, [e_3, e_4] = e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>-1/2 + $\frac{\sqrt{3}}{2} i$</th>
<th>-1/2 - $\frac{\sqrt{3}}{2} i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>2</th>
<th>1/2 - $\frac{\sqrt{3}}{2} i$</th>
<th>1/2 + $\frac{\sqrt{3}}{2} i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

(g-26) $g_{4,5}(i, -1) : [e_1, e_4] = i e_1, [e_2, e_4] = -e_2, [e_3, e_4] = e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>i</th>
<th>-i</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>-1 + i</th>
<th>-1 - i</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>13</td>
<td>13</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>2</th>
<th>1 + i</th>
<th>0</th>
<th>1 - i</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

(g-27) $g_{4,7} : [e_2, e_3] = e_1, [e_1, e_4] = 2 e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\frac{3}{2}$</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

(g-28) $g_{4,8}(a) : [e_2, e_3] = e_1, [e_1, e_4] = (1 + a) e_1, [e_2, e_4] = e_2, [e_3, e_4] = ae_3 a \neq 0, \pm 1, \pm 2, \pm 1/2, -1/2 \pm \sqrt{3} i/2$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>2</th>
<th>a</th>
<th>$\frac{1}{a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0</th>
<th>1</th>
<th>1 + 2a</th>
<th>$1 + \frac{a}{5}$</th>
<th>$\frac{7}{2} + \frac{a}{2}(a + \frac{1}{a})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\frac{3}{2} + \frac{a}{2}$</th>
<th>$\frac{3}{2} + \frac{a}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

(g-29) $g_{4,8}(1) : [e_2, e_3] = e_1, [e_1, e_4] = 2 e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>12</td>
<td>12</td>
<td>14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>2</th>
<th>$\frac{3}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

(g-30) $g_{4,8}(2) : [e_2, e_3] = e_1, [e_1, e_4] = 3 e_1, [e_2, e_4] = e_2, [e_3, e_4] = 2 e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>2</th>
<th>$\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\alpha)$</td>
<td>5</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0</th>
<th>1</th>
<th>5</th>
<th>2</th>
<th>$\frac{10}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>2</th>
<th>$\frac{3}{2}$</th>
<th>$\frac{4}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^0(\alpha)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Appendix B. Proof of Theorem 3.6

Lemma 5.1. For the following complex four-dimensional Lie algebras defined in Appendix A it holds:

(g-31) $g_{4,8}(0)$ : $[e_2, e_3] = e_1$, $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>12</td>
<td>13</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>13</td>
<td>14</td>
<td>12</td>
<td>10</td>
</tr>
</tbody>
</table>

(g-32) $g_{4,8}(-1)$ : $[e_2, e_3] = e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = -e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>12</td>
<td>14</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>13</td>
<td>14</td>
<td>12</td>
<td>0</td>
</tr>
</tbody>
</table>

(g-33) $g_{4,8}(-2)$ : $[e_2, e_3] = e_1$, $[e_1, e_4] = -e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = -2e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>11</td>
</tr>
</tbody>
</table>

(g-34) $g_{4,8}\left(\frac{-1}{2} + \frac{\sqrt{2}}{2}i\right)$ : $[e_2, e_3] = e_1$, $[e_1, e_4] = \left(\frac{1}{2} + \frac{\sqrt{2}}{2}i\right)e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = \left(-\frac{1}{2} + \frac{\sqrt{2}}{2}i\right)e_3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>-\frac{1}{2} + \frac{\sqrt{3}}{2}i</td>
<td>-\frac{1}{2} + \frac{\sqrt{3}}{2}i</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
<th>$\alpha$</th>
<th>$\psi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-\frac{3}{2} + \frac{\sqrt{3}}{2}i</td>
<td>-\frac{3}{2} - \frac{\sqrt{3}}{2}i</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>12</td>
<td>11</td>
</tr>
</tbody>
</table>

(g-8) $g_{3,4}(a) \oplus g_1, a \neq 0, \pm 1$

- $\psi g_{3,4}(a) \oplus g_1 : 71,63,5$
- $\phi g_{3,4}(a) \oplus g_1 : 133, 12$

(g-11) $g_{4,2}(a), a \neq 0, \pm 1, -2$

- $\psi g_{4,2}(a) : 61,52,4$
- $\phi g_{4,2}(a) : 132,12$

(g-17) $g_{4,5}(a,b), a \neq 0, \pm 1, \pm b, \frac{1}{b}, b^2, -1-b, b \neq 0, \pm 1, \pm a, \frac{1}{a}, a^2, -1-a$

- $\psi g_{4,5}(a,b) : 61,56,4$
- $\phi g_{4,5}(a,b) : 133,12$

(g-18) $g_{4,5}(a,-1-a), a \neq 0, \pm 1, -2, -\frac{1}{2}, -\frac{1}{2} \pm \frac{\sqrt{2}}{2}i$

- $\psi g_{4,5}(a,-1-a) : 61,56,4$
- $\phi g_{4,5}(a,-1-a) : 151,12$. 
Proof. Let us give the detailed proof for the case (g-18). We have to check for solutions each of 15 possible equalities
\[ a = \frac{1}{a}, \quad a = -1 - a, \quad a = -\frac{a}{a + 1}, \ldots, \quad -\frac{a}{1 + a} = -\frac{a + 1}{a}. \]
These equations have all solutions in the set \( \{0, \pm 1, -2, -\frac{1}{2} \pm \sqrt{3} i\} \) – these values we excluded from the beginning. The rest of the proof is analogous. \( \square \)

Lemma 5.2. For the four-dimensional Lie algebras from Lemma 5.1 it holds:

(g-8) If \( \psi_{g_{3,4}}(a) \oplus g_1 = \psi_{g_{3,4}}(a') \oplus g_1 \) then \( a' = a \cdot \frac{1}{a} \).
(g-11) If \( \varphi_{g_{4,2}}(a) = \varphi_{g_{4,2}}(a') \) then \( a' = a \).
(g-17) If \( \varphi_{g_{4,5}}(a, b) = \varphi_{g_{4,5}}(a', b') \) then
\[ (a', b') = (a, b), (b, a), \left( \frac{1}{a}, \frac{b}{a} \right), \left( \frac{b}{a}, \frac{1}{a} \right), \left( \frac{1}{b}, \frac{a}{b} \right), \left( \frac{a}{b}, \frac{1}{b} \right). \]
(g-18) If \( \psi_{g_{4,5}}(a, -1 - a) = \psi_{g_{4,5}}(a', -1 - a') \) then
\[ a' = a \cdot \frac{1}{a}, \quad -\frac{a}{1 + a}, -1 - a, -1 - a, -\frac{a}{1 + a}. \]
(g-19) If \( \psi_{g_{4,5}}(a, a') = \psi_{g_{4,5}}(a', a) \) then \( a' = a \cdot \frac{1}{a} \).
(g-20) If \( \varphi_{g_{4,5}}(a, 1) = \varphi_{g_{4,5}}(a', 1) \) then \( a' = a \).
(g-21) If \( \varphi_{g_{4,5}}(a, -1) = \varphi_{g_{4,5}}(a', -1) \) then \( a' = a, -a \).
(g-28) If \( \psi_{g_{4,8}}(a) = \psi_{g_{4,8}}(a') \) then \( a' = a \cdot \frac{1}{a} \).
Proof. Cases (g-8), (g-19), (g-20), (g-21) and (g-28) are obvious.

Case (g-11). The function $\varphi$ of $g_{4,2}(a')$ has the form

\[
\begin{array}{c|ccc}
\alpha & 1 + a' & \frac{a}{a} & \frac{a}{a} \\
\varphi_{g_{4,2}}(a')(\alpha) & 13 & 13 & 12 \\
\end{array}
\]

(24)

and there are two possibilities:

(12) If $a + 1 = a' + 1$, $\frac{2}{a} = \frac{2}{a'}$, then $a' = a$.

(21) If $a + 1 = \frac{2}{a}$, $a' + 1 = \frac{2}{a'}$ then $a = a' = 1, -2$, which is not possible.

Case (g-17). The function $\varphi$ of $g_{4,5}(a', b')$ has the form

\[
\begin{array}{c|ccc}
\alpha & a' + b' & \frac{1 + a}{b'} & \frac{1 + b}{a'} \\
\varphi_{g_{4,5}}(a')(\alpha) & 13 & 13 & 12 \\
\end{array}
\]

(25)

and there are six possible correspondences between this table and (g-17). We obtain:

(123) If $z_2 = \frac{1+a}{b}$, $z_3 = a + b$ then $a' = a, b' = b$.
(213) If $z_2 = \frac{1+a}{b}$, $z_3 = a + b$ then $a' = b, b' = a$.
(132) If $z_2 = a + b, z_3 = \frac{1+a}{b}$ then $a' = \frac{1}{a}, b' = b$.
(212) If $z_2 = \frac{1+a}{b}$, $z_3 = \frac{1+b}{a}$ then $a' = \frac{1}{a}, b' = \frac{1}{b}$.
(321) If $z_2 = a + 1, z_3 = \frac{1+a}{b}$ then $a' = \frac{1}{b}, b' = b$.
(312) If $z_2 = \frac{1+b}{a}, z_3 = \frac{1+b}{a}$ then $a' = \frac{1}{b}, b' = \frac{1}{b}$.

Case (g-18). It is convenient to note that six values in the table (g-18) can be arranged in the triple of pairs $(a, \frac{1}{a})$, $(-1 - a, -\frac{1}{1+a})$, and $(-\frac{1+a}{a}, -\frac{a}{1+a})$. Then one checks directly only $6 \cdot 2^3 = 48$ permutations and obtains the solutions like in the previous case. □

Corollary 5.3. For the four-dimensional Lie algebras from Lemma 5.1 it holds:

\begin{align*}
(\text{g-8}) & \text{ If } \psi g_{3,4}(a) \oplus g_1 = \psi g_{3,4}(a') \oplus g_1 \text{ then } g_{3,4}(a) \oplus g_1 \cong g_{3,4}(a') \oplus g_1. \\
(\text{g-17}) & \text{ If } \psi g_{4,5}(a,b) = \varphi g_{4,5}(a', b') \text{ then } g_{4,5}(a,b) \cong g_{4,5}(a', b'). \\
(\text{g-18}) & \text{ If } \psi g_{4,5}(a, -1 - a) = \psi g_{4,5}(a', -1 - a') \text{ then } g_{4,5}(a, -1 - a) \cong g_{4,5}(a', -1 - a'). \\
(\text{g-19}) & \text{ If } \psi g_{4,5}(a, a^2) = \psi g_{4,5}(a', a^2) \text{ then } g_{4,5}(a, a^2) \cong g_{4,5}(a', a^2). \\
(\text{g-21}) & \text{ If } \varphi g_{4,5}(a, -1) = \varphi g_{4,5}(a', -1) \text{ then } g_{4,5}(a, -1) \cong g_{4,5}(a', -1). \\
(\text{g-28}) & \text{ If } \psi g_{4,8}(a) = \psi g_{4,8}(a') \text{ then } g_{4,8}(a) \cong g_{4,8}(a').
\end{align*}

Proof. The statement follows from Lemma 5.2 and from the relations

\begin{align}
\text{g}_{3,4}(a) \oplus g_1 & \cong g_{3,4}(1/a) \oplus g_1 \\
\text{g}_{4,5}(a,b) & \cong \text{g}_{4,5}(b,a) \cong g_{4,5} \left(\frac{1}{a} \cdot \frac{b}{a}\right) \cong g_{4,5} \left(\frac{b}{a} \cdot \frac{1}{a}\right) \cong g_{4,5} \left(\frac{1}{b} \cdot \frac{a}{b}\right) \cong g_{4,5} \left(\frac{a}{b} \cdot \frac{1}{b}\right) \\
\text{g}_{4,8}(a) & \cong \text{g}_{4,8}(1/a).
\end{align}

which hold for all $a, b \neq 0$ and can be directly verified. □

Proof of Theorem 3.6. ⇒: See Corollary 3.3.

⇐: According to Lemmas 5.1, 5.2, Corollary 5.3 and observing the tables in Appendix A, we conclude that all non-isomorphic four-dimensional complex Lie algebras differ at least in one of the functions $\psi$ or $\varphi$. 
References


