A Property of the Hyperplanes of a Matroid and an Extension of Dilworth's Theorem

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1. Introduction

In this paper, I show that in any combinatorial geometry, that is a matroid without loops or parallel edges, there exist at least as many hyperplanes as elements. This is equivalent to there being at least as many copoints as points in a geometric lattice, and hence extends Dilworth's theorem [4] which proves equality only for modular lattices. Since proving this result I have discovered that both Basterfield and Kelly [1] and Greene [5] have previously established it. However my proof is totally different as it is incorporated in settling the case of equality. For I shall also show that the only geometries which have precisely as many hyperplanes as elements are the union of disjoint connected matroids, each of which is either a projective geometry or is a geometry of rank two or less. This is not true of matroids in general. The equivalent result for matroids is that there are at least as many hyperplanes as closed sets of rank one.

In [3], Dembowski and Wagner essentially prove that if the blocks of a symmetric block design form the hyperplanes of a matroid, then this matroid is a projective geometry. This result may be deduced as a corollary of the main result of this paper. Lastly I show that the conditions of the Dembowski–Wagner theorem may be relaxed, inasmuch as I prove that if the hyperplanes of a combinatorial geometry are such that any two intersect in \( \lambda \) elements, then it is either a projective geometry or has a single base or is the union of a single element and a geometry of rank two.

2. Notation

Throughout \( S \) will denote a finite set. The cardinality of a set \( X \) is denoted by \( |X| \). Let \( x \in X \), then \( (X - x) \) will denote \( (X - \{x\}) \).
A matroid on a set $S$ is a family $\mathbf{M}$ of subsets of $S$ such that:

(i) $\emptyset \in \mathbf{M}$.

(ii) If $A \in \mathbf{M}$ and $B \subseteq A$, then $B \in \mathbf{M}$.

(iii) If $A \subset \mathbf{M}$, $B \subset \mathbf{M}$, and $|A| = |B| + 1$, there exists an element $x \in (A - B)$ such that $B \cup \{x\} \in \mathbf{M}$.

The members of $\mathbf{M}$ are the independent sets of the matroid. A set is dependent if it is not independent. A circuit is a minimal dependent set, and a base is a maximal independent subset of $S$. If $Z \subseteq S$, the rank $r(Z)$ of $Z$ is the cardinality of a maximal independent subset of $Z$. I shall call $r(S)$ the rank of $\mathbf{M}$. A subset $X$ of $S$ is closed in $\mathbf{M}$ if for any $x \in S - X$,

$$r(X \cup \{x\}) = r(X) + 1.$$ 

I shall call $X$ a $k$-closed set of $\mathbf{M}$ if $X$ is closed in $\mathbf{M}$ and $r(X) = k$. A maximal proper closed subset of $S$ is a hyperplane of $\mathbf{M}$. The closure of $X$, denoted by $\mathbf{C}(X)$, is the minimal closed set containing $X$. I shall denote by $h(\mathbf{M})$ the number of hyperplanes of $\mathbf{M}$.

The matroid $\mathbf{M}^*$ dual to $\mathbf{M}$ is the matroid on $S$ which has as its bases all sets of the form $S - B$, where $B$ is a base of $\mathbf{M}$. A base of $\mathbf{M}^*$ is a cobase of $\mathbf{M}$, a circuit of $\mathbf{M}^*$ is a cocircuit of $\mathbf{M}$ and so on. If $T \subseteq S$, denote by $\mathbf{M} \times T$, the reduction of $\mathbf{M}$ to $T$, that is $X$ is an independent set of $\mathbf{M} \times T$, if $X \subseteq T$ and $X$ is independent in $\mathbf{M}$. The contraction $\mathbf{M} \cdot T$ of $\mathbf{M}$ to $T$ is defined by

$$(\mathbf{M} \cdot T) = (\mathbf{M}^* \times T)^*.$$ 

Alternatively the circuits of $\mathbf{M} \cdot T$ are the minimal nonnull members of the family $(C_i \cap T: C_i$ is a circuit of $\mathbf{M})$. A geometry is a matroid in which every two-element subset of $S$ is independent. If $\mathbf{M}_1$, $\mathbf{M}_2$ are matroids on $S$, their union $\mathbf{M}_1 \vee \mathbf{M}_2$ is a matroid on $S$ defined to have as its independent sets, all sets of the form $X_1 \cup X_2$, where $X_1 \in \mathbf{M}_1$, $X_2 \in \mathbf{M}_2$.

A matroid is connected or nonseparable if there exists no proper subset $A$ of $S$, such that

$$r(A) + r(S - A) = r(S).$$

Alternatively a matroid is connected if every pair of elements of $S$ are contained in a circuit. Let $\mathbf{M}$ be a matroid on $S$. Then if $A \subseteq S$ such that

$$r(S) + r(S - A) = r(S),$$

and $A$ is minimal nonnull with respect to this property, then $\mathbf{M} \times A$ is called a connected component of $\mathbf{M}$. Clearly $\mathbf{M} = \mathbf{M}_1 \vee \mathbf{M}_2 \vee \cdots \vee \mathbf{M}_k$, where each $\mathbf{M}_i$ is a connected component of $\mathbf{M}$. This matroid notation has been that of Harary and Welsh [7].
My block design terminology is that of Hall's book [6]. In particular a symmetric \((v, k, \lambda)\) design on the set of elements \(X = \{x_i: 1 \leq i \leq v\}\) is a set \((X_i: 1 \leq i \leq v)\) of subsets of \(X\) such that:

(iv) \(|X_i| = k, 1 \leq i \leq v;\)

(v) \(|X_i \cap X_j| = \lambda, i \neq j.\)

(vi) The integers \(v, k, \lambda\) satisfy \(0 < \lambda < k < v - 1.\)

The subsets \(X_i\) are called the blocks of the design.

Call the parameter set \((v, k, \lambda)\) matroidal if there exists a design with these parameters such that the blocks of the design form the hyperplanes of a matroid. That is they satisfy the following hyperplane axioms for a matroid.

**HYPERPLANE AXIOM.** If \(H_1, H_2\) are distinct hyperplanes and \(x \in (S - (H_1 \cap H_2)),\)

there exists a hyperplane \(H_3\) such that

\[\{x\} \cup (H_1 \cap H_2) \subseteq H_3.\]

If the hyperplanes of a matroid form the blocks of a design, then the matroid is called a design matroid.

Again following Hall [6], a projective geometry is a set of points and a set of subsets of the points called lines such that:

**P.G.1** There is one and only one line containing two distinct points.

**P.G.2** If \(A, B, C\) are three points not on a line, and if \(D \neq A\) is a point on the line through \(A\) and \(B\), and if \(E \neq A\) is a point on the line through \(A\) and \(C\), then there is a point \(F\) on a line with \(D\) and \(E\) and also on a line with \(B\) and \(C\).

**P.G.3** Every line contains at least three distinct points.

Thus considering the two-closed sets of a matroid \(M\) as lines and the elements as points, these form a projective geometry iff:

(vii) \(M\) is a geometry.

(viii) Let \(K\) be a three-closed set of \(M\), and let \(\{x_1, x_2, x_3\}\) be an independent set of \(M\) contained in \(K\). Let \(x_4 \in C(\{x_1, x_2\}), x_4 \neq x_1,\) and \(x_5 \in C(\{x_2, x_3\}), x_5 \neq x_1,\) then

\((C(\{x_1, x_3\})) \cap (C(\{x_2, x_3\})) \neq \emptyset.\)

(ix) Every two-closed set contains at least three elements.

Veblen and Young [11] define the dimension of a projective space recursively, starting from a point being of dimension zero and a line of dimension 1.
If $X_{n-1}$ is a space of dimension $n - 1$ and $P$ a point not contained in $X_{n-1}$, the set of all points in all lines $PB$, $B$ a point of $X_{n-1}$, is a space $X_n$ of dimension $n$. The set of all points of a projective geometry is clearly a space of the projective geometry. Its dimension is the dimension of the projective geometry. Thus a matroid $M$ is a projective geometry of dimension $n$, if its elements and 2-closed sets form a projective geometry $PG$, and a set is a $k$-closed set of $M$ iff it is a $(k - 1)$-dimensional subspace of $PG$.

Let $PG(n, q)$ be a projective geometry of dimension $n \geq 3$ over a finite field of size $q = p^r$, where $p$ is a prime. Then

(x) There are precisely $(q^{n+1} - 1)/(q - 1)$ points.

(xi) Every hyperplane contains $(q^n - 1)/(q - 1)$ points.

(xii) Every $k$-dimensional subspace contains $(q^{k+1} - 1)/(q - 1)$ points.

These are well known properties of a projective geometry of dimension three or more. If it is of dimension two then it is a projective plane. An important property of projective planes is that the hyperplanes, that is the lines, are all of the same cardinality. Perhaps the most important property of a projective geometry, as far as this paper is concerned, is that there are the same number of hyperplanes as points. Also to be used is that every pair of hyperplanes intersect in a $(n - 2)$ dimensional subspace. That is the hyperplanes form the blocks of a $(v, K, h)$-design.

A geometric lattice (or matroid lattice) is a finite dimensional semimodular point lattice. By considering points of a geometric lattice as elements and the rank of a subset of elements as the height of the join of the corresponding points, we get a rank function of a geometry. The elements of the lattice of height $k$ correspond to the $k$-closed sets of the matroid. The reverse process of defining a geometric lattice from a geometry, is most easily seen as follows. Construct each level of the lattice successively as follows. The level of height $k$ consists of elements which correspond to the $k$-closed sets of the matroid, and an element of height $k - 1$ is joined to an element of height $k$ if the corresponding closed sets are such that the $(k - 1)$-closed set is a subset of the $k$-closed set. Clearly the copoints of the geometric lattice correspond to the hyperplanes of the geometry. Further lattice theory may be found in Birkhoff [2], which has provided the terminology already used.

3. Preliminary Lemmas

It is easy to see that $H$ is a hyperplane of $M$ on $S$ iff $S - H$ is a cocircuit of $M$. We also need to observe that if $M$ is connected, then $M^*$ is a connected matroid.
**Lemma 1.** If $M_1, M_2$ are matroids on disjoint sets $S_1, S_2$, then the hyperplanes of $M_1 \vee M_2$ are precisely those sets of the form $(S_1 \cup H_2)$ and $(S_2 \cup H_1)$, where $H_1, H_2$ are hyperplanes of $M_1$ and $M_2$ respectively.

**Lemma 2.** If $M$ is a geometry on $S$, and $T \subseteq S$. Then $M \times T$ is not only a matroid but a geometry.

The proof of the first two lemmas is elementary from the definitions and is left to the reader.

**Lemma 3.** Let $M$ be a matroid on $S$, and $T \subseteq S$. Then $h(M \times T) \leq h(M)$. Moreover if $H$ is a hyperplane of $M \times T$, then $H = K \cap T$, where $K$ is a hyperplane of $M$.

**Proof.** $M \times T = (M^* \cdot T)^*$. The circuits of $M^* \cdot T$ are precisely the minimal nonnull members of the family $(C_i \cap T; C_i$ is a circuit of $M^*)$. Thus $M^* \cdot T$ has at most as many circuits as $M^*$. Since a hyperplane is the complement of a cocircuit the result follows.

**Lemma 4.** Let $M$ be a matroid on $S$. Let $C$ be a cocircuit of $M$, and let $x \in S$ be such that $x \in C$. Let $|C| = k$, and let $x$ belongs to $d$ cocircuits. Then $M \times (S - C)$ has $(n - k)$ elements and

$$h(M \times (S - C)) \leq h(M) - d.$$

**Proof.** Consider $M^*$, then $C$ is a circuit of $M^*$. Let

$$M_2 = M^* \cdot (S - (C - x)).$$

The circuits of $M_2$ are precisely those minimal nonnull members of the family $(C_i \cap (S - (C - x)); C_i$ is a circuit of $M^*)$. Clearly $x$ is a circuit of $M_2$. Thus if $x \in C_j$, then $C_j \cap (S - (C - x))$ is not a circuit of $M_2$ unless $C_j = C$. Thus $M_2$ has at least $(d - 1)$ circuits fewer than $M^*$, which implies $M^* \cdot (S - C)$ has at least $d$ circuits less than $M^*$. Again since the complement of a hyperplane is a cocircuit the result follows.

The following lemma is the most important result required to show that $h(M) = 1_S$ implies that $M$ is a projective geometry. To see this I refer the reader to the description of Theorem 1 just prior to its proof.

**Lemma 5.** Let $M$ be a connected matroid. Let $C$ be a cocircuit of $M$, such that every element of $C$ belongs to exactly $k$ cocircuits of $M$. Then if $M \times (S - C)$ has precisely $k$ fewer cocircuits than $M$, it is connected.
Proof. Since $M \times (S - C)$ has exactly $k$ fewer cocircuits than $M$, for each $x \in C$, the proof of Lemma 4 with $k = d$ demonstrates that the cocircuits of $M$ are described by

$$(C_i \cap (S - C): x \notin C_i, C_i \text{ a cocircuit of } M).$$

Since $x$ is an arbitrary member of $C$, it follows that, in this special case with $k = d$ the cocircuits of $M \times (S - C)$ are given by

$$(C_i \cap (S - C): C_i \text{ a cocircuit of } M, C_i \neq C).$$

Since $C_i \cap (S - C)$ can avoid being a cocircuit only if it contains $x$ for all $x \in C$. It follows that $M \times (S - C)$ must be connected. For if $a, b \in S - C$, then there is a cocircuit $C'$ containing $a$ and $b$ since $M$ (and hence $M^*$) is connected, and so $a, b \in C' \cap (S - C)$ which is a cocircuit of $M \times (S - C)$.

4. MAIN RESULTS

It is obvious that a connected matroid $M$ on $S$ has all its hyperplanes of the same cardinality and exactly the same number of hyperplanes as elements if

(i) $M$ is a projective geometry.

(ii) $M$ is of rank two (every element is a hyperplane).

(iii) $M$ consists of just a single independent element.

The main result is to show that these are the only connected geometries which have the property that $h(M) = |S|$, and that for any connected geometry $M$ $h(M) \geq |S|$. This result, called Theorem 1, has a long proof, and therefore I offer this synopsis so that the reader may know where the proof is headed.

It is first assumed true for geometries with $n - 1$ elements or less. Then assume that $M$ is a geometry on $S$ such that $h(M) \leq |S|$ and $|S| = n$. It is a relatively easy matter, using Lemmas 1–3, to show that $M \times (S - e)$ contradicts the induction hypothesis.

Next suppose $M$ is a connected geometry on $S$, and that $|S| = n = h(M)$. Then, by employing Lemma 4, it will be shown that $M$ has cocircuits all of the same cardinality $k$, and every element belongs to precisely $k$ cocircuits. Let $H_i \subseteq S - C_i$, $C_i$ a cocircuit of $M$. Then, by Lemma 5, $M \times H_i$ is a connected geometry on $(n - k)$ elements with $(n - k)$ hyperplanes. Thus the induction hypothesis implies that $M \times H_i$ is a projective geometry. Clearly since the cocircuits of $M$ are all of cardinality $k$, $|H_i| = n - k = t$.
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1 \leq i \leq n. Also every element of \( M \) belongs to \( t \) hyperplanes. It is a fairly easy matter to show that the hyperplanes of \( M \), regarded as blocks of a design, form a \((v, k, \lambda)\) design, using the fact that \( M \times H_i \) is a projective geometry. By employing this again, together with the conditions of a \((v, k, \lambda)\) design, it can be shown that \( M \) is a projective geometry.

**Theorem 1.** If \( M \) is a connected geometry on \( S \), then \( h(M) \geq |S| \), with equality if and only if \( M \) is either a projective geometry or a geometry of rank two or less.

**Proof.** Clearly the theorem is true if \(|S| = 1\). Assume that the theorem is true for \(|S| \leq n - 1\). If \( M \) is a disconnected geometry on \( S \), \(|S| \leq n - 1\), then by combining Lemma 1 with the induction hypothesis for the connected components of \( M \), \( h(M) \geq |S| \).

Suppose that \( M \) is a connected geometry on \( S \), \(|S| = n \), such that \( h(M) < |S| \). Let \( e \in S \), then consider \( M \times (S - e) = (M^* \cdot (S - e))^* \). Since \( M \) is connected, \( M^* \) is connected. If \( M^* \cdot (S - e) \) is not connected then at least one circuit of \( M^* \) must have been destroyed by contracting \( e \). That is \( h(M \times (S - e)) < h(M) \). But by the induction hypothesis \( h(M \times (S - e)) \geq n - 1 \), thus \( h(M) \geq n \), which contradicts the initial supposition.

Therefore \( M^* \cdot (S - e) \) must be connected. Combining with Lemma 2, this implies that \( M \times (S - e) \) is a connected geometry on \((n - 1)\) elements. By Lemma 3, \( h(M \times (S - e)) \leq h(M) \), which together with the induction hypothesis implies \( h(M \times (S - e)) = h(M) = n - 1 \). Suppose \( M \) has rank three and that \( M \times (S - e) \) has rank two. Thus \((S - e)\) is a hyperplane of \( M \), which means \( e \) is a cocircuit of \( M \). The cocircuit \( e \) implies \( M^* \) is not connected, which contradicts the initial hypothesis that \( M \) is connected. Therefore both \( M \) and \( M \times (S - e) \) have ranks greater than two. Thus combining with the induction hypothesis \( M \times (S - e) \) is a projective geometry on \( n - 1 \) elements. A projective geometry on \((q^{t+1} - 1)/(q - 1)\) points has hyperplanes of cardinality \((q^t - 1)/(q - 1)\). Suppose \( H_i \) is a hyperplane of \( M \). Since \( h(M \times (S - e)) = h(M) \), \( H_i \cap (S - e) \) is a hyperplane of \( M \times (S - e) \) by Lemma 3. But the hyperplanes of \( M \times (S - e) \) are all of the same cardinality \( k \), since \( M \times (S - e) \) is a projective geometry on \( n - 1 \) elements and hence \( k \) is independent of \( e \). \( k > 0 \) since \( M \times (S - e) \) has rank greater than two. Thus the hyperplanes of \( M \) which do not contain \( e \) are of cardinality \( k \), and those which do contain \( e \) are of cardinality \( k + 1 \). But \( e \) was an arbitrary element of \( S \). Thus any hyperplane of \( M \), since it contains an element, contains \( S \). This contradicts the definition of a hyperplane. Thus using the induction hypothesis, connected geometries on \( n \) elements have at least \( n \) hyperplanes.
Next suppose $M$ is a connected geometry on $S$, $|S| = n$, such that $h(M) = n$. Let the cocircuits of $M$ be $(C_i: 1 \leq i \leq n)$ and let 

$$k_i = |C_i|, \quad 1 \leq i \leq n.$$ 

Also suppose that $k_1 \leq k_2 \leq \cdots \leq k_n$.

Let $S = \{x_j: 1 \leq j \leq n\}$ and let $x_j$ belong to the cocircuits of $M$. Suppose that $d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_n$.

By elementary counting

$$\sum_{i=1}^{n} k_i = \sum_{j=1}^{n} d_j. \quad (4.1)$$

Let cocircuit $C_i$ contain the element $x_j$. Suppose $d_j > k_i$, then by Lemma 4, $M \times (S - C_i)$ is a geometry on $(n - k_i)$ elements and has at most $(n - d_j) < (n - k_i)$ hyperplanes. This contradicts the induction hypothesis whether it is connected or not. Thus $d_j \leq k_i$ for all $C_i$, such that $x_j \in C_i$.

Clearly $d_1 \leq k_1$, for otherwise $C_1$ could have no members; and suppose

$$d_r \leq k_r, \quad 1 \leq r \leq p - 1,$$

but that $d_p > k_p$. This implies that no member of $\{x_i: p \leq i \leq n\}$ belongs to a member of $(C_i: 1 \leq i \leq p)$. Thus the cocircuits $(C_1, C_2, \ldots, C_p)$ are contained in the set $\{x_1, x_2, \ldots, x_{p-1}\}$. Let $S - T = \{x_1, x_2, \ldots, x_{p-1}\}$. Consider $M \times T$. This is a geometry with $(n - p + 1)$ elements and only $(n - p)$ hyperplanes, which contradicts the induction hypothesis. Thus $d_p \leq k_p$, and therefore by induction

$$d_i \leq k_i, \quad 1 \leq i \leq n.$$ 

But from (4.1),

$$\sum_{i=1}^{n} k_i = \sum_{i=1}^{n} d_i.$$

Thus combining the two,

$$d_i = k_i, \quad 1 \leq i \leq n.$$ 

Therefore $d_1 = k_1$, and assume that $d_i = k_i = d_1 = k_1, 1 \leq i \leq q - 1$. Now suppose that, $d_q = k_q > k_1 = d_1$. Since $d_j \leq k_i$ for all $C_i$, such that $x_j \in C_i$, then

$$x_j \notin C_i \quad (1 \leq i \leq q - 1; q \leq j \leq n).$$

Thus

$$\bigcup_{i=1}^{q-1} C_i \subset \bigcup_{i=1}^{q-1} \{x_i\}.$$
Let $d_j'$ be the number of $C_i$, $1 \leq i \leq q - 1$, to which $x_j$ belongs. Again by counting repetitions,

$$
\sum_{j=1}^{q-1} d_j' = \sum_{i=1}^{q-1} k_i.
$$

But

$$
\sum_{j=1}^{q-1} d_j = \sum_{i=1}^{q-1} k_i = (q - 1) k_1.
$$

Combining these and the fact that $d_j' \leq d_j$, implies that

$$
d_j = d_j' = d_1, \quad 1 \leq j \leq q - 1.
$$

Thus $x_1$ does not belong to any member of $(C_i; q \leq i \leq n)$. But $x_q$ does not belong to any member of $(C_i; 1 \leq i \leq q - 1)$, as these cocircuits only contain the elements $\{x_i; 1 \leq i \leq q - 1\}$. Therefore $x_1$ and $x_q$ cannot be contained in a cocircuit, which implies $M^*$ is not connected. Thus $M$ is not connected which is a contradiction. Thus by induction

$$
d_i = k_i = k_1, \quad 1 \leq i \leq n.
$$

Thus $M$ is a geometry with $n$ hyperplanes all of the same cardinality $t$, and each element of $M$ belongs to exactly $t$ hyperplanes. It will now be shown that each hyperplane intersects every other in precisely the same number of elements.

Let $H_i = S - C_i$. By Lemma 5, $M \times H_i$ is a connected geometry on $n - k_1$ elements with $n - d_j = n - k_1$ hyperplanes, and therefore by the induction hypothesis is a projective geometry or has rank two or less. Thus all its hyperplanes have the same cardinality. Now every hyperplane $H_j$ of $M$ ($j \neq i$) meets $C_i$ in some point $x$. But the $t$ distinct hyperplanes of $M$ containing $x$ arise from the $t$ distinct hyperplanes of $M \times H_i$, by taking their closure with $x$. Thus $H_j \cap H_i$ is a hyperplane of $M \times H_i$ (all of which have the same cardinality). Therefore each hyperplane $H_i$ intersects every other hyperplane in $a_i$ elements. This implies that $|H_i \cap H_j| = a_i = a_j$.

Thus every member of $(H_i; 1 \leq i \leq n)$ is of the same cardinality and intersects every other in precisely the same number of elements $a$. That is the hyperplanes regarded as the blocks of a design, form a $(v, k, \lambda)$ design with $v = n, k = t, \lambda = a$, provided $0 < a < t < n - 1$.

If $t = (n - 1)$ then clearly $M$ consists of a single element, since every element is a cocircuit of $M$, but $M$ is supposed to be connected.

If $t < (n - 1)$, but $a = 0$, then it follows that $M$ must be of rank two.

This leaves the case where the hyperplanes of $M$ form a $(v, k, \lambda)$-design.
If $\mathbf{M}$ has rank three, then the hyperplanes of $\mathbf{M}$ are two-closed sets, and every pair intersects in a single element. Clearly $\mathbf{M}$ will satisfy the axioms of a projective geometry. It is also easy to see that the dimension is two and is called a projective plane. Suppose $\mathbf{M}$ has rank greater than three. Consider $K$ a 3-closed set of $\mathbf{M}$. Let $H$ be a hyperplane of $\mathbf{M}$ such that $K \subset H$. This is always possible since $r(H) \geq 3$. But $\mathbf{M} \times H$ is a connected geometry, such that $h(\mathbf{M} \times H) = |H|$, as was shown earlier, thus $\mathbf{M} \times H$ is a projective geometry. Since $\mathbf{M} \times H$ is a projective geometry, the axioms of a projective geometry apply for the 3-closed set $K$. But $K$ was any 3-closed set of $\mathbf{M}$. Thus the 2-closed sets and elements of $\mathbf{M}$ form the lines and points of a projective geometry. Since $\mathbf{M} \times H$ is a projective geometry of dimension $m$, clearly the $k$-dimensional subspaces of $\mathbf{M}$ are the $(k + 1)$-closed sets of $\mathbf{M}$ provided $k \leq m$. This means to ensure $\mathbf{M}$ is a projective geometry, all that is required is to confirm that its dimension is $m + 1$. This will be the case if every 2-closed set of $\mathbf{M}$ intersects every hyperplane. Let $L$ be a 2-closed set of $\mathbf{M}$, and $H_1$ a hyperplane of $\mathbf{M}$, such that $L \subset H_1$. Since $\mathbf{M} \times H_1$ is a projective geometry $L$ intersects every $(m - 1)$ dimensional subspace. That is, $L$ intersects every $m$-closed set of $\mathbf{M}$ contained by $H_1$. Consider any hyperplane $H_2$ of $\mathbf{M}$, $H_1 \cap H_2$ is an $m$-closed set of $\mathbf{M}$ contained by $H_1$. Thus $L$ intersects $H_2$. This implies $\mathbf{M}$ is of dimension $m + 1$. Therefore $\mathbf{M}$ is a projective geometry. This establishes Theorem 1 by induction.

**Dembowski-Wagner Theorem.** The parameter set $(v, k, \lambda)$ is matroidal if and only if it is one of the following types:

(a) $\lambda = 1$, that is the parameters correspond to a projective not necessarily desarguesian plane.

(b) It is the parameter set of a design derived from a projective geometry $PG(n, q)$; that is, for some prime $p$ and positive integer $n$,

$$v = \frac{(q^{n+1} - 1)}{(q - 1)}, \quad k = \frac{(q^n - 1)}{(q - 1)}, \quad \lambda = \frac{(q^{n-1} - 1)}{(q - 1)},$$

where $q = p^r$, for some positive integer $r$.

The proof of this theorem, which may be found quoted in this form in Welsh [12], appears in Dembowski-Wagner [3], and later presented by Kantor [9]. The sufficiency of the theorem is obvious. Its necessity may either be deduced from Theorem 1, or the last part of the proof of Theorem 1 provides an easy direct proof.

**Theorem 2.** If $\mathbf{M}$ is a geometry on $S$. Then $h(\mathbf{M}) \geq |S|$ with equality iff $\mathbf{M} = \mathbf{M}_1 \vee \mathbf{M}_2 \vee \cdots \vee \mathbf{M}_k$, where $\mathbf{M}_i$ is a matroid on $S_i$, $(S_i: 1 \leq i \leq k)$.
forming a partition of $S$, and each $M_i$ is either a projective space or a geometry of rank two or less.

Proof. Consider the connected components of $M$ on $S$. Each of these contains at least as many hyperplanes as elements, which by Lemma 1 implies the first part of the theorem. Equality will only occur for $M$, if for every connected component $M_i$, $h(M_i) = |S_i|$. Thus Lemma 1 and Theorem 1 applied to each $M_i$ give Theorem 2.

In his paper [12], Welsh conjectured that if $M$ is a geometry such that for any two distinct hyperplanes $H_i$, $H_j$, $|H_i \cap H_j| = \lambda > 1$, and $|H_i| = |H_j|$, then $M$ is a design matroid. I shall prove the following stronger result.

**Theorem 3.** Let $M$ be a geometry on $S$, such that if $H_i$, $H_j$ are hyperplanes of $M$, then $|H_i \cap H_j| = \lambda \geq 1$, $i \neq j$. Then $M$ is either a projective geometry or the union of a single independent element and a geometry of rank two or just consists of a single base.

Proof. This result depends on a theorem by Isbell [8]. He showed that if $(B_i, l \leq i \leq m)$ are subsets of $S, |S| = n, with |B_i \cap B_j| = \lambda$, for $i \neq j$, and $1 \leq i, j \leq m$, then $m \leq n$. The same result in dual form was obtained by Majumdar [10] earlier.

Combining the above condition with the fact that I have proved any geometry has at least as many hyperplanes as elements, gives $h(M) = |S|$. Theorem 2 tells us that $M = M_1 \lor M_2 \lor \cdots \lor M_k$, in which each $M_i$ is either a projective geometry or a geometry of rank two or less. By Lemma 3 and the fact that the complement of a hyperplane is a cocircuit, the cocircuits of $M$ are precisely the cocircuits of each $M_i$. Since the intersection of any pair of hyperplanes of $M$ has cardinality $\lambda$, then the union of any pair of cocircuits of $M$ has cardinality $|S| - \lambda$.

Now suppose that $M_1$ has rank greater than two. Thus $M_1$ is a projective geometry. Hence its cocircuits are of cardinality $q', q, r \geq 2$, and the cardinality of the union of a pair of its cocircuits is $q' + q'^{-1} = |S| - \lambda$. Consider $M_k$ then the union of a cocircuit of $M_2$ and a cocircuit of $M_k$ has cardinality $q' + k$, where $k$ is the cardinality of a cocircuit of $M_2$. Thus $k = q'^{-1}$. But $M_k$ is either a projective geometry or a geometry of rank two or less. If $M_k$ is a projective geometry with cocircuits of cardinality $q'^{-1}$, the union of a pair of cocircuits of $M_k$ has cardinality $q'^{-1} + q'^{-2} \neq |S| - \lambda$, which contradicts the original condition. Next suppose that $M_2$ has rank two. Since the elements of $M_2$ are hyperplanes, the cardinality of a cocircuit of $M_2$ is $|S_2| - 1 = k = q'^{-1}$, and the cardinality of the union of a pair of cocircuits of $M_2$ is $|S_2| \neq |S| - \lambda$. This contradiction leaves only the case of $M_2$ being a single element. This element is a cocircuit of $M_2$, that
is \( k = 1 \), and thus \( k \) cannot equal \( q^r - 1 \). Therefore if \( M_1 \) is a projective
geometry then \( M = M_1 \). Clearly the same result will follow if any of the \( M_i \)
is a projective geometry.

Now suppose that \( M_i \) has rank two. As just shown its cocircuits are of
Cardinality \( |S_1| - 1 \), and the union of a pair of its cocircuits has cardinality
\( |S_1| \). Thus \( |S_1| = |S| - \lambda \). Since \( \lambda \geq 1 \), there exists \( M_2 \), a geometry
which has rank two or less. Suppose \( M_2 \) has rank two, then again
\( |S_2| = |S| - \lambda \), and the cocircuits of \( M_2 \) have cardinality \( |S_2| - 1 \).
Thus the union of a cocircuit of \( M_2 \) and a cocircuit of \( M_1 \) has cardinality
\( |S_1| + |S_2| - 2 = 2|S| - 2\lambda - 2 \) which must equal \( |S| - \lambda \).

Thus \( |S| - \lambda = 2 \), therefore \( |S_1| = |S_2| = 2 \). This implies that both
\( M_1 \) and \( M_2 \) consist of just a single base of two elements. Following a similar
argument any other \( M_i, 3 \leq i \leq k \), must either consist of a pair of elements
which form a base or consists of just a single independent element. Thus \( M \)
consists of a single base. Alternatively \( M_i \) may be the only one of \( M_i \),
\( 1 \leq i \leq k \), which has rank greater than one. Thus \( M_1 \lor M_2 \lor \cdots \lor M_k \)
is a geometry which consists of just a single base. Therefore the cocircuits of
\( M \) are those of \( M_1 \), which have cardinality \( |S_1| - 1 \), and those of
\( M_2 \lor M_3 \lor \cdots \lor M_k = M \), which have cardinality one. The union of a pair
of cocircuits of \( M_1 \) has cardinality \( |S_1| \). The cardinality of the union of a pair
of cocircuits of \( M_i \) is 2. Thus again \( |S_1| = 2 \), and \( M \) will again be
found to be a single base. This just leaves the case of \( M = M_1 \lor M_2 \),
where \( M_i \) is of rank two and \( M_2 \) is just a single element. Clearly by taking
\( M_i \) there has been no loss of generality and the same result would follow for
any \( M_i, 1 \leq i \leq k \).

This just leaves the case in which each \( M_i, 1 \leq i \leq k \), is just a single
element. Hence \( M \) consists of just a single base. Thus the truth of Theorem 3
has been established.

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