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Minimal cubature formulas exact for Haar polynomials

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Abstract

The cubature formulas we consider are exact for spaces of Haar polynomials in one or two variables. Among all cubature formulas, being exact for the same class of Haar polynomials, those with a minimal number of nodes are of special interest. We outline here the research and construction of such cubature formulas.

Keywords: Cubature formulas; Haar polynomials; Minimal number of nodes

0. Introduction

The problem of constructing and analyzing cubature formulas, which integrate exactly a given collection of functions, has been mainly considered before in the cases when these functions are algebraic or trigonometric polynomials (see, for instance, [11,12]). The approximate integration formulas, exact for finite Haar sums, can be found in the monograph [13] and articles [1,2]. Nevertheless, in these works the authors did not consider the question of minimizing the number of nodes while preserving exactness of the formula on such sums. We will use a notion of the Haar polynomials [9,5] and the corresponding definition of the exactness of approximate integration formulas on the mentioned polynomials. It allows us to introduce a notion of a minimal approximate integration formula, exact for the Haar polynomials. The research and construction of such quadrature and cubature formulas was established in [9,10]. In the present article we will outline the main results of these works. Let us mention that in this article we only consider the one- and two-dimensional cases.

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1. The Haar polynomials

We recall the definition of the functions $\chi_{m,j}(x)$, introduced by A. Haar [3].

By binary intervals $l_{m,j}$ we denote the intervals with the ends at $(j-1)/2^{m-1}$, $j/2^{m-1}$, $m = 1, 2, \dots,\ j = 1, 2, \dots, 2^{m-1}$. If the left end of a binary interval coincides with 0, then we will consider this interval as closed from the left. If the right end of a binary interval coincides with 1, then we will consider this interval as closed from the right. The remaining intervals we consider as open ones. The left (right) half $l_{m,j}$ (taking out the midpoint of) we will denote by $l^-_{m,j}$ ($l^+_{m,j}$) so that

$$l^-_{m,1} = \left[0, \frac{1}{2^m}\right), \qquad l^-_{m,j} = \left(\frac{j-1}{2^{m-1}}, \frac{2j-1}{2^m}\right), \quad j = 2, \dots, 2^{m-1},$$
$$l^+_{m,j} = \left(\frac{2j-1}{2^m}, \frac{j}{2^{m-1}}\right), \quad j = 1, \dots, 2^{m-1}-1, \qquad l^+_{m,2^{m-1}} = \left(1 - \frac{1}{2^m}, 1\right].$$

It is convenient to construct the Haar system of functions by classes: the m-th class contains 2^{m-1} functions $\chi_{m,j}(x)$, where $m = 1, 2, \dots,\ j = 1, 2, \dots, 2^{m-1}$. The Haar functions $\chi_{m,j}(x)$ are defined by:

$$\chi_{m,j}(x) = \begin{cases} 2^{\frac{m-1}{2}}, & \text{if } x \in l^-_{m,j}, \\ -2^{\frac{m-1}{2}}, & \text{if } x \in l^+_{m,j}, \\ 0, & \text{if } x \in [0,1] \setminus \overline{l_{m,j}}, \\ \dfrac{\chi_{m,j}(x-0) + \chi_{m,j}(x+0)}{2}, & \text{if } x \text{ is an interior break point} \end{cases} \tag{1.1}$$

with $\overline{l_{m,j}} = [\frac{j-1}{2^{m-1}}, \frac{j}{2^{m-1}}]$, $m = 1, 2, \dots,\ j = 1, 2, \dots, 2^{m-1}$. The Haar system of functions contains the function $\chi_1(x) \equiv 1$, too, which does not belong to any class.

The following properties of the Haar functions pointed out in [13] will be important for us.

1. The system $\{\chi_{m,j}(x)\}$ is orthogonal w.r.t. the inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)\mathrm{d}x$.
2. Any function, continuous on [0, 1], can be expanded into a series that converges uniformly with respect to functions of the system $\{\chi_{m,j}(x)\}$.

The second property implies that approximate integration formulas, exact for all Haar polynomials with degrees not greater than some sufficiently large number d, have a relatively small error [13].

In the one-dimensional case, the Haar polynomials of degree d are by definition the functions

$$P_d(x) = a_0\chi_1(x) + \sum_{m=1}^{d}\sum_{j=1}^{2^{m-1}} a_m^{(j)}\chi_{m,j}(x),$$

where $d = 1, 2, \dots, a_0,\ a_m^{(j)} \in \mathbb{R}, m = 1, \dots, d,\ j = 1, \dots, 2^{m-1}$, and

$$\sum_{j=1}^{2^{d-1}} \left(a_d^{(j)}\right)^2 \neq 0.$$

By 0-degree Haar polynomials we will consider real constants.

In the two-dimensional case, the Haar polynomials of degree d are the functions

$$P_d(x_1, x_2) = a_0 + \sum_{l=1}^{d} \sum_{i=1}^{2^{l-1}} a_l^{(i)} \chi_{l,i}(x_1) + \sum_{m=1}^{d} \sum_{j=1}^{2^{m-1}} b_m^{(j)} \chi_{m,j}(x_2) + \sum_{2 \leqslant l+m \leqslant d} \sum_{i=1}^{2^{l-1}} \sum_{j=1}^{2^{m-1}} c_{l,m}^{(i,j)} \chi_{l,i}(x_1) \chi_{m,j}(x_2),$$

where $d = 2, 3, a_0,\ a_l^{(i)},\ b_m^{(j)},\ c_{l,m}^{(i,j)} \in \mathbb{R}$, and

$$\sum_{i=1}^{2^{d-1}} \left[\left(a_d^{(i)}\right)^2 + \left(b_d^{(i)}\right)^2 \right] + \sum_{l+m=d} \sum_{i=1}^{2^{l-1}} \sum_{j=1}^{2^{m-1}} \left(c_{l,m}^{(i,j)}\right)^2 \neq 0.$$

Haar polynomials of degree 1 are the functions

$$P_1(x_1, x_2) = a_0 + a_1 \chi_{1,1}(x_1) + b_1 \chi_{1,1}(x_2),$$

where $a_0,\ a_1,\ b_1 \in \mathbb{R}$, $a_1^2 + b_1^2 \neq 0$. The same way as in the one-dimensional case, by 0-degree Haar polynomials we will consider real constants.

2. Minimal quadrature formulas of Haar degree *d*

2.1. The construction of minimal quadrature formulas

We consider the following quadrature formulas

$$I[f] = \int_0^1 g(x)\ f(x) \mathrm{d}x \approx \sum_{i=1}^{N} C_i\ f(x^{(i)}) = Q[f], \tag{2.1}$$

where the nodes $x^{(i)} \in [0, 1]$, the coefficients C_i on the nodes are real, $i = 1, \ldots, N$, the functions $g(x),\ f(x)$ are defined and summable on [0, 1], and the so called weight function $g(x) \neq 0$ is almost everywhere on [0, 1].

We say that such a formula is a formula of Haar degree d, if it is exact for every Haar polynomial $P(x)$ of degree at most d, but not exact for at least one Haar polynomial $P_{d+1}(x)$ of degree $d+1$, i.e. $Q[P] = I[P]$, but $Q[P_{d+1}] \neq I[P_{d+1}]$. If a formula among all formulas (2.1) of Haar degree d has the least possible number of nodes, then we call such a formula a minimal formula of Haar degree d.

We recall how to characterize quadrature formulas (2.1) of Haar degree d by the location of the nodes of such a formula [9].

Theorem 1. 1. *If a quadrature formula (2.1) has a Haar degree d then it satisfies the condition:*
(a) *each of the binary intervals*

$$\overline{l_{d+1,1}} = \left[0, \frac{1}{2^d}\right],\ \overline{l_{d+1,2}} = \left[\frac{1}{2^d}, \frac{2}{2^d}\right],\ \ldots,$$
$$\overline{l_{d+1,2^d-1}} = \left[\frac{2^d-2}{2^d}, \frac{2^d-1}{2^d}\right],\ \overline{l_{d+1,2^d}} = \left[\frac{2^d-1}{2^d}, 1\right]$$

contains at least one node of (2.1).

2. *If the quadrature formula (2.1) is a minimal formula of Haar degree d then it satisfies the condition:*

(b) *each of the binary intervals*

$$l_{d+1,1} = \left[0, \frac{1}{2^d}\right), l_{d+1,2} = \left(\frac{1}{2^d}, \frac{2}{2^d}\right), \dots, l_{d+1,2^d-1} = \left(\frac{2^d-2}{2^d}, \frac{2^d-1}{2^d}\right),$$
$$l_{d+1,2^d} = \left(\frac{2^d-1}{2^d}, 1\right]$$

contains at most one node of the formula.

Let k be a fixed positive integer. The interval $\left[\frac{p-1}{2^d}, \frac{p+k}{2^d}\right]$ is called a d-singular set if

$$\sum_{j=0}^{k}\left[(-1)^j \int_{(p+j-1)/2^d}^{(p+j)/2^d} g(x)\mathrm{d}x\right] = 0,$$

and

$$\int_{(p+j-1)/2^d}^{(p+j)/2^d} g(x)\mathrm{d}x \neq 0, \quad j = 0, \dots, k.$$

The size of a d-singular set is $(k+1)2^{-d}$, from which k can be reconstructed, if necessary.

Example 2.1. Let $g(x) = \sin 2\pi x$ and $k = 1$. Then there exist two 3-singular sets:

$$\left[\frac{1}{8}, \frac{3}{8}\right], \qquad \left[\frac{5}{8}, \frac{7}{8}\right].$$

Example 2.2. Let $g(x) \equiv 1$ and $k = 1$. Then there exist four 3-singular sets:

$$\left[0, \frac{1}{4}\right], \qquad \left[\frac{1}{4}, \frac{1}{2}\right], \qquad \left[\frac{1}{2}, \frac{3}{4}\right], \qquad \left[\frac{3}{4}, 1\right].$$

The following estimates hold for the number of nodes of the quadrature formula (2.1) of Haar degree d [9].

Theorem 2. *Let the weight function $g(x)$ be such that it is possible to assign (not more than) m pairwise disjoint d-singular sets. Then the number of nodes of a quadrature formula (2.1) of Haar degree d satisfies the inequality:*

$$N \geqslant 2^d - m.$$

If it is impossible to assign any d-singular set, then the number of nodes of a quadrature formula (2.1) of Haar degree d satisfies the inequality:

$$N \geqslant 2^d.$$

We outline the statement that establishes the necessary and sufficient conditions for a quadrature formula (2.1) to be minimal of Haar degree d [9].

Theorem 3. *Let the weight function $g(x)$ be such that it is possible to assign (not more than) m pairwise disjoint d-singular sets. The quadrature formula (2.1) is a minimal formula of Haar degree d if and only if the following four conditions are satisfied:*

(1) *the number of nodes of the formula is $N = 2^d - m$,*

(2) *there exists a family of d-singular sets*

$$\left[\frac{p_1 - 1}{2^d}, \frac{p_1 + k_1}{2^d}\right], \left[\frac{p_2 - 1}{2^d}, \frac{p_2 + k_2}{2^d}\right], \ldots, \left[\frac{p_m - 1}{2^d}, \frac{p_m + k_m}{2^d}\right]$$
$$(0 \leqslant p_1 - 1 < p_1 + k_1 < p_2 - 1 < p_2 + k_2 < \cdots < p_m - 1 < p_m + k_m \leqslant 1),$$

such that the nodes that belong to every one of the sets $\left[\frac{p_j-1}{2^d}, \frac{p_j+k_j}{2^d}\right]$, are located at the points $\frac{p_j}{2^d}, \frac{p_j+1}{2^d}, \ldots, \frac{p_j+k_j-1}{2^d}$, $j = 1, \ldots, m$,

(3) *in any set*

$$\left[0, \frac{p_1 - 1}{2^d}\right], \left[\frac{p_1 + k_1}{2^d}, \frac{p_2 - 1}{2^d}\right], \ldots, \left[\frac{p_{m-1} + k_{m-1}}{2^d}, \frac{p_m - 1}{2^d}\right], \left[\frac{p_m + k_m}{2^d}, 1\right]$$
$$(p_1 > 1,\ p_m + k_m < 1)$$

there are $p_1 - 1$, $p_2 - p_1 - k_1 - 1, \ldots,$ $p_m - p_{m-1} - k_{m-1} - 1$, $2^d - p_m - k_m$ nodes (resp.), located according to the conditions (a) *and* (b) *of Theorem* 1,

(4) *the coefficients of the nodes are defined by the system:*

$$\begin{cases} y_1 + \frac{1}{2} y'_1 = \int_0^{1/2^d} g(x)\mathrm{d}x, \\ \frac{1}{2} y'_1 + y_2 + \frac{1}{2} y'_2 = \int_{1/2^d}^{2/2^d} g(x)\mathrm{d}x, \\ \frac{1}{2} y'_2 + y_3 + \frac{1}{2} y'_3 = \int_{2/2^d}^{3/2^d} g(x)\mathrm{d}x, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{2} y'_{2^d-2} + y_{2^d-1} + \frac{1}{2} y'_{2^d-1} = \int_{(2^d-2)/2^d}^{(2^d-1)/2^d} g(x)\mathrm{d}x, \\ \frac{1}{2} y'_{2^d-1} + y_{2^d} = \int_{(2^d-1)/2^d}^{1} g(x)\mathrm{d}x, \end{cases} \tag{2.2}$$

where

$$y_j = \begin{cases} 0, & \text{if there is no node in } l_{d+1,j}, \\ C_{i_j}, & \text{if there is a node } x^{(i_j)} \in l_{d+1,j}, \end{cases} \quad j = 1, \ldots, 2^d,$$

$$y'_j = \begin{cases} 0, & \text{if there is no node at } \dfrac{j}{2^d}, \\ C_{i_j}, & \text{if there is a node } x^{(i_j)} = \dfrac{j}{2^d}, \end{cases} \quad j = 1, \ldots, 2^{d-1}.$$

Theorem 3 implies an algorithm for the construction of minimal quadrature formulas of Haar degree d.

Step 1. Find the maximal number of pairwise disjoint d-singular sets. Let this number be equal to m and let the sets be:

$$\left[\frac{p_1 - 1}{2^d}, \frac{p_1 + k_1}{2^d}\right], \left[\frac{p_2 - 1}{2^d}, \frac{p_2 + k_2}{2^d}\right], \ldots, \left[\frac{p_m - 1}{2^d}, \frac{p_m + k_m}{2^d}\right].$$

Step 2. On the j-th d-singular set (numbering from left to right)

$$\left[\frac{p_j - 1}{2^d}, \frac{p_j + k_j}{2^d}\right]$$

we select nodes according to the following rule:

$$x^{(i_j)} = \frac{p_j}{2^d}, x^{(i_j+1)} = \frac{p_j + 1}{2^d}, \ldots, x^{(i_j+k_j-1)} = \frac{p_j + k_j - 1}{2^d}, \quad j = 1, \ldots, m,$$

the coefficients of these nodes can be computed using the formula:

$$C_{i_j+s-1} = 2\sum_{l=0}^{s-1}\left[(-1)^{s-l-1}\int_{(p_j+l-1)/2^d}^{(p_j+l)/2^d} g(x)\mathrm{d}x\right],$$

where $x^{(i_j+s-1)}$ is the s-th node on the set $\left[\frac{p_j-1}{2^d}, \frac{p_j+k_j}{2^d}\right]$ (numbering from left to right), $j = 1, \ldots, m,\ s = 1, \ldots, k_j$.

Step 3. On every binary interval $l_{d+1,j}$ which is not a subset of any of the sets

$$\left[\frac{p_1 - 1}{2^d}, \frac{p_1 + k_1}{2^d}\right], \left[\frac{p_2 - 1}{2^d}, \frac{p_2 + k_2}{2^d}\right], \ldots, \left[\frac{p_m - 1}{2^d}, \frac{p_m + k_m}{2^d}\right],$$

we arbitrarily select exactly one point as a node i.e. $x^{(i_j)} \in l_{d+1,j}$. We compute the coefficient C_{i_j} of this node using the formula:

$$C_{i_j} = \int_{(j-1)/2^d}^{j/2^d} g(x)\mathrm{d}x.$$

Note that if the function $g(x)$ is such that it is possible to select (not more than) m pairwise disjoint d-singular sets, then nodes of the minimal formula (2.1) of Haar degree d that belong to the sets

$$\left[0, \frac{p_1 - 1}{2^d}\right], \left[\frac{p_1 + k_1}{2^d}, \frac{p_2 - 1}{2^d}\right], \ \ldots, \ \left[\frac{p_{m-1} + k_{m-1}}{2^d}, \frac{p_m - 1}{2^d}\right], \left[\frac{p_m + k_m}{2^d}, 1\right]$$

can be defined in different ways. If it is impossible to select any d-singular set, then all nodes of the minimal formula can be defined in different ways. In this algorithm we specified the most simple way of selection of such nodes.

Now some examples of minimal quadrature formulas of Haar degree d.

Example 2.3. Let $g(x) \equiv 1$. Then $N = 2^{d-1}$,

$$x^{(j)} = \frac{2j - 1}{2^d}, \qquad C_j = \frac{1}{2^{d-1}}, \quad j = 1, \ldots, 2^{d-1}.$$

It is interesting to compare the constructed quadrature formula with those considered in [13]. We recall the definition of the Π_0-nets [13] in the one-dimensional case. A Π_0-net is a net which consists of $N = 2^d$ nodes such that each of the binary intervals of length 2^{-d} contains one node. In the one-dimensional case the formulas with 2^d nodes, generating the Π_0-net have the greatest Haar degree equal to d among the formulas from [13]. Comparing the number of nodes of these formulas and of the quadrature formulas in Example 2.3 with same Haar degree d, we see that the number of nodes in Example 2.3 is half as much.

Example 2.4. Let the weight function $g(x)$ be such that

$$I[\chi_{d,j}] = 0, \quad \text{i.e.} \int_{(2j-2)/2^d}^{(2j-1)/2^d} g(x)\mathrm{d}x = \int_{(2j-1)/2^d}^{2j/2^d} g(x)\mathrm{d}x, \quad j = 1, \ldots, 2^{d-1}.$$

As in Example 2.3, $N = 2^{d-1}$,

$$x^{(j)} = \frac{2j-1}{2^d}, \qquad C_j = \int_{(j-1)/2^{d-1}}^{j/2^{d-1}} g(x)\mathrm{d}x, \quad j = 1, \ldots, 2^{d-1}.$$

The function

$$g(x) = |\sin\, 2^{d-1}\pi x|$$

may serve as an example of such a weight function.

Example 2.5. Let the weight function $g(x)$ be such that on the interval [0, 1] there is no d-singular set. Then $N = 2^d$,

$$x^{(j)} \text{ is any point of the interval } l_{d+1,j}, \quad C_j = \int_{(j-1)/2^d}^{j/2^d} g(x)\mathrm{d}x, \quad j = 1, \ldots, 2^d.$$

Example 2.6. Let $g(x) = \sin 2\pi x$, $d = 3$. Then there exist two d-singular sets, namely the intervals [1/8, 3/8] and [5/8, 7/8] (see Example 2.1), consequently, $N = 6$. In this case

$$x^{(1)} \in [0, 1/8), x^{(3)} \in (3/8, 1/2), x^{(4)} \in (1/2, 5/8), x^{(6)} \in (7/8, 1],$$
$$x^{(2)} = 1/4, x^{(5)} = 3/4,$$
$$C_1 = C_3 = (2 - \sqrt{2})/4\pi, C_4 = C_6 = (\sqrt{2} - 2)/4\pi, C_2 = \sqrt{2}/2\pi, C_5 = -\sqrt{2}/2\pi.$$

2.2. *The norm of the error functional of minimal quadrature formulas of Haar degree d on the spaces S_p*

Let p be a fixed number with $1 < p < \infty$. For arbitrary positive A one can define a class $S_p(A)$ as the set of the functions $f(x)$, defined on [0, 1] that can be represented as the Fourier–Haar series

$$f(x) = c_1 + \sum_{m=1}^{\infty} \sum_{j=1}^{2^{m-1}} c_m^{(j)} \chi_{m,j}(x)$$

with real coefficients c_1, $c_m^{(j)}$ $(m = 1, 2, \ldots, j = 1, 2, \ldots, 2^{m-1})$, satisfying the condition

$$A_p(f) := \sum_{m=1}^{\infty} 2^{\frac{m-1}{2}} \Big[\sum_{j=1}^{2^{m-1}} |c_m^{(j)}|^p\Big]^{\frac{1}{p}} \leqslant A.$$

In [13] it is proved that $\bigcup_{A>0} S_p(A)$ with the norm

$$\|f\|_{S_p} = A_p(f)$$

forms a linear normed space denoted by S_p.

To establish an estimate for the norm of the error functional of the quadrature formula (2.1)

$$\delta_N := I[f] - Q[f],$$

we need to recall the definition of the space L^∞ [4]. It consists of all measurable almost everywhere finite functions $g(x)$, for each of which there exists a number C_g, such that $|g(x)| \leqslant C_g$ almost everywhere. We call such functions essentially bounded (on the interval in question). We will consider the functions, essentially bounded on [0, 1]. For a function $g \in L^\infty$ one defines the proper (essential) supremum of its absolute value

$$\operatorname*{ess\,sup}_{x\in[0,1]} |g(x)|$$

as an infimum of the set of numbers $\alpha \in \mathbb{R}$ such that the measure of the set

$$\{x \in [0, 1] : |g(x)| > \alpha\}$$

is zero. L^∞ is a linear subset in the set of measurable almost everywhere finite functions. The norm on L^∞ can be introduced using the formula

$$\|g\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in[0,1]} |g(x)| .$$

We now establish a two-sided estimate for the norm of the error functional of the quadrature formulas considered in Example 2.5. We assume that in the formulas mentioned the function $f(x)$ belongs to the space S_p, the weight function $g(x)$ is nonnegative and essentially bounded on [0, 1], the nodes of the formula are not binary rational points, i.e. cannot be represented as

$$x^{(i_j)} = \frac{j}{2^n}, \quad n = 1, 2, \ldots, \ j = 0, 1, \ldots, 2^n.$$

For the formulas from Example 2.5 the norm of the error functional can be estimated as follows

$$GN^{-1/p} \leqslant \|\delta_N\|_{S_p^*} \leqslant (2G)^{1-\frac{1}{p}} \left(\|g\|_{L^\infty}\right)^{\frac{1}{p}} N^{-\frac{1}{p}}, \tag{2.3}$$

where $G = \int_0^1 g(x)\mathrm{d}x$ [8].

For formulas of the type (2.1) with an arbitrary summable weight function $g(x)$ and a function $f(x)$, that belongs to S_p under the assumption that all nodes of the formula are pairwise different and all coefficients on the nodes are positive and satisfy the relationship

$$\sum_{i=1}^{N} C_i = G,$$

the norm of the functional $\delta_N(f)$ has the lower estimate

$$\|\delta_N\|_{S_p^*} \geqslant GN^{-1/p}, \quad \text{see [14]}.$$

In the case when $g(x) \geqslant 0$ in [14] the formula for which

$$\|\delta_N\|_{S_p^*} \leqslant 2GN^{-1/p} \tag{2.4}$$

is constructed.

Therefore for the formulas (2.1), considered in [8], the value $\|\delta_N\|_{S_p^*}$ has the same order $N^{-\frac{1}{p}}$ (that is the best one) as for the formulas studied by I.M. Sobol. At the same time the quadrature formulas, studied in [8], being minimal formulas of approximate integration, provide the best pointwise convergence $\delta_N(f)$ to zero. Obviously for weight functions, satisfying the inequality $G \leqslant \|g\|_{L^\infty} < 2G$, the constant $(2G)^{1-\frac{1}{p}}\,(\|g\|_{L^\infty})^{\frac{1}{p}}$ in the estimate (2.3) is less than the constant $2G$ in the estimate (2.4).

3. Minimal cubature formulas of Haar degree *d* in the two-dimensional case

3.1. The construction of minimal cubature formulas

Consider cubature formulas

$$I[f] = \int_0^1 \int_0^1 f(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \approx \sum_{i=1}^{N} C_i f(x_1^{(i)}, x_2^{(i)}) = Q[f], \tag{3.1}$$

with nodes $(x_1^{(i)}, x_2^{(i)}) \in [0,1]^2$ and corresponding coefficients $C_i \in \mathbb{R}$, $i = 1, \ldots, N$, $f(x_1, x_2)$ is a function defined and summable on $[0,1]^2$.

We say that such a formula is a formula of Haar degree d, if it is exact for every Haar polynomial $P(x_1, x_2)$ of degree at most d, but not exact for at least one Haar polynomial $P_{d+1}(x_1, x_2)$ of degree $d+1$, i.e. $Q[P] = I[P]$, but $Q[P_{d+1}] \neq I[P_{d+1}]$. If a formula among all formulas (3.1) of Haar degree d has the least possible number of nodes, then we call such formula a minimal formula of Haar degree d.

In [6] the following property of cubature formulas of Haar degree d is proved, which characterizes the location of nodes of these formulas.

Theorem 4. *If the cubature formula (3.1) is a formula of Haar degree d then any of the binary rectangles with area of* 2^{-d}

$$\overline{l_{m_1,j_1}} \times \overline{l_{m_2,j_2}}, \quad m_1 + m_2 = d + 2, \quad j_k = 1, \ldots, 2^{m_k - 1}, \quad k = 1, 2,$$

contains at least one node of (3.1).

A lower estimate of the number of nodes of the cubature formula (3.1) of Haar degree d is established in the following theorem.

Theorem 5 (*[5]*). *If the cubature formula (3.1) is a formula of Haar degree d, then the number N of its nodes satisfies the inequality:*

$$N \geqslant 2^{d-1} + 1, \quad d \geqslant 2. \tag{3.2}$$

This estimate can be adjusted in the following way [6].

Theorem 6. *If the cubature formula (3.1) is a formula of Haar degree d, then the number N of its nodes satisfies the inequality:*

$$N \geqslant 2^d - \lambda(d), \tag{3.3}$$

where

$$\lambda(d) = \begin{cases} 2^{\frac{d}{2}+1} - 2 & \text{for } d = 2k, \\ 3 \times 2^{\frac{d-1}{2}} - 2 & \text{for } d = 2k+1, \end{cases} \qquad k = 1, 2, \dots$$

Example 3.1. The formula (3.1) with the only node $(x_1^{(1)}, x_2^{(1)}) = (1/2, 1/2)$ and the coefficient $C_1 = 1$ on this node is a Haar degree 1 formula. Obviously, this formula is minimal.

Now consider examples of cubature formulas of 2-nd and 3-rd Haar degree [5]. Their number of nodes equals 3 or 5 respectively, and therefore (3.2) implies their minimality. The construction of such formulas is obvious by Theorem 4: The first node ($d = 2$) or the first three nodes ($d = 3$) are selected to belong to two different binary rectangles each. Then the other nodes are selected to belong to the remaining binary rectangles.

Example 3.2. $d = 2: \ N = 3, \ C_1 = 1/2, \ C_2 = C_3 = 1/4,$

$$(x_1^{(1)}, x_2^{(1)}) = (1/4, 1/2), \ (x_1^{(2)}, x_2^{(2)}) = (5/8, 1/8), \ (x_1^{(3)}, x_2^{(3)}) = (7/8, 7/8).$$

Example 3.3. $d = 3: \ N = 5, \ C_1 = C_2 = C_3 = 1/4, \ C_4 = C_5 = 1/8,$

$$(x_1^{(1)}, x_2^{(1)}) = (1/8, 1/2), \ (x_1^{(2)}, x_2^{(2)}) = (1/2, 7/8), \ (x_1^{(3)}, x_2^{(3)}) = (3/4, 1/4),$$
$$(x_1^{(4)}, x_2^{(4)}) = (5/16, 1/16), (x_1^{(5)}, x_2^{(5)}) = (15/16, 11/16).$$

It is convenient to represent the nodes of cubature formulas of Haar degrees $d \geqslant 4$ in the following way:

$$(x_1^{(i)}, x_2^{(i)}) = (2^{-d-1}a^{(i)}, 2^{-d-1}b^{(i)}).$$

Below we write down the coefficients on the nodes and tables of values of parameters $a^{(i)}, \ b^{(i)}$ of the corresponding cubature formulas of Haar degree $d = 4, \ 5, \ 6$. For $d = 5, \ 6$ the number of nodes of these formulas is $N = 2^d - \lambda(d)$, and therefore they are minimal because of (3.3). For $d = 4$ the number of nodes of the cubature formula is $2^d - \lambda(d) + 1$. The formulas mentioned were obtained by experiment.

Example 3.4. $d = 4: \ N = 11, \ C_1 = \dots = C_5 = 2^{-3}, \ C_6 = \dots = C_{11} = 2^{-4}.$

i	$a^{(i)}$	$b^{(i)}$	i	$a^{(i)}$	$b^{(i)}$	i	$a^{(i)}$	$b^{(i)}$	i	$a^{(i)}$	$b^{(i)}$	i	$a^{(i)}$	$b^{(i)}$
1	2	16	**2**	8	4	**3**	12	24	**4**	20	8	**5**	24	28
6	5	31	**7**	15	11	**8**	17	21	**9**	27	1	**10**	29	19
11	31	13												

Example 3.5. $d = 5 : N = 22, C_1 = \dots = C_{10} = 2^{-4}, C_{11} = \dots = C_{22} = 2^{-5}.$

i	$a^{(i)}$	$b^{(i)}$	i	$a^{(i)}$	$b^{(i)}$	i	$a^{(i)}$	$b^{(i)}$	i	$a^{(i)}$	$b^{(i)}$	i	$a^{(i)}$	$b^{(i)}$
1	2	32	**2**	8	8	**3**	16	44	**4**	20	16	**5**	24	56
6	32	64	**7**	40	40	**8**	48	12	**9**	52	48	**10**	56	24
11	5	63	**12**	63	5	**13**	11	51	**14**	13	21	**15**	27	27
16	29	37	**17**	35	53	**18**	37	29	**19**	43	19	**20**	45	61
21	59	35	**22**	61	59									

Example 3.6. $d = 6 : N = 50, C_1 = \cdots = C_{14} = 2^{-5}, C_{15} = \cdots = C_{50} = 2^{-6}$.

i	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$	**i**	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$	**i**	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$	**i**	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$	**i**	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$
1	6	64	**2**	12	32	**3**	16	88	**4**	32	116	**5**	40	16
6	48	56	**7**	56	80	**8**	64	6	**9**	72	48	**10**	80	72
11	88	112	**12**	96	12	**13**	112	40	**14**	116	96	**15**	9	109
16	19	9	**17**	21	43	**18**	23	99	**19**	25	51	**20**	27	75
21	29	23	**22**	35	37	**23**	37	93	**24**	43	107	**25**	45	69
26	51	103	**27**	53	27	**28**	59	45	**29**	61	123	**30**	67	121
31	69	83	**32**	75	101	**33**	77	25	**34**	83	59	**35**	85	21
36	91	35	**37**	93	91	**38**	99	105	**39**	101	53	**40**	103	77
41	105	29	**42**	107	85	**43**	109	119	**44**	119	19	**45**	121	61
46	123	67	**47**	1	3	**48**	125	1	**49**	3	127	**50**	127	125

In [7] the author obtained relations which allow the computation of the coordinates of nodes of minimal cubature formulas of Haar degree $d \geqslant 7$. The number of nodes of such formulas is $N = 2^d - \lambda(d)$. For odd (resp. even) values of d, one uses the $\lambda(d)$ nodes of a minimal 5-th (or 6-th) degree Haar formula. The coordinates of these nodes are multiples of 2^{-d}, their coefficients are always equal to 2^{-d}. The coordinates of the remaining $2^d - 2\lambda(d)$ nodes are odd multiples of 2^{-d-1}. The coefficients of such nodes are always 2^{-d}. The relations mentioned for computing the coordinates of nodes are too bulky to be written down here. We show just an example of a minimal formula of Haar degree 8, obtained by the minimal formula of Haar degree 5 (Example 3.5).

Example 3.7. $d = 7\ :\ N = 106, C_1 = \cdots = C_{22} = 2^{-6}, C_{23} = \cdots = C_{106} = 2^{-7}$.

i	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$	**i**	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$	**i**	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$	**i**	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$	**i**	$\mathbf{a^{(i)}}$	$\mathbf{b^{(i)}}$
1	4	64	**2**	10	128	**3**	16	16	**4**	32	88	**5**	40	32
6	48	112	**7**	64	4	**8**	80	80	**9**	96	24	**10**	104	96
11	112	48	**12**	128	10	**13**	144	208	**14**	152	160	**15**	160	232
16	176	176	**17**	192	252	**18**	208	144	**19**	216	224	**20**	224	168
21	240	240	**22**	252	192	**23**	7	195	**24**	13	237	**25**	19	243
26	21	153	**27**	23	103	**28**	25	213	**29**	27	43	**30**	29	165
31	35	171	**32**	37	221	**33**	43	227	**34**	45	141	**35**	51	147
36	53	201	**37**	55	55	**38**	57	181	**39**	59	75	**40**	61	249
41	69	149	**42**	71	107	**43**	73	107	**44**	75	59	**45**	77	173
46	83	179	**47**	85	217	**48**	87	39	**49**	89	133	**50**	91	123
51	93	229	**52**	99	235	**53**	101	157	**54**	107	163	**55**	109	205
56	115	211	**57**	117	185	**58**	119	71	**59**	121	137	**60**	123	119
61	125	245	**62**	131	247	**63**	133	117	**64**	135	139	**65**	137	69
66	139	187	**67**	141	45	**68**	147	51	**69**	149	93	**70**	155	99
71	157	21	**72**	163	27	**73**	165	121	**74**	167	135	**75**	169	37
76	171	219	**77**	173	77	**78**	179	83	**79**	181	57	**80**	183	199
81	185	105	**82**	187	151	**83**	195	7	**84**	197	73	**85**	199	183
86	201	53	**87**	203	203	**88**	205	109	**89**	211	115	**90**	213	29
91	219	35	**92**	221	85	**93**	227	91	**94**	229	41	**95**	231	215
96	233	101	**97**	235	155	**98**	237	13	**99**	243	19	**100**	245	125
101	247	131	**102**	249	61	**103**	1	189	**104**	189	1	**105**	67	255
106	255	67												

Note that the number of nodes of cubature formulas shown in the considered examples is less than the number of nodes, constructed in [13] of cubature formulas of Haar degree d based on Π_0-nets, i.e., nets which consist of $N = 2^d$ nodes such that each of the binary rectangles with area 2^{-d} contains one node [13].

3.2. The norm of the error functional of cubature formulas of Haar degree d on the spaces S_p

In [10] the norm of error functionals for cubature formulas (3.1) of Haar degree d on the spaces S_p ($p \geqslant 1$) is studied. These spaces were introduced in [13] for arbitrary dimensions n. These spaces can be considered as a generalization of the one-dimensional spaces S_p (see Section 2.2) to the multidimensional case.

The error functional δ_N of the mentioned formulas can be estimated as follows [10]:

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{1}{p}}(2^d)^{-\frac{1}{p}}.$$

It is also proved that in the case of $N \sim 2^d$ when $d \to \infty$ the following formula holds

$$\|\delta_N\|_{S_p^*} \sim 2^{\frac{1}{p}} N^{-\frac{1}{p}}, \quad N \to \infty.$$

In [13] one considered the following cubature formulas

$$\int_0^1 \int_0^1 \dots \int_0^1 f(x_1, x_2, \dots, x_n) \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \approx \frac{1}{N} \sum_{i=1}^{N} f(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}) \tag{3.4}$$

with 2^d nodes, generating Π_τ-nets ($0 \leqslant \tau < d$), i.e., nets that consist of $N = 2^d$ nodes such that each of the binary parallelepipeds of volume $2^{\tau-d}$ contains 2^τ nodes [13]. In [13] it is proved that these formulas are formulas of Haar degree $(d - \tau)$ and their error functional δ_N satisfies the following inequality:

$$N^{-\frac{1}{p}} \leqslant \|\delta_N\|_{S_p^*} \leqslant 2^{\frac{n-1+\tau}{p}} N^{-\frac{1}{p}}.$$

Thus in the case of $N \sim 2^d$ when $d \to \infty$ the cubature formulas (3.1) of Haar degree d, studied in [10], have the best order of convergence $\|\delta_N\|_{S_p^*}$, equal to $N^{-\frac{1}{p}}$. In particular, minimal cubature formulas, constructed in [7] satisfy the condition $N \sim 2^d$ when $d \to \infty$. At the same time these formulas, being minimal formulas of approximate integration, provide the best pointwise convergence of $\delta_N(f)$ to zero.

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