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Gibbs cluster measures on configuration spaces

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Abstract

The probability distribution g_{cl} of a Gibbs cluster point process in $X = \mathbb{R}^d$ (with i.i.d. random clusters attached to points of a Gibbs configuration with distribution g) is studied via the projection of an auxiliary Gibbs measure $\hat{g}$ in the space of configurations $\hat{\gamma} = \{(x, \bar{y})\} \subset X \times \mathfrak{X}$, where $x \in X$ indicates a cluster "center" and $\bar{y} \in \mathfrak{X} := \bigsqcup_n X^n$ represents a corresponding cluster relative to x. We show that the measure g_{cl} is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$ of compactly supported diffeomorphisms of X, and prove an integration-by-parts formula for g_{cl}. The associated equilibrium stochastic dynamics is then constructed using the method of Dirichlet forms.

Keywords: Cluster point process; Configuration space; Gibbs measure; Poisson measure; Quasi-invariance; Integration by parts; Dirichlet form; Stochastic dynamics

1. Introduction

The concept of particle configurations is instrumental in mathematical modeling of multi-component stochastic systems. Rooted in statistical mechanics and theory of point processes, the development of the general mathematical framework for suitable classes of configurations has been a recurrent research theme fostered by widespread applications across the board,

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including quantum physics, astrophysics, chemical physics, biology, ecology, computer science, economics, finance, etc. (see an extensive bibliography in [13]).

In the past 15 years or so, there has been a more specific interest in the *analysis* on configuration spaces. To fix basic notation, let X be a topological space (e.g., a Euclidean space $X = \mathbb{R}^d$), and let $\Gamma_X = \{\gamma\}$ be the configuration space over X, that is, the space of countable subsets (called *configurations*) $\gamma \subset X$ without accumulation points. Albeverio, Kondratiev and Röckner [2,3] have proposed an approach to configuration spaces Γ_X as *infinite-dimensional manifolds*, based on the choice of a suitable probability measure μ on Γ_X which is quasi-invariant with respect to $\mathrm{Diff}_0(X)$, the group of compactly supported diffeomorphisms of X. Providing that the measure μ can be shown to satisfy an integration-by-parts formula, one can construct, using the theory of Dirichlet forms, an associated equilibrium dynamics (stochastic process) on Γ_X such that μ is its invariant measure [2,3,31] (see [1,4,11,15,22,23,34,39] and references therein for further discussion of various theoretical aspects and applications).

This general program has been first implemented in [2] for the *Poisson* measure μ on Γ_X, and then extended in [3] to a wider class of *Gibbs* measures, which appear in statistical mechanics of classical continuous gases. In the Poisson case, the canonical equilibrium dynamics is given by the well-known independent particle process, that is, an infinite family of independent (distorted[3]) Brownian motions started at the points of a random Poisson configuration. In the Gibbsian case, the equilibrium dynamics is much more complex due to interaction between the particles.

In our earlier papers [8,9], a similar analysis was developed for a different class of random spatial structures, namely *Poisson cluster point processes*, featured by spatial grouping ("clustering") of points around the background random (Poisson) configuration of invisible "centers". Cluster models are well known in the general theory of random point processes [12,13] and are widely used in numerous applications ranging from neurophysiology (nerve impulses) and ecology (spatial aggregation of species) to seismology (earthquakes) and cosmology (constellations and galaxies); see [9,12,13] for some references to original papers.

Our technique in [8,9] was based on the representation of a given Poisson cluster measure on the configuration space Γ_X as the projection image of an auxiliary Poisson measure on a more complex configuration space $\Gamma_{\mathfrak{X}}$ over the disjoint-union space $\mathfrak{X} := \bigsqcup_n X^n$, with "droplet" points $\bar{y} \in \mathfrak{X}$ representing individual clusters (of variable size). The principal advantage of this construction is that it allows one to apply the well-developed apparatus of Poisson measures to the study of the Poisson cluster measure.

In the present paper,[4] our aim is to extend this approach to a more general class of *Gibbs cluster measures* on the configuration space Γ_X, where the distribution of cluster centers is given by a Gibbs (grand canonical) measure $\mathrm{g} \in \mathscr{G}(\theta, \Phi)$ on Γ_X, with a reference measure θ on X and an interaction potential Φ. We focus on Gibbs cluster processes in $X = \mathbb{R}^d$ with independent random clusters of random size. Let us point out that we do not require the uniqueness of the Gibbs measure, so our results are not influenced by the possible occurrence of phase transition (i.e., where the class $\mathscr{G}(\theta, \Phi)$ contains more than one measure). Under some natural smoothness conditions on the reference measure θ and the distribution η of the generic cluster, we prove the $\mathrm{Diff}_0(X)$-quasi-invariance of the corresponding Gibbs cluster measure g_{cl} (Section 3.2), establish the integration-by-parts formula (Section 3.3) and construct the associated Dirichlet operator,

[3] The term "distorted" refers to a special space-dependent drift generated by a non-flat intensity measure.

[4] Some of our results have been announced in [7] (in the case of clusters of fixed size).

which leads to the existence of the equilibrium stochastic dynamics on the configuration space Γ_X (Section 4).

Unlike the Poisson cluster case, it is now impossible to work with the measure arising in the space $\Gamma_{\mathfrak{X}}$ of droplet configurations $\bar{\gamma} = \{\bar{y}\}$, which is hard to characterize for Gibbs cluster measures. Instead, in order to be able to pursue our projection approach while still having a tractable pre-projection measure, we choose the configuration space $\Gamma_{\mathcal{Z}}$ over the set $\mathcal{Z} := X \times \mathfrak{X}$, where each configuration $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$ is a (countable) set of pairs $z = (x, \bar{y})$, with $x \in X$ indicating a cluster center and $\bar{y} \in \mathfrak{X}$ representing a cluster attached to x. A crucial step is to show that the corresponding measure $\hat{\mathrm{g}}$ on $\Gamma_{\mathcal{Z}}$ is again Gibbsian, with the reference measure $\hat{\theta} = \theta \otimes \eta$ and a "cylinder" interaction potential $\hat{\Phi}(\hat{\gamma}) := \Phi(\mathfrak{p}(\hat{\gamma}))$, where Φ is the original interaction potential associated with the background Gibbs measure g and $\mathfrak{p}$ is the operator on the configuration space $\Gamma_{\mathcal{Z}}$ projecting a configuration $\hat{\gamma} = \{(x, \bar{y})\}$ to the configuration of cluster centers, $\gamma = \{x\}$. We then project the Gibbs measure $\hat{\mathrm{g}}$ from the "higher floor" $\Gamma_{\mathcal{Z}}$ directly to the configuration space Γ_X (thus skipping the "intermediate floor" $\Gamma_{\mathfrak{X}}$), and show that the resulting measure coincides with the original Gibbs cluster measure g_{cl} (Section 2).

In fact, it can be we shown (Section 2.3) that *any* cluster measure μ_{cl} on Γ_X can be obtained by a similar projection from $\Gamma_{\mathcal{Z}}$. Even though it may not always be possible to find an intrinsic characterization of the corresponding lifted measure $\hat{\mu}$ on the configuration space $\Gamma_{\mathcal{Z}}$ (unlike the Poisson and Gibbs cases), the projection approach is instrumental in the study of more general cluster point processes by a reduction to point processes in more complex phase spaces but with a simpler correlation structure. These ideas are further developed in our recent paper [10].

2. Gibbs cluster measures via projections

In this section, we start by recalling some basic concepts and notation for random point processes and associated probability measures in configuration spaces (Section 2.1), followed in Section 2.2 by a definition of a general cluster point process (CPP). In Section 2.3, we explain our main "projection" construction allowing one to represent CPPs in the phase space X in terms of auxiliary measures on a more complex configuration space involving Cartesian powers of X. The implications of such a description are discussed in greater detail for the particular case of Gibbs CPPs (Sections 2.4, 2.5).

2.1. Probability measures on configuration spaces

Let X be a locally compact Polish space equipped with the Borel σ-algebra $\mathcal{B}(X)$ generated by the open sets. Let $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, and consider the space $\mathfrak{X}$ built from all Cartesian powers of X, that is, the disjoint union

$$\mathfrak{X} := \bigsqcup_{n \in \mathbb{Z}_+} X^n, \tag{2.1}$$

including $X^0 = \{\emptyset\}$. That is, $\bar{x} = (x_1, x_2, \dots) \in \mathfrak{X}$ if and only if $\bar{x} \in X^n$ for some $n \in \mathbb{Z}_+$. We take the liberty to write $x_i \in \bar{x}$ if x_i is a coordinate of the "vector" $\bar{x}$. The space $\mathfrak{X}$ is endowed with the natural disjoint union topology induced by the topology in X.

Remark 2.1. Note that a set $K \subset \mathfrak{X}$ is compact if and only if $K = \bigsqcup_{n=0}^{N} K_n$, where $N < \infty$ and K_n are compact subsets of X^n, respectively.

Remark 2.2. $\mathfrak{X}$ is a Polish space as a disjoint union of Polish spaces.

Denote by $\mathcal{N}(X)$ the space of $\mathbb{Z}_+$-valued measures on $\mathcal{B}(X)$ with countable (i.e., finite or countably infinite) support. Consider the natural projection

$$\mathfrak{X} \ni \bar{x} \mapsto \mathfrak{p}(\bar{x}) := \sum_{x_i \in \bar{x}} \delta_{x_i} \in \mathcal{N}(X), \tag{2.2}$$

where δ_x is the Dirac measure at point $x \in X$. That is to say, under the map $\mathfrak{p}$ each vector from $\mathfrak{X}$ is "unpacked" into its components to yield a countable aggregate of (possibly multiple) points in X, which can be interpreted as a *generalized configuration* γ,

$$\mathfrak{p}(\bar{x}) \leftrightarrow \gamma := \bigsqcup_{x_i \in \bar{x}} \{x_i\}, \quad \bar{x} = (x_1, x_2, \ldots) \in \mathfrak{X}. \tag{2.3}$$

In what follows, we interpret the notation γ either as an aggregate of points in X or as a $\mathbb{Z}_+$-valued measure or both, depending on the context. Even though generalized configurations are not, strictly speaking, subsets of X (because of possible multiplicities), it is convenient to use set-theoretic notation, which should not cause any confusion. For instance, we write $\gamma \cap B$ for the restriction of configuration γ to a subset $B \in \mathcal{B}(X)$. For a (measurable) function $f : X \to \mathbb{R}$ we denote

$$\langle f, \gamma \rangle := \sum_{x \in \gamma} f(x) \equiv \int_X f(x)\gamma(\mathrm{d}x) \tag{2.4}$$

whenever the right-hand side is well defined. In particular, if $\mathbf{1}_B(x)$ is the indicator function of a set $B \in \mathcal{B}(X)$ then $\langle \mathbf{1}_B, \gamma \rangle = \gamma(B)$ is the total number of points (counted with their multiplicities) in $\gamma \cap B$.

Definition 2.1. A *configuration space* $\Gamma_X^\sharp$ is the set of generalized configurations γ in X, endowed with the *cylinder σ-algebra* $\mathcal{B}(\Gamma_X^\sharp)$ generated by the class of cylinder sets $C_B^n := \{\gamma \in \Gamma_X^\sharp\colon \gamma(B) = n\}$, $B \in \mathcal{B}(X)$, $n \in \mathbb{Z}_+$.

Remark 2.3. It is easy to see that the map $\mathfrak{p} : \mathfrak{X} \to \Gamma_X^\sharp$ defined by formulas (2.2), (2.3) is measurable.

Let us denote by $\mathrm{M}_+(X)$ the class of non-negative measurable functions on X. The next simple fact follows from Definition 2.1 by a standard approximation and monotone class argument (see, e.g., [13, §A1.1]).

Lemma 2.1. *For any function $f \in \mathrm{M}_+(X)$, the pairing $\langle f, \cdot \rangle$ defined in* (2.4) *is measurable.*

Conventional theory of point processes (and their distributions as probability measures on configuration spaces) usually rules out the possibility of accumulation points or multiple points (see, e.g., [13]).

Definition 2.2. A configuration $\gamma \in \Gamma_X^\sharp$ is said to be *locally finite* if $\gamma(B) < \infty$ for any compact set $B \subset X$. A configuration $\gamma \in \Gamma_X^\sharp$ is called *simple* if $\gamma(\{x\}) \leqslant 1$ for each $x \in X$. A configuration $\gamma \in \Gamma_X^\sharp$ is called *proper* if it is both locally finite and simple. The set of proper configurations will be denoted by Γ_X and called the *proper configuration space* over X. The corresponding σ-algebra $\mathcal{B}(\Gamma_X)$ is generated by the cylinder sets $\{\gamma \in \Gamma_X\colon \gamma(B) = n\}$ $(B \in \mathcal{B}(X), n \in \mathbb{Z}_+)$.

Remark 2.4. The measurable (cylinder) structure in $(\Gamma_X, \mathcal{B}(\Gamma_X))$ coincides with that induced by the measurable space $(\Gamma_X^\sharp, \mathcal{B}(\Gamma_X^\sharp))$, that is,

$$\mathcal{B}(\Gamma_X) = \big\{B \cap \Gamma_X,\ B \in \mathcal{B}\big(\Gamma_X^\sharp\big)\big\}.$$

Definition 2.3. We shall also need the notation Γ_X^0 for the space of finite configurations in X,

$$\Gamma_X^0 := \big\{\gamma \in \Gamma_X\colon \gamma(X) < \infty\big\}.$$

In what follows, we shall often use various mappings of configuration spaces. To prepare some general ground for such considerations, let Y be a Polish space equipped with the Borel σ-algebra $\mathcal{B}(Y)$. For an arbitrary map $\phi : X \to Y$ consider its pointwise lifting to the configuration space $\Gamma_X^\sharp$ defined (preserving the same notation ϕ) as follows

$$\Gamma_X^\sharp \ni \gamma \mapsto \phi(\gamma) := \bigsqcup_{x \in \gamma} \big\{\phi(x)\big\} \in \Gamma_Y^\sharp. \tag{2.5}$$

Equivalently, treating configurations as $\mathbb{Z}_+$-valued measures, we can interpret (2.5) as the push-forward measure $\phi^*\gamma$, that is,

$$\phi(\gamma)(B) = \phi^*\gamma(B) := \gamma\big(\phi^{-1}(B)\big), \quad B \in \mathcal{B}(Y).$$

Lemma 2.2. *Let $\phi : X \to Y$ be a measurable map. Then its lifting $\phi : \Gamma_X^\sharp \to \Gamma_Y^\sharp$ and restriction $\phi|_{\Gamma_X}$ on Γ_X are also measurable.*

Proof. For any cylinder set $C_B^n \in \mathcal{B}(\Gamma_Y^\sharp)$ $(B \in \mathcal{B}(Y), n \in \mathbb{Z}_+)$, we have

$$\begin{aligned}\phi^{-1}\big(C_B^n\big) &= \big\{\gamma \in \Gamma_X^\sharp\colon \phi(\gamma) \in C_B^n\big\} = \big\{\gamma \in \Gamma_X^\sharp\colon \phi^*\gamma(B) = n\big\} \\ &= \big\{\gamma \in \Gamma_X^\sharp\colon \gamma\big(\phi^{-1}(B)\big) = n\big\} = C_{\phi^{-1}(B)}^n \in \mathcal{B}\big(\Gamma_X^\sharp\big),\end{aligned}$$

which proves the first assertion of the lemma. The second one follows by the fact that the measurable structure on Γ_X is induced from $\Gamma_X^\sharp$ (see Remark 2.4). □

Like in the standard theory based on proper configuration spaces (see, e.g., [13, §6.1]), *every probability measure μ on the generalized configuration space $\Gamma_X^\sharp$ can be characterized by its Laplace functional*

$$L_\mu(f) := \int_{\Gamma_X^\sharp} \mathrm{e}^{-\langle f,\gamma\rangle} \mu(\mathrm{d}\gamma), \quad f \in \mathrm{M}_+(X) \tag{2.6}$$

(the integral in (2.6) is well defined, since $f \geqslant 0$ and hence $0 \leqslant \mathrm{e}^{-\langle f,\gamma\rangle} \leqslant 1$). To see this, note that if $B \in \mathcal{B}(X)$ then $L_\mu(s\mathbf{1}_B)$ as a function of $s > 0$ gives the Laplace–Stieltjes transform of the distribution of the random variable $\gamma(B)$ and as such determines the values of the measure μ on the cylinder sets $C_B^n \in \mathcal{B}(\Gamma_X^\sharp)$ ($n \in \mathbb{Z}_+$). In particular, $L_\mu(s\mathbf{1}_B) = 0$ if and only if $\gamma(B) = \infty$ (μ-a.s.). Similarly, using linear combinations $\sum_{i=1}^k s_i \mathbf{1}_{B_i}$ we can recover the values of μ on the cylinder sets

$$C_{B_1,\dots,B_k}^{n_1,\dots,n_k} := \bigcap_{i=1}^k C_{B_i}^{n_i} = \bigl\{\gamma \in \Gamma_X^\sharp \colon \gamma(B_i) = n_i,\ i = 1,\dots,k\bigr\},$$

and hence on the ring $\mathcal{C}(X)$ of finite disjoint unions of such sets. Since the ring $\mathcal{C}(X)$ generates the cylinder σ-algebra $\mathcal{B}(\Gamma_X^\sharp)$, the extension theorem (see, e.g., [20, §13, Theorem A] or [13, Theorem A1.3.III]) ensures that the measure μ on $\mathcal{B}(\Gamma_X^\sharp)$ is determined uniquely.

2.2. *Cluster point processes*

Let us recall the notion of a cluster point process with independent clusters (see, e.g., [12,13]). Heuristically, its realizations are constructed in two steps: (i) a background random configuration of (invisible) "centers" is obtained as a realization of some point process γ_{c} governed by a probability measure μ_{c} on Γ_X, and (ii) relative to each center $x \in \gamma_{\mathrm{c}}$, a set of observable secondary points (referred to as a *cluster* centered at x) is generated, independently of all other clusters, according to a point process γ'_x with distribution μ_x on the space of finite configurations Γ_X^0 (see Definition 2.3). The resulting (countable) assembly of random points, called the *cluster point process* (*CPP*), can be expressed symbolically as

$$\gamma = \bigsqcup_{x \in \gamma_{\mathrm{c}}} \gamma'_x \in \Gamma_X^\sharp, \tag{2.7}$$

where the disjoint union signifies that multiplicities of points should be taken into account. Note that CPP configurations (2.7) may in principle have accumulation and/or multiple points due to the overlapping contributions from different clusters.

In what follows, we assume that (i) X is a topological vector space (e.g., $X = \mathbb{R}^d$), so that translations $X \ni y \mapsto y + x \in X$ ($x \in X$) are defined and continuous, and (ii) random clusters are independent and identically distributed (i.i.d.), being governed by a common probability law translated to the cluster centers, so that $\mu_x(A) = \mu_0(A - x)$ ($x \in X$, $A \in \mathcal{B}(\Gamma_X^0)$). In turn, the measure μ_0 on Γ_X^0 determines a probability distribution η in $\mathfrak{X}$, which (i) is symmetric with respect to permutations of coordinates and (ii) does not charge any coordinate diagonals, i.e., $\eta(\{\bar{y} = (y_1, y_2, \dots) \in \mathfrak{X} \colon \exists i \neq j \text{ such that } y_i = y_j\}) = 0$. Conversely, μ_0 is the push-forward of the measure η under the projection map $\mathfrak{p}$ defined by (2.3),

$$\mu_0 = \mathfrak{p}^*\eta \equiv \eta \circ \mathfrak{p}^{-1}. \tag{2.8}$$

Remark 2.5. For the sake of technical convenience, we prefer to work with the measure η on the vector space $\mathfrak{X}$, rather than with the original measure μ_0 on the configuration space Γ_X^0.

Let μ_{cl} denote a probability measure on $(\Gamma_X^\sharp, \mathcal{B}(\Gamma_X^\sharp))$ determined by the CPP configurations (2.7). Rigorous construction of such a measure can be carried out via a general approach developed for models based on conditioning (see [13, Chapter 6]) by defining the corresponding Laplace functional

$$L_{\mu_{\mathrm{cl}}}(f) = \int_{\Gamma_X} L_{\mu_{\mathrm{cl}}}(f|\gamma)\,\mu_{\mathrm{c}}(\mathrm{d}\gamma), \qquad f \in \mathrm{M}_+(X), \tag{2.9}$$

where $L_{\mu_{\mathrm{cl}}}[f|\gamma]$ is the conditional Laplace functional of the hypothetical cluster measure μ_{cl} (conditioned on the configuration $\gamma \in \Gamma_X$ of the cluster centers). Furthermore, according to general theory (see [13, §6.1, Proposition 6.1.II, p. 165, and Lemma 6.1.III, p. 166]) one has to identify $L_{\mu_{\mathrm{cl}}}[f|\gamma]$ and to verify that this is a measurable function of γ. Using the i.i.d. structure of clusters, we obtain

$$\begin{aligned}
L_{\mu_{\mathrm{cl}}}[f|\gamma] &= \int_{(\Gamma_X^0)^\gamma} \exp\Big\{-\sum_{x\in\gamma}\langle f, \gamma_x'\rangle\Big\} \bigotimes_{x\in\gamma} \mu_x\big(\mathrm{d}\gamma_x'\big) \\
&= \int_{\mathfrak{X}^\gamma} \exp\Big\{-\sum_{x\in\gamma}\sum_{y_i\in\bar{y}_x+x} f(y_i)\Big\} \bigotimes_{x\in\gamma} \eta(\mathrm{d}\bar{y}_x) \\
&= \prod_{x\in\gamma}\int_{\mathfrak{X}} \exp\Big\{-\sum_{y_i\in\bar{y}_x} f(y_i+x)\Big\}\,\eta(\mathrm{d}\bar{y}_x) \\
&= \exp\Big\{-\sum_{x\in\gamma}\bar{f}(x)\Big\} = \exp\big\{-\langle\bar{f},\gamma\rangle\big\},
\end{aligned}$$

where $\mathfrak{X}^\gamma = \prod_{x\in\gamma}\mathfrak{X}_x$ ($\mathfrak{X}_x = \mathfrak{X}$) and

$$\bar{f}(x) := -\log\bigg(\int_{\mathfrak{X}} \exp\Big\{-\sum_{y_i\in\bar{y}} f(y_i+x)\Big\}\,\eta(\mathrm{d}\bar{y})\bigg) \geqslant 0, \qquad x \in X.$$

Since the function $\bar{f}$ is obviously $\mathcal{B}(X)$-measurable, Lemma 2.1 implies that $L_{\mu_{\mathrm{cl}}}[f|\gamma]$ is $\mathcal{B}(\Gamma_X^\sharp)$-measurable, as required.

Thus, we have established that the cluster measure μ_{cl} exists, and in particular its Laplace functional is given by

$$L_{\mu_{\mathrm{cl}}}(f) = \int_{\Gamma_X}\prod_{x\in\gamma}\bigg(\int_{\mathfrak{X}} \exp\Big(-\sum_{y_i\in\bar{y}} f(y_i+x)\Big)\eta(\mathrm{d}\bar{y})\bigg)\mu_{\mathrm{c}}(\mathrm{d}\gamma). \tag{2.10}$$

Remark 2.6. Infinite product of the form $\prod_{x\in\gamma} a_x$ with $a_x \in [0,1]$ (see (2.10)) is well defined as $\exp\big\{\sum_{x\in\gamma}\ln a_x\big\} \in [0,1]$.

Remark 2.7. Formula (2.10) is well known in the case of CPPs without accumulation points (see, e.g., [13, §6.3]).

Remark 2.8. In the standard theory of point processes, CPP sample configurations are, by definition, *presumed* to be almost surely (a.s.) locally finite (see, e.g., [13, Definition 6.3.I]). As we have seen, this is not necessary for the existence of the cluster point process as a measure μ_{cl} on the generalized configuration space $\Gamma_X^\sharp$. However, developing the differential analysis on configuration spaces in the spirit of [2,3] demands that the measure μ_{cl} be supported on the *proper* configuration space Γ_X; conditions for the latter to be true are also of general interest. We shall address this issue in Section 2.4 below for the Gibbs CPPs (see [9] for the case of the Poisson CPPs).

2.3. A projection construction of cluster measures on configurations

Denote $\mathcal{Z} := X \times \mathfrak{X}$ and consider the space $\Gamma_{\mathcal{Z}} = \{\hat\gamma\}$ of (proper) configurations in $\mathcal{Z}$. Let $p_X : \mathcal{Z} \to X$ and $p_{\mathfrak{X}} : \mathcal{Z} \to \mathfrak{X}$ be the natural projections to the first and second coordinate, respectively,

$$\mathcal{Z} \ni z = (x, \bar y) \mapsto p_X(z) := x \in X, \tag{2.11}$$

$$\mathcal{Z} \ni z = (x, \bar y) \mapsto p_{\mathfrak{X}}(z) := \bar y \in \mathfrak{X}, \tag{2.12}$$

and consider their lifting to the configuration space $\Gamma_{\mathcal{Z}}$ (cf. (2.5))

$$\Gamma_{\mathcal{Z}} \ni \hat\gamma \mapsto p_X(\hat\gamma) := \bigsqcup_{z\in\hat\gamma} \{p_X(z)\} \in \Gamma_X^\sharp, \tag{2.13}$$

$$\Gamma_{\mathcal{Z}} \ni \hat\gamma \mapsto p_{\mathfrak{X}}(\hat\gamma) := \bigsqcup_{z\in\hat\gamma} \{p_{\mathfrak{X}}(z)\} \in \Gamma_{\mathfrak{X}}^\sharp. \tag{2.14}$$

Let us define a probability measure $\hat\mu$ on $\Gamma_{\mathcal{Z}}$ as the distribution of the marked point process (see [13, §§6.1, 6.4]) with configurations $\hat\gamma := \bigsqcup_{x\in\gamma} \{(x, \bar y_x)\}$ in $\mathcal{Z}$, obtained from configurations $\gamma \in \Gamma_X$ by attaching to each point $x \in \gamma$ an i.i.d. random vector $\bar y_x$ with distribution η:

$$\Gamma_X \ni \gamma \mapsto \hat\gamma := \bigsqcup_{x\in\gamma} \{(x, \bar y_x)\} \in \Gamma_{\mathcal{Z}}. \tag{2.15}$$

The measure $\hat\mu$ may be expressed in differential form as a skew product

$$\hat\mu(\mathrm{d}\hat\gamma) = \mu_{\mathrm{c}}\big(p_X(\mathrm{d}\hat\gamma)\big) \bigotimes_{z\in\hat\gamma} \eta\big(p_{\mathfrak{X}}(\mathrm{d}z)\big), \quad \hat\gamma \in \Gamma_{\mathcal{Z}}. \tag{2.16}$$

Equivalently, for any function $F \in \mathrm{M}_+(\Gamma_{\mathcal{Z}})$,

$$\int_{\Gamma_{\mathcal{Z}}} F(\hat\gamma)\, \hat\mu(\mathrm{d}\hat\gamma) = \int_{\Gamma_X} \bigg(\int_{\mathfrak{X}^\gamma} F\bigg(\bigcup_{x\in\gamma} \{(x, \bar y_x)\} \bigg) \bigotimes_{x\in\gamma} \eta(\mathrm{d}\bar y_x) \bigg)\, \mu_{\mathrm{c}}(\mathrm{d}\gamma). \tag{2.17}$$

In particular, general construction ensures (see details in [13, §§6.1, 6.4]) that the internal integral on the right-hand side of (2.17) is measurable in $\gamma \in \Gamma_X$.

Remark 2.9. Given the probability measures μ_c on Γ_X and η on $\mathfrak{X}$, we can construct two point processes, the *cluster process* (with distribution μ_{cl}) and the *marked process* (with distribution $\hat{\mu}$). The crucial difference between them is that sample configurations of the latter are a.s.-proper, in contrast to the former process which, in general, is supported on the space of generalized configurations. Moreover, many analytical properties of $\hat{\mu}$ are simpler than those of μ_{cl}. In fact, as will be explained below, there is a natural link between the two processes, in that the cluster measure μ_{cl} can be represented as a certain "projection" of the marked measure $\hat{\mu}$, which in turn paves the way for the study of μ_{cl} using $\hat{\mu}$. In a nutshell, this is the main idea of our approach.

Recall that the "unpacking" map $\mathfrak{p} : \mathfrak{X} \to \Gamma_X^\sharp$ is defined in (2.3), and consider a map $\mathfrak{q} : \mathcal{Z} \to \Gamma_X^\sharp$ acting by the formula

$$\mathfrak{q}(x, \bar{y}) := \mathfrak{p}(\bar{y} + x) = \bigsqcup_{y_i \in \bar{y}} \{y_i + x\}, \quad (x, \bar{y}) \in \mathcal{Z}. \tag{2.18}$$

Here and below, we use the shift notation ($x \in X$)

$$\bar{y} + x := (y_1 + x, y_2 + x, \ldots), \quad \bar{y} = (y_1, y_2, \ldots) \in \mathfrak{X}. \tag{2.19}$$

The map $\mathfrak{q}$ can be lifted to the configuration space $\Gamma_{\mathcal{Z}}$,

$$\Gamma_{\mathcal{Z}} \ni \hat{\gamma} \mapsto \mathfrak{q}(\hat{\gamma}) := \bigsqcup_{z \in \hat{\gamma}} \mathfrak{q}(z) \in \Gamma_X^\sharp. \tag{2.20}$$

Proposition 2.3. *The map $\mathfrak{q} : \Gamma_{\mathcal{Z}} \to \Gamma_X^\sharp$ defined by (2.20) is measurable.*

Proof. Observe that $\mathfrak{q}$ can be represented as the composition

$$\mathfrak{q} = \mathfrak{p} \circ q : \Gamma_{\mathcal{Z}} \xrightarrow{q} \Gamma_{\mathfrak{X}}^\sharp \xrightarrow{\mathfrak{p}} \Gamma_X^\sharp, \tag{2.21}$$

where $\mathfrak{p}$ is defined by

$$\Gamma_{\mathfrak{X}}^\sharp \ni \bar{\gamma} \mapsto \mathfrak{p}(\bar{\gamma}) := \bigsqcup_{\bar{y} \in \bar{\gamma}} \mathfrak{p}(\bar{y}) \in \Gamma_X^\sharp, \tag{2.22}$$

whereas q is the lifting (see (2.5)) of the map

$$\mathcal{Z} \ni (x, \bar{y}) \mapsto \bar{y} + x \in \mathfrak{X}$$

and is therefore measurable by Lemma 2.2. Hence, it remains to show that the map (2.22) is measurable, that is, $\mathfrak{p}^{-1}(C_B^n) \in \mathcal{B}(\Gamma_{\mathfrak{X}}^\sharp)$ for any cylinder set $C_B^n = \{\gamma \in \Gamma_X^\sharp\colon\ \gamma(B) = n\} \subset \Gamma_X^\sharp$ ($B \in \mathcal{B}(X)$, $n \in \mathbb{Z}_+$). Setting

$$\mathfrak{X}_B^n := \left\{ \bar{x} \in \mathfrak{X}\colon \sum_{x_i \in \bar{x}} \mathbf{1}_B(x_i) = n \right\} \in \mathcal{B}(\mathfrak{X}), \tag{2.23}$$

it is easy to see that the pre-image $\mathfrak{p}^{-1}(C_B^n)$ may be represented as a measurable combination of various cylinder sets $\bar{C}_{\bar{B}}^k$ in $\Gamma_{\mathfrak{X}}^\sharp$, for instance,

$$\begin{aligned}
\mathfrak{p}^{-1}(C_B^0) &= \{\bar{\gamma} \in \Gamma_{\mathfrak{X}}^\sharp\colon \bar{\gamma}(\mathfrak{X} \setminus \mathfrak{X}_B^0) = 0\} = \bar{C}^0_{\mathfrak{X} \setminus \mathfrak{X}_B^0}, \\
\mathfrak{p}^{-1}(C_B^1) &= \{\bar{\gamma} \in \Gamma_{\mathfrak{X}}^\sharp\colon \bar{\gamma}(\mathfrak{X}_B^1) = 1\} = \bar{C}^1_{\mathfrak{X}_B^1}, \\
\mathfrak{p}^{-1}(C_B^2) &= \{\bar{\gamma} \in \Gamma_{\mathfrak{X}}^\sharp\colon \bar{\gamma}(\mathfrak{X}_B^2) = 1 \text{ or } \bar{\gamma}(\mathfrak{X}_B^1) = 2\} = \bar{C}^1_{\mathfrak{X}_B^2} \cup \bar{C}^2_{\mathfrak{X}_B^1},
\end{aligned}$$

and more generally

$$\mathfrak{p}^{-1}(C_B^n) = \bigcup_{(n_k)} \bigcap_{k=1}^{\infty} \{\bar{\gamma} \in \Gamma_{\mathfrak{X}}^\sharp\colon \bar{\gamma}(\mathfrak{X}_B^k) = n_k\} = \bigcup_{(n_k)} \bigcap_{k=1}^{\infty} \bar{C}^{n_k}_{\mathfrak{X}_B^k} \in \mathcal{B}(\Gamma_{\mathfrak{X}}^\sharp),$$

where the union is taken over integer arrays $(n_k) = (n_1, n_2, \ldots)$ such that $n_k > 0$ and $\sum_k k n_k = n$. This proves that the map $\mathfrak{p}$ is measurable, and hence the composition of maps in (2.21) is measurable as well. □

Let us define a measure on $\Gamma_X^\sharp$ as the push-forward of $\hat{\mu}$ (see (2.16), (2.17)) under the map $\mathfrak{q}$ defined in (2.18), (2.20):

$$\mathfrak{q}^*\hat{\mu}(A) \equiv \hat{\mu}\big(\mathfrak{q}^{-1}(A)\big), \quad A \in \mathcal{B}\big(\Gamma_X^\sharp\big), \tag{2.24}$$

or equivalently

$$\int_{\Gamma_X^\sharp} F(\gamma)\, \mathfrak{q}^*\hat{\mu}(\mathrm{d}\gamma) = \int_{\Gamma_{\mathcal{Z}}} F\big(\mathfrak{q}(\hat{\gamma})\big)\, \hat{\mu}(\mathrm{d}\hat{\gamma}), \quad F \in \mathrm{M}_+\big(\Gamma_X^\sharp\big). \tag{2.25}$$

The next general result shows that this measure may be identified with the original cluster measure μ_{cl} introduced in Section 2.2.

Theorem 2.4. *The measure* (2.24) *coincides with the cluster measure* μ_{cl}*,*

$$\mu_{\mathrm{cl}} = \mathfrak{q}^*\hat{\mu} \equiv \hat{\mu} \circ \mathfrak{q}^{-1}. \tag{2.26}$$

Proof. Let us evaluate the Laplace functional of the measure $\mathfrak{q}^*\hat{\mu}$. For any function $f \in \mathrm{M}_+(X)$, we obtain, using (2.25), (2.20) and (2.17),

$$L_{\mathfrak{q}^*\hat{\mu}}(f) = \int_{\Gamma_X^\sharp} \exp\big(-\langle f, \xi\rangle\big)\, \mathfrak{q}^*\hat{\mu}(\mathrm{d}\xi) = \int_{\Gamma_{\mathcal{Z}}} \exp\big(-\big\langle f, \mathfrak{q}(\hat{\gamma})\big\rangle\big)\, \hat{\mu}(\mathrm{d}\hat{\gamma})$$

$$= \int_{\Gamma_X} \biggl(\int_{\mathfrak{X}^\gamma} \prod_{x\in\gamma} \exp\biggl(-\sum_{y_i\in\bar{y}_x} f(y_i+x) \biggr) \bigotimes_{x\in\gamma} \eta(\mathrm{d}\bar{y}_x) \biggr) \mu_{\mathrm{c}}(\mathrm{d}\gamma)$$

$$= \int_{\Gamma_X} \prod_{x\in\gamma} \biggl(\int_{\mathfrak{X}} \exp\biggl(-\sum_{y_i\in\bar{y}} f(y_i+x) \biggr) \eta(\mathrm{d}\bar{y}) \biggr) \mu_{\mathrm{c}}(\mathrm{d}\gamma),$$

which coincides with the Laplace functional (2.10) of the cluster measure μ_{cl}. □

2.4. *Gibbs cluster measure via an auxiliary Gibbs measure*

In this paper, we are concerned with *Gibbs cluster point processes*, for which the distribution of cluster centers is given by some Gibbs measure $\mathrm{g} \in \mathscr{G}(\theta, \Phi)$ on the configuration space Γ_X (see Appendix A), specified by a *reference measure* θ on X and an *interaction potential* $\Phi : \Gamma_X^0 \to \mathbb{R} \cup \{+\infty\}$, where $\Gamma_X^0 \subset \Gamma_X$ is the subspace of finite configurations in X. We assume that the set $\mathscr{G}(\theta, \Phi)$ of all Gibbs measures on Γ_X associated with θ and Φ is non-empty.[5]

Specializing to the Gibbs case the definition of the measure $\hat{\mu}$ given in Section 2.3, let us consider the corresponding auxiliary measure $\hat{\mathrm{g}}$ given by formulas (2.16), (2.17) with $\mu_{\mathrm{c}} = \mathrm{g}$. Owing to the general Theorem 2.4 (see (2.26)), the corresponding Gibbs cluster measure g_{cl} on the configuration space Γ_X is represented as a push-forward of $\hat{\mathrm{g}}$ on $\Gamma_{\mathcal{Z}}$ under the map $\mathfrak{q}$ defined in (2.18), (2.20):

$$\mathrm{g}_{\mathrm{cl}} = \mathfrak{q}^* \hat{\mathrm{g}} \equiv \hat{\mathrm{g}} \circ \mathfrak{q}^{-1}. \tag{2.27}$$

Our next goal is to show that $\hat{\mathrm{g}}$ is a *Gibbs measure* on $\Gamma_{\mathcal{Z}}$, with the reference measure $\hat{\theta}$ defined as a product measure on the space $\mathcal{Z} = X \times \mathfrak{X}$,

$$\hat{\theta} := \theta \otimes \eta, \tag{2.28}$$

and with the interaction potential $\hat{\Phi} : \Gamma_{\mathcal{Z}}^0 \to \mathbb{R} \cup \{+\infty\}$ given by

$$\hat{\Phi}(\hat{\gamma}) := \begin{cases} \Phi(p_X(\hat{\gamma})), & \hat{\gamma} \in \Gamma_{\mathcal{Z}}^0 \cap \Gamma_X(\mathfrak{X}), \\ +\infty, & \hat{\gamma} \in \Gamma_{\mathcal{Z}}^0 \setminus \Gamma_X(\mathfrak{X}), \end{cases} \tag{2.29}$$

where p_X is the projection defined in (2.13). The corresponding functionals of energy $\hat{E}(\hat{\xi})$ and interaction energy $\hat{E}(\hat{\xi}, \hat{\gamma})$ ($\hat{\xi} \in \Gamma_{\mathcal{Z}}^0$, $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$) are then given by (see (A.1) and (A.2))

$$\hat{E}(\hat{\xi}) := \sum_{\hat{\xi}' \subset \hat{\xi}} \hat{\Phi}(\hat{\xi}'), \tag{2.30}$$

$$\hat{E}(\hat{\xi}, \hat{\gamma}) := \begin{cases} \sum_{\hat{\gamma} \supset \hat{\gamma}' \in \Gamma_{\mathcal{Z}}^0} \hat{\Phi}(\hat{\xi} \cup \hat{\gamma}') & \text{if } \sum_{\hat{\gamma} \supset \hat{\gamma}' \in \Gamma_{\mathcal{Z}}^0} |\hat{\Phi}(\hat{\xi} \cup \hat{\gamma}')| < \infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.31}$$

The following "projection" property of the energy is obvious from the definition (2.29) of the potential $\hat{\Phi}$.

[5] For various sufficient conditions, consult [33,35]; see also references in Appendix A.

Lemma 2.5. *For any configurations $\hat{\xi} \in \Gamma_{\mathcal{Z}}^0$ and $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$, we have*

$$\hat{E}(\hat{\xi}) = E\big(p_X(\hat{\xi})\big), \qquad \hat{E}(\hat{\xi}, \hat{\gamma}) = E\big(p_X(\hat{\xi}), p_X(\hat{\gamma})\big).$$

Theorem 2.6. (a) *Let $\mathrm{g} \in \mathscr{G}(\theta, \Phi)$ be a Gibbs measure on the configuration space Γ_X, and let $\hat{\mathrm{g}}$ be the corresponding probability measure on the configuration space $\Gamma_{\mathcal{Z}}$ (see* (2.16), (2.17)). *Then $\hat{\mathrm{g}} \in \mathscr{G}(\hat{\theta}, \hat{\Phi})$, i.e., $\hat{\mathrm{g}}$ is a Gibbs measure on $\Gamma_{\mathcal{Z}}$ with the reference measure $\hat{\theta}$ and the interaction potential $\hat{\Phi}$ defined by* (2.28) *and* (2.29), *respectively.*

(b) *If the measure $\mathrm{g} \in \mathscr{G}(\theta, \Phi)$ has finite correlation function κ_{g}^n of some order $n \in \mathbb{N}$ (see Definition* A.3 *in Appendix* A), *then the correlation function $\kappa_{\hat{\mathrm{g}}}^n$ of the measure $\hat{\mathrm{g}} \in \mathscr{G}(\hat{\theta}, \hat{\Phi})$ is also finite, being given by*

$$\kappa_{\hat{\mathrm{g}}}^n(z_1, \dots, z_n) = \kappa_{\mathrm{g}}^n\big(p_X(z_1), \dots, p_X(z_n)\big), \quad z_1, \dots, z_n \in \mathcal{Z}. \tag{2.32}$$

Proof. (a) In order to show that $\hat{\mathrm{g}} \in \mathscr{G}(\hat{\theta}, \hat{\Phi})$, it suffices to check that $\hat{\mathrm{g}}$ satisfies the GNZ equation on $\Gamma_{\mathcal{Z}}$ (see Eq. (A.3) in Appendix A), that is, for any non-negative, $\mathcal{B}(\mathcal{Z}) \times \mathcal{B}(\Gamma_{\mathcal{Z}})$-measurable function $H(z, \hat{\gamma})$ it holds

$$\int_{\Gamma_{\mathcal{Z}}} \sum_{z \in \hat{\gamma}} H(z, \hat{\gamma})\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_{\mathcal{Z}}} \bigg(\int_{\mathcal{Z}} H\big(z, \hat{\gamma} \cup \{z\}\big) \mathrm{e}^{-\hat{E}(\{z\}, \hat{\gamma})}\, \hat{\theta}(\mathrm{d}z) \bigg) \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}). \tag{2.33}$$

Using formula (2.17), the left-hand side of (2.33) can be represented as

$$\begin{aligned} &\int_{\Gamma_X} \bigg(\int_{\mathfrak{X}^\gamma} \sum_{x \in \gamma} H\bigg((x, \bar{y}_x), \bigcup_{x' \in \gamma} \big\{ (x', \bar{y}_{x'}) \big\} \bigg) \bigotimes_{x' \in \gamma} \eta(\mathrm{d}\bar{y}_{x'}) \bigg) \mathrm{g}(\mathrm{d}\gamma) \\ &\quad = \int_{\Gamma_X} \sum_{x \in \gamma} H_0(x, \gamma)\, \mathrm{g}(\mathrm{d}\gamma), \end{aligned} \tag{2.34}$$

where

$$H_0(x, \gamma) := \int_{\mathfrak{X}^\gamma} \mathbf{1}_\gamma(x) H\bigg((x, \bar{y}_x), \bigcup_{x' \in \gamma} \big\{ (x', \bar{y}_{x'}) \big\} \bigg) \bigotimes_{x' \in \gamma} \eta(\mathrm{d}\bar{y}_{x'}).$$

Applying the GNZ equation to the Gibbs measure g with the function $H_0(x, \gamma)$, we see that (2.34) takes the form

$$\int_{\Gamma_X} \bigg(\int_X H_0\big(x, \gamma \cup \{x\}\big) \mathrm{e}^{-E(\{x\}, \gamma)}\, \theta(\mathrm{d}x) \bigg) \mathrm{g}(\mathrm{d}\gamma). \tag{2.35}$$

Similarly, recalling that $\hat{\theta} = \theta \otimes \eta$ and using Lemma 2.5, the right-hand side of (2.33) is reduced to

$$
\begin{aligned}
&\int_{\Gamma_X}\biggl(\int_{\mathfrak{X}^\gamma}\biggl(\int_{X\times\mathfrak{X}} H\Bigl((x,\bar y_x),\bigcup_{x'\in\gamma}\bigl\{(x',\bar y_{x'})\bigr\}\cup\bigl\{(x,\bar y_x)\bigr\}\Bigr)\\
&\qquad\times \mathrm{e}^{-E(\{x\},\gamma)}\,\eta(\mathrm{d}\bar y_x)\,\theta(\mathrm{d}x)\biggr)\bigotimes_{x'\in\gamma}\eta(\mathrm{d}\bar y_{x'})\biggr)\,\mathrm{g}(\mathrm{d}\gamma)\\
&\quad=\int_{\Gamma_X}\biggl(\int_X\biggl(\int_{\mathfrak{X}^\gamma} H\Bigl((x,\bar y_x),\bigcup_{x''\in\gamma\cup\{x\}}\bigl\{(x'',\bar y_{x''})\bigr\}\Bigr)\,\mathrm{e}^{-E(\{x\},\gamma)}\bigotimes_{x''\in\gamma\cup\{x\}}\eta(\mathrm{d}\bar y_{x''})\biggr)\,\theta(\mathrm{d}x)\biggr)\,\mathrm{g}(\mathrm{d}\gamma)\\
&\quad=\int_{\Gamma_X}\biggl(\int_X H_0\bigl(x,\gamma\cup\{x\}\bigr)\mathrm{e}^{-E(\{x\},\gamma)}\,\theta(\mathrm{d}x)\biggr)\,\mathrm{g}(\mathrm{d}\gamma),
\end{aligned}
$$

thus coinciding with (2.35). This proves Eq. (2.33), hence $\hat{\mathrm{g}}\in\mathscr{G}(\hat\theta,\hat\Phi)$.

(b) Let $f\in\mathrm{M}_+(\mathcal{Z}^n)$ be a symmetric function. According to formula (2.17) applied to the function

$$F(\hat\gamma):=\sum_{\{z_1,\dots,z_n\}\subset\hat\gamma} f(z_1,\dots,z_n),$$

we have

$$\int_{\Gamma_{\mathcal{Z}}} F(\hat\gamma)\,\hat{\mathrm{g}}(\mathrm{d}\hat\gamma)=\int_{\Gamma_X}\sum_{\{x_1,\dots,x_n\}\subset\gamma}\phi(x_1,\dots,x_n)\,\mathrm{g}(\mathrm{d}\gamma), \tag{2.36}$$

where

$$\phi(x_1,\dots,x_n):=\int_{\mathfrak{X}^n} f\bigl((x_1,\bar y_1),\dots,(x_n,\bar y_n)\bigr)\bigotimes_{i=1}^n\eta(\mathrm{d}\bar y_i)\in\mathrm{M}_+\bigl(X^n\bigr).$$

Applying the definition of the correlation function κ_{g}^n (see (A.7)) and using that $\theta(\mathrm{d}x)\otimes\eta(\mathrm{d}\bar y)=\hat\theta(\mathrm{d}x\times\mathrm{d}\bar y)$, we obtain from (2.36)

$$\int_{\Gamma_{\mathcal{Z}}} F(\hat\gamma)\,\hat{\mathrm{g}}(\mathrm{d}\hat\gamma)=\frac{1}{n!}\int_{\mathcal{Z}^n} f(z_1,\dots,z_n)\kappa_{\mathrm{g}}^n\bigl(p_X(z_1),\dots,p_X(z_n)\bigr)\bigotimes_{i=1}^n\hat\theta(\mathrm{d}z_i),$$

and equality (2.32) follows. □

In the rest of this subsection, $\mathscr{G}_L$ denotes the subclass of Gibbs measures in $\mathscr{G}$ (with a given reference measure and interaction potential) that satisfy the so-called *Lenard bound* (see Appendix A, formula (A.11)).

Corollary 2.7. *We have* $\mathrm{g}\in\mathscr{G}_L(\theta,\Phi)$ *if and only if* $\hat{\mathrm{g}}\in\mathscr{G}_L(\hat\theta,\hat\Phi)$.

Proof. Follows directly from formula (2.32). □

The following statement is, in a sense, converse to Theorem 2.6(a).

Theorem 2.8. *If* $\varpi \in \mathscr{G}(\hat{\theta}, \hat{\Phi})$ *then* $\mathrm{g} := p_X^* \varpi \in \mathscr{G}(\theta, \Phi)$. *Moreover, if* $\mathrm{g} \in \mathscr{G}_L(\theta, \Phi)$ *then* $\varpi = \hat{\mathrm{g}}$.

Proof. Applying the GNZ equation (A.3) to the measure ϖ and using the cylinder structure of the interaction potential $\hat{\Phi}$, we have

$$\begin{aligned}
\int_{\Gamma_X} \sum_{x\in\gamma} H(x,\gamma) p_X^* \varpi(\mathrm{d}\gamma) &= \int_{\Gamma_{\mathcal{Z}}} \sum_{x\in p_X\hat{\gamma}} H(x, p_X\hat{\gamma})\, \varpi(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_{\mathcal{Z}}} \left(\int_{\mathcal{Z}} H\big(p_X z, p_X\big(\hat{\gamma}\cup\{z\}\big)\big)\, \mathrm{e}^{-E(\{p_X z\}, p_X\hat{\gamma})} \theta\otimes\eta(\mathrm{d}z) \right) \varpi(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_{\mathcal{Z}}} \left(\int_X H\big(x, p_X\hat{\gamma}\cup\{x\}\big)\, \mathrm{e}^{-E(\{x\}, p_X\hat{\gamma})} \theta(\mathrm{d}x) \right) \varpi(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_X} \left(\int_X H\big(x, \gamma\cup\{x\}\big)\, \mathrm{e}^{-E(\{x\},\gamma)}\, \theta(\mathrm{d}x) \right) p_X^* \varpi(\mathrm{d}\gamma).
\end{aligned}$$

Thus, the measure $p_X^*\varpi$ satisfies the GNZ equation and so, by Theorem A.1, belongs to the Gibbs class $\mathscr{G}(\theta, \Phi)$.

Next, in order to prove that $\varpi = \hat{\mathrm{g}}$, by Proposition A.2 it suffices to show that the measures ϖ and $\hat{\mathrm{g}}$ have the same correlation functions. Note that the correlation function κ_ϖ^n can be written in the form [25, §2.3, Lemma 2.3.8]

$$\begin{aligned}
\kappa_\varpi^n(z_1,\dots,z_n) &= \mathrm{e}^{-\hat{E}(\{z_1,\dots,z_n\})} \int_{\Gamma_{\mathcal{Z}}} \mathrm{e}^{-\hat{E}(\{z_1,\dots,z_n\},\hat{\gamma})}\, \varpi(\mathrm{d}\hat{\gamma}) \\
&= \mathrm{e}^{-E(\{p_X(z_1),\dots,p_X(z_n)\})} \int_{\Gamma_X} \mathrm{e}^{-E(\{p_X(z_1),\dots,p_X(z_n)\},\gamma)} p_X^*\varpi(\mathrm{d}\gamma) \\
&= \mathrm{e}^{-E(\{p_X(z_1),\dots,p_X(z_n)\})} \int_{\Gamma_X} \mathrm{e}^{-E(\{p_X(z_1),\dots,p_X(z_n)\},\gamma)}\, \mathrm{g}(\mathrm{d}\gamma) \\
&= \kappa_{\mathrm{g}}^n\big(p_X(z_1),\dots,p_X(z_n)\big).
\end{aligned}$$

Therefore, on account of Theorem 2.6(b) we get $\kappa_\varpi^n(z_1,\dots,z_n) = \kappa_{\hat{\mathrm{g}}}^n(z_1,\dots,z_n)$ for all $z_1,\dots,z_n \in \mathcal{Z}$ $(z_i \neq z_j)$, as required. □

In the next corollary, $\operatorname{ext}\mathscr{G}$ denotes the set of *extreme points* of the class $\mathscr{G}$ of Gibbs measures with the corresponding reference measure and interaction potential (see Appendix A).

Corollary 2.9. *Suppose that* $\mathrm{g} \in \mathscr{G}_L(\theta, \Phi)$. *Then* $\mathrm{g} \in \operatorname{ext}\mathscr{G}(\theta, \Phi)$ *if and only if* $\hat{\mathrm{g}} \in \operatorname{ext}\mathscr{G}(\hat{\theta}, \hat{\Phi})$.

Proof. Let $\mathrm{g} \in \mathscr{G}_L(\theta, \Phi) \cap \operatorname{ext}\mathscr{G}(\theta, \Phi)$. Assume that $\hat{\mathrm{g}} = \frac{1}{2}(\mu_1 + \mu_2)$ with some $\mu_1, \mu_2 \in \mathscr{G}(\hat{\theta}, \hat{\Phi})$. Then $\mathrm{g} = \frac{1}{2}(\mathrm{g}_1 + \mathrm{g}_2)$, where $\mathrm{g}_i = p_X^*\mu_i \in \mathscr{G}(\theta, \Phi)$. Since $\mathrm{g} \in \operatorname{ext}\mathscr{G}(\theta, \Phi)$, this implies that $\mathrm{g}_1 = \mathrm{g}_2 = \mathrm{g}$. In particular, $\mathrm{g}_1, \mathrm{g}_2 \in \mathscr{G}_L(\theta, \Phi)$ and by Theorem 2.8 we obtain that $\mu_1 = \hat{\mathrm{g}}_1 = \hat{\mathrm{g}} = \hat{\mathrm{g}}_2 = \mu_2$, which implies $\hat{\mathrm{g}} \in \operatorname{ext}\mathscr{G}(\hat{\theta}, \hat{\Phi})$.

Conversely, let $\hat{\mathrm{g}} \in \operatorname{ext}\mathscr{G}(\hat{\theta}, \hat{\Phi})$ and $\mathrm{g} = \frac{1}{2}(\mathrm{g}_1 + \mathrm{g}_2)$ with $\mathrm{g}_1, \mathrm{g}_2 \in \mathscr{G}(\theta, \Phi)$. Then $\hat{\mathrm{g}} = \frac{1}{2}(\hat{\mathrm{g}}_1 + \hat{\mathrm{g}}_2)$, hence $\hat{\mathrm{g}}_1 = \hat{\mathrm{g}}_2 = \hat{\mathrm{g}} \in \mathscr{G}_L(\hat{\theta}, \hat{\Phi})$, which implies by Theorem 2.8 that $\mathrm{g}_1 = p_X^*\hat{\mathrm{g}}_1 = p_X^*\hat{\mathrm{g}}_2 = \mathrm{g}_2$. Thus, $\mathrm{g} \in \operatorname{ext}\mathscr{G}(\theta, \Phi)$. □

2.5. Criteria of local finiteness and simplicity of the Gibbs cluster process

Let us give conditions sufficient for the Gibbs CPP to be (a) locally finite and (b) simple. For a given Borel set $B \in \mathcal{B}(X)$, consider a set-valued function (referred to as the *droplet cluster*)

$$D_B(\bar{y}) := \bigcup_{y_i \in \bar{y}} (B - y_i), \quad \bar{y} \in \mathfrak{X}. \tag{2.37}$$

Let us also denote by $N_B(\bar{y})$ the number of coordinates of the vector $\bar{y} = (y_i)$ that belong to the set $B \in \mathcal{B}(X)$ (cf. (2.23)),

$$N_B(\bar{y}) := \sum_{y_i \in \bar{y}} \mathbf{1}_B(y_i), \quad \bar{y} \in \mathfrak{X}. \tag{2.38}$$

In particular, $N_X(\bar{y}) < \infty$ is the "dimension" of $\bar{y}$, that is, the total number of its coordinates (recall that $\bar{y} \in \mathfrak{X} = \bigsqcup_{n=0}^{\infty} X^n$, see (2.1)). Recall from Section 2.2 that, under the distribution η, random vector $\bar{y} \in \mathfrak{X}$ has no coinciding coordinates, that is, if $\bar{y} \in X^n$ for some $n \in \mathbb{Z}_+$ then $N_X(\bar{y}) = n$ (η-a.s.).

Theorem 2.10. *Let* g_{cl} *be a Gibbs cluster measure on the generalized configuration space* $\Gamma_X^\sharp$.

(a) *Assume that the correlation function* κ_{g}^1 *of the measure* $\mathrm{g} \in \mathscr{G}(\theta, \Phi)$ *is bounded. Then, in order that* g_{cl}*-a.a. configurations* $\gamma \in \Gamma_X^\sharp$ *be locally finite, it is sufficient that for any compact set* $B \in \mathcal{B}(X)$,

$$\int_{\mathfrak{X}} \theta\big(D_B(\bar{y})\big)\, \eta(\mathrm{d}\bar{y}) < \infty. \tag{2.39}$$

(b) *In order that* g_{cl}*-a.a. configurations* $\gamma \in \Gamma_X^\sharp$ *be simple, it is sufficient that*

$$\theta\big(D_{\{x\}}(\bar{y})\big) = 0 \quad \textit{for } (\theta \otimes \eta)\textit{-a.a. } (x, \bar{y}) \in X \times \mathfrak{X}. \tag{2.40}$$

For the proof of part (a) of this theorem, we need a reformulation (stated as Proposition 2.11 below) of condition (2.39), which will also play an important role in utilizing the projection construction of the Gibbs cluster measure (see Section 3). For any Borel subset $B \in \mathcal{B}(X)$, denote

$$\mathcal{Z}_B := \mathfrak{q}^{-1}(B) \equiv \big\{z \in \mathcal{Z}\colon\ \mathfrak{q}(z) \cap B \neq \emptyset\big\} \in \mathcal{B}(\mathcal{Z}), \tag{2.41}$$

where $\mathfrak{q}(z) = \bigsqcup_{y_i \in p_{\mathfrak{X}}(z)} \{y_i + p_X(z)\}$ (see (2.18)). That is to say, the set $\mathcal{Z}_B$ consists of all points $z = (x, \bar{y}) \in \mathcal{Z}$ such that, under the "projection" $\mathfrak{q}$ onto the space X, at least one coordinate $y_i + x$ ($y_i \in \bar{y}$) belongs to the set $B \subset X$.

Proposition 2.11. *For any $B \in \mathcal{B}(X)$, condition (2.39) of Theorem 2.10(a) is necessary and sufficient in order that $\hat{\theta}(\mathcal{Z}_B) < \infty$, where $\hat{\theta} = \theta \otimes \eta$.*

Proof of Proposition 2.11. By definition (2.41), $(x, \bar{y}) \in \mathcal{Z}_B$ if and only if $x \in \bigcup_{y_i \in \bar{y}} (B - y_i) \equiv D_B(\bar{y})$ (see (2.37)). Hence,

$$\hat{\theta}(\mathcal{Z}_B) = \int_{\mathfrak{X}} \biggl(\int_X \mathbf{1}_{D_B(\bar{y})}(x)\, \theta(\mathrm{d}x) \biggr)\, \eta(\mathrm{d}\bar{y}) = \int_{\mathfrak{X}} \theta\bigl(D_B(\bar{y})\bigr)\, \eta(\mathrm{d}\bar{y}),$$

and we see that the bound $\hat{\theta}(\mathcal{Z}_B) < \infty$ is nothing else but condition (2.39). □

Proof of Theorem 2.10. (a) Let $B \subset X$ be a compact set. By Proposition 2.11, condition (2.39) is equivalent to $\hat{\theta}(\mathcal{Z}_B) < \infty$. On the other hand, by Theorem 2.6(b) we have $\kappa^1_{\hat{\mathrm{g}}}(x, \bar{y}) = \kappa^1_{\mathrm{g}}(x)$. Hence, $\kappa^1_{\hat{\mathrm{g}}}$ is bounded, and by Remark A.5 (see Appendix A) it follows that $\hat{\gamma}(\mathcal{Z}_B) < \infty$ ($\hat{\mathrm{g}}$-a.s.). Furthermore, according to the representation $\mathrm{g}_{\mathrm{cl}} = \mathfrak{q}^* \hat{\mathrm{g}}$ (see (2.27)) and formula (2.41), we have

$$\begin{aligned} \mathrm{g}_{\mathrm{cl}}\bigl\{\gamma \in \Gamma_X\colon\ \gamma(B) < \infty\bigr\} &= \hat{\mathrm{g}}\bigl\{\hat{\gamma} \in \Gamma_{\mathcal{Z}}\colon\ \mathfrak{q}(\hat{\gamma})(B) < \infty\bigr\} \\ &= \hat{\mathrm{g}}\bigl\{\hat{\gamma} \in \Gamma_{\mathcal{Z}}\colon\ \hat{\gamma}(\mathcal{Z}_B) < \infty\bigr\} = 1, \end{aligned}$$

that is, $\gamma(B) < \infty$ (g_{cl}-a.s.), which completes the proof of part (a).

(b) It suffices to prove that, for any compact set $\Lambda \subset X$, there are g_{cl}-a.s. no cross-ties between the clusters whose centers belong to Λ. Due to the formula $\mathrm{g}_{\mathrm{cl}} = \mathfrak{q}^* \hat{\mathrm{g}}$ (see (2.27)), this will follow if we show that $\hat{\mathrm{g}}(A_\Lambda) = 0$, where the set $A_\Lambda \in \mathcal{B}(\Gamma_{\mathcal{Z}})$ is defined by

$$A_\Lambda := \bigl\{\hat{\gamma} = \bigl\{(x, \bar{y}_x)\bigr\}\colon\ \exists x_1, x_2 \in p_X(\hat{\gamma}) \cap \Lambda,\ \exists y_1 \in \bar{y}_{x_1} \text{ such that} x_1 + y_1 - x_2 \in \bar{y}_{x_2}\bigr\}. \tag{2.42}$$

Applying formula (2.17), we obtain

$$\hat{\mathrm{g}}(A_\Lambda) = \int_{\Gamma_X} F(\gamma)\, \mathrm{g}(\mathrm{d}\gamma), \tag{2.43}$$

where

$$F(\gamma) := \int_{\mathfrak{X}^\gamma} \mathbf{1}_{A_\Lambda} \biggl(\bigcup_{x \in \gamma} \bigl\{(x, \bar{y}_x)\bigr\} \biggr) \bigotimes_{x \in \gamma} \eta(\mathrm{d}\bar{y}_x), \qquad \gamma \in \Gamma_X. \tag{2.44}$$

Furthermore, using the definition (2.42) of the set A_Λ and integrating out superfluous variables $\bar{y}_x$'s, formula (2.44) can be simplified to

$$F(\gamma)=\int_{\mathfrak{X}^2}\mathbf{1}_{\check{A}_\Lambda}(\gamma,\bar{y}_1,\bar{y}_2)\,\eta(\mathrm{d}\bar{y}_1)\,\eta(\mathrm{d}\bar{y}_2),\qquad \gamma\in\Gamma_X,\tag{2.45}$$

where the set $\check{A}_\Lambda\in\mathcal{B}(\Gamma_X\times\mathfrak{X}^2)$ is given by

$$\check{A}_\Lambda:=\bigl\{(\gamma,\bar{y}_1,\bar{y}_2)\colon\ \exists x_1,x_2\in\gamma\cap\Lambda,\ \exists y_1\in\bar{y}_1\text{ such that }x_1+y_1-x_2\in\bar{y}_2\bigr\}.\tag{2.46}$$

According to formulas (2.45), (2.46), $F(\gamma)\equiv F(\gamma\cap\Lambda)$ ($\gamma\in\Gamma_X$), hence by Proposition A.3 we can rewrite (2.43) in the form

$$\hat{\mathrm{g}}(A_\Lambda)=\int_{\Gamma_\Lambda}F(\gamma)\mathrm{g}_\Lambda(\mathrm{d}\gamma)=\int_{\Gamma_\Lambda}F(\xi)S_\Lambda(\xi)\,\lambda_\theta(\mathrm{d}\xi),\tag{2.47}$$

with $S_\Lambda(\cdot)\in L^1(\Gamma_\Lambda,\lambda_\theta)$. Therefore, in order to show that the right-hand side of (2.47) vanishes, it suffices to check that

$$\int_{\Gamma_\Lambda}F(\xi)\,\lambda_\theta(\mathrm{d}\xi)=0.\tag{2.48}$$

To this end, substituting here representation (2.45) and changing the order of integration, we can rewrite the integral in (2.48) as

$$\int_{\mathfrak{X}^2}\theta^{\otimes 2}\bigl(B_\Lambda(\bar{y}_1,\bar{y}_2)\bigr)\,\eta(\mathrm{d}\bar{y}_1)\,\eta(\mathrm{d}\bar{y}_2),$$

where

$$B_\Lambda(\bar{y}_1,\bar{y}_2):=\bigl\{(x_1,x_2)\in\Lambda^2\colon\ x_1+y_1=x_2+y_2\text{ for some }y_1\in\bar{y}_1,\ y_2\in\bar{y}_2\bigr\}.$$

It remains to note that

$$\begin{aligned}\theta^{\otimes 2}\bigl(B_\Lambda(\bar{y}_1,\bar{y}_2)\bigr)&=\int_\Lambda\theta\Biggl(\bigcup_{y_1\in\bar{y}_1}\bigcup_{y_2\in\bar{y}_2}\{x_1+y_1-y_2\}\Biggr)\theta(\mathrm{d}x_1)\\&\leqslant\sum_{y_1\in\bar{y}_1}\int_\Lambda\theta\Biggl(\bigcup_{y_2\in\bar{y}_2}\{x_1+y_1-y_2\}\Biggr)\theta(\mathrm{d}x_1)\\&=\sum_{y_1\in\bar{y}_1}\int_\Lambda\theta\bigl(D_{\{x_1+y_1\}}(\bar{y}_2)\bigr)\theta(\mathrm{d}x_1)=0\quad(\eta\text{-a.s.})\end{aligned}$$

by assumption (2.40). Hence, (2.48) follows and so part (b) is proved. □

Remark 2.10. In the Poisson cluster case (see [9, Theorem 2.7(a)]), condition (2.39) of Theorem 2.10(a) is necessary for the local finiteness of cluster configurations, and there may be a question as to whether this remains true in the case of a Gibbs cluster measure g_{cl}. Inspection of the proof of Theorem 2.10(a) shows that the difficulty here lies in the questionable relationship between the conditions $\hat{\theta}(\mathcal{Z}_B) < \infty$ and $\hat{\gamma}(\mathcal{Z}_B) < \infty$ ($\hat{g}$-a.s.) (which are equivalent in the Poisson cluster case). According to Remark A.5 (see Appendix A), under the hypothesis of boundedness of the first-order correlation function κ_g^1, the former implies the latter, but the converse may not always be true. Simple counter-examples can be constructed by considering translation-invariant pair interaction potentials $\Phi(\{x_1, x_2\}) = \phi_0(x_1 - x_2)$ such that $\phi_0(x) = +\infty$ on some subset $\Lambda_\infty \subset X$ with $\theta(\Lambda_\infty) = \infty$. However, if κ_g^1 is bounded away from zero and the mean number of configuration points in a set B is finite then the measure $\theta(B)$ must be finite (see Remark A.5).

Remark 2.11. Similarly to Remark 2.10, it is of interest to ask whether condition (2.40) of Theorem 2.10(b) is necessary for the simplicity of the cluster measure g_{cl} (as is the case for the Poisson cluster measure, see [9, Theorem 2.7(b)]). However, in the Gibbs cluster case this is not so. For a simple counter-example, let the in-cluster measure η be concentrated on a single-point vector $\bar{y} = (0)$, so that the droplet cluster $D_{\{x\}}(\bar{y})$ is reduced to a single-point set $\{x\}$. Here, any measure θ with atoms will not satisfy condition (2.40). On the other hand, consider a Gibbs measure g with a hard-core translation-invariant pair interaction potential $\Phi(\{x_1, x_2\}) = \phi_0(x_1 - x_2)$, where $\phi_0(x) = +\infty$ for $|x| < r_0$ and $\phi_0(x) = 0$ for $|x| \geqslant r_0$; then in each admissible configuration γ any two points are at least at a distance r_0, and in particular any such γ is simple.

Remark 2.12. As suggested by Remarks 2.10 and 2.11, it is plausible that conditions (2.39) and (2.40) of Theorem 2.10 *are* necessary for the claims (a) and (b), respectively, if the interaction potential of the underlying Gibbs measure g is finite on all finite configurations, i.e., $\Phi(\xi) < +\infty$ for all $\xi \in \Gamma_X^0$.

In conclusion of this section, let us state some criteria sufficient for conditions (2.39) and (2.40) of Theorem 2.10.

Proposition 2.12. *Either of the following conditions is sufficient for* (2.39):

(a′) *For any compact set $B \in \mathcal{B}(X)$,*

$$C_B := \sup_{x \in X} \theta(B + x) < \infty, \tag{2.49}$$

and, moreover,

$$\int_{\mathfrak{X}} N_X(\bar{y})\, \eta(\mathrm{d}\bar{y}) < \infty. \tag{2.50}$$

(a″) *There is a compact set $B_0 \in \mathcal{B}(X)$ such that $N_{B_0}(\bar{y}) = N_X(\bar{y})$ for η-a.a. $\bar{y} \in \mathfrak{X}$.*

Proposition 2.13. *Either of the following conditions is sufficient for* (2.40):

(b′) *For each $x \in X$, $\theta(\{x\}) = 0$.*
(b″) *For each $x \in X$, $N_{\{x\}}(\bar{y}) = 0$ for η-a.a. $\bar{y} \in \mathfrak{X}$.*

3. Quasi-invariance and integration by parts

From now on, we restrict ourselves to the case $X=\mathbb{R}^d$. We assume throughout that the correlation function $\kappa_{\mathrm{g}}^1(x)$ is bounded, which implies by Theorem 2.10 that the same is true for the correlation function $\kappa_{\hat{\mathrm{g}}}^1(z)$. Let us also impose conditions (2.49) and (2.50) which, by Proposition 2.12, ensure that condition (2.39) of Theorem 2.10(a) is fulfilled and so g_{cl}-a.a. configurations $\gamma\in\Gamma_X^\sharp$ are locally finite. According to Proposition 2.11, condition (2.39) also implies that $\hat{\theta}(\mathcal{Z}_B)<\infty$ whenever $\theta(B)<\infty$, where the set $\mathcal{Z}_B\subset\mathcal{Z}$ is defined in (2.41).

Finally, we require the probability measure η on $\mathfrak{X}$ to be absolutely continuous with respect to the Lebesgue measure $\mathrm{d}\bar{y}$,

$$\eta(\mathrm{d}\bar{y})=h(\bar{y})\mathrm{d}\bar{y},\qquad \bar{y}=(y_1,\dots,y_n)\in X^n\quad (n\in\mathbb{Z}_+).\tag{3.1}$$

By Proposition 2.13(b″), this implies that Gibbs CPP configurations γ are g_{cl}-a.s. simple (i.e., have no multiple points). Altogether, the above assumptions ensure that g_{cl}-a.a. configurations γ belong to the proper configuration space Γ_X.

Our aim in this section is to prove the quasi-invariance of the measure g_{cl} with respect to compactly supported diffeomorphisms of X (Section 3.2), and to establish an integration-by-parts formula (Section 3.3). We begin in Section 3.1 with a brief description of some convenient "manifold-like" concepts and notation first introduced in [2] (see also [9, §4.1]), which furnish a suitable framework for analysis on configuration spaces.

3.1. Differentiable functions on configuration spaces

Let T_xX be the tangent space of $X=\mathbb{R}^d$ at point $x\in X$. It can be identified in the natural way with $\mathbb{R}^d$, with the corresponding (canonical) inner product denoted by a "fat" dot $\cdot$. The gradient on X is denoted by ∇. Following [2], we define the "tangent space" of the configuration space Γ_X at $\gamma\in\Gamma_X$ as the Hilbert space $T_\gamma\Gamma_X:=L^2(X\to TX;\mathrm{d}\gamma)$, or equivalently $T_\gamma\Gamma_X=\bigoplus_{x\in\gamma}T_xX$. The scalar product in $T_\gamma\Gamma_X$ is denoted by $\langle\cdot,\cdot\rangle_\gamma$, with the corresponding norm $|\cdot|_\gamma$. A vector field V over Γ_X is a map $\Gamma_X\ni\gamma\mapsto V(\gamma)=(V(\gamma)_x)_{x\in\gamma}\in T_\gamma\Gamma_X$. Thus, for vector fields V_1, V_2 over Γ_X we have

$$\big\langle V_1(\gamma),V_2(\gamma)\big\rangle_\gamma=\sum_{x\in\gamma}V_1(\gamma)_x\cdot V_2(\gamma)_x,\quad \gamma\in\Gamma_X.$$

For $\gamma\in\Gamma_X$ and $x\in\gamma$, denote by $\mathcal{O}_{\gamma,x}$ an arbitrary open neighborhood of x in X such that $\mathcal{O}_{\gamma,x}\cap\gamma=\{x\}$. For any measurable function $F:\Gamma_X\to\mathbb{R}$, define the function $F_x(\gamma,\cdot):\mathcal{O}_{\gamma,x}\to\mathbb{R}$ by $F_x(\gamma,y):=F((\gamma\setminus\{x\})\cup\{y\})$, and set

$$\nabla_xF(\gamma):=\nabla F_x(\gamma,y)|_{y=x},\quad x\in X,$$

provided that $F_x(\gamma,\cdot)$ is differentiable at x.

Recall that for a function $\phi:X\to\mathbb{R}$ its support $\operatorname{supp}\phi$ is defined as the closure of the set $\{x\in X:\ \phi(x)\neq 0\}$. Denote by $\mathcal{FC}(\Gamma_X)$ the class of functions on Γ_X of the form

$$F(\gamma)=f\big(\langle\phi_1,\gamma\rangle,\dots,\langle\phi_k,\gamma\rangle\big),\quad \gamma\in\Gamma_X,\tag{3.2}$$

where $k \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^k)$ (:= the set of C^∞-functions on $\mathbb{R}^k$ bounded together with all their derivatives), and $\phi_1, \dots, \phi_k \in C_0^\infty(X)$ (:= the set of C^∞-functions on X with compact support). Each $F \in \mathcal{FC}(\Gamma_X)$ is local, that is, there is a compact $K \subset X$ (which may depend on F) such that $F(\gamma) = F(\gamma \cap K)$ for all $\gamma \in \Gamma_X$. Thus, for a fixed γ, $\nabla_x F(\gamma) \neq 0$ for at most finitely many $x \in \gamma$.

For a function $F \in \mathcal{FC}(\Gamma_X)$ its Γ-gradient $\nabla^\Gamma F \equiv \nabla_X^\Gamma F$ is defined as

$$\nabla^\Gamma F(\gamma) := \big(\nabla_x F(\gamma)\big)_{x\in\gamma} \in T_\gamma \Gamma_X, \quad \gamma \in \Gamma_X, \tag{3.3}$$

so the directional derivative of F along a vector field V is given by

$$\nabla_V^\Gamma F(\gamma) := \big\langle \nabla^\Gamma F(\gamma), V(\gamma)\big\rangle_\gamma = \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot V(\gamma)_x, \quad \gamma \in \Gamma_X.$$

Note that the sum here contains only finitely many non-zero terms.

Further, let $\mathcal{FV}(\Gamma_X)$ be the class of cylinder vector fields V on Γ_X of the form

$$V(\gamma)_x = \sum_{i=1}^k A_i(\gamma) v_i(x) \in T_x X, \quad x \in X, \tag{3.4}$$

where $A_i \in \mathcal{FC}(\Gamma_X)$ and $v_i \in \mathrm{Vect}_0(X)$ (:= the space of compactly supported C^∞-smooth vector fields on X), $i = 1, \dots, k$ ($k \in \mathbb{N}$). Any vector field $v \in \mathrm{Vect}_0(X)$ generates a constant vector field V on Γ_X defined by $V(\gamma)_x := v(x)$. We shall preserve the notation v for it. Thus,

$$\nabla_v^\Gamma F(\gamma) = \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot v(x), \quad \gamma \in \Gamma_X. \tag{3.5}$$

The approach based on "lifting" the differential structure from the underlying space X to the configuration space Γ_X as described above can also be applied to the spaces $\mathfrak{X} = \bigsqcup_{n=0}^\infty X^n$, $\mathcal{Z} = X \times \mathfrak{X}$ and $\Gamma_{\mathfrak{X}}$, $\Gamma_{\mathcal{Z}}$. For these spaces, we will use the analogous notation as above without further explanation.

3.2. Diff_0-*quasi-invariance*

In this section, we discuss the property of quasi-invariance of the measure g_{cl} with respect to diffeomorphisms of X. Let us start by describing how diffeomorphisms of X act on configuration spaces. For a measurable map $\varphi : X \to X$, its *support* $\operatorname{supp}\varphi$ is defined as the closure of the set $\{x \in X\colon\ \varphi(x) \neq x\}$. Let $\mathrm{Diff}_0(X)$ be the group of diffeomorphisms of X with *compact support*. For any $\varphi \in \mathrm{Diff}_0(X)$, consider the corresponding "diagonal" diffeomorphism $\bar\varphi : \mathfrak{X} \to \mathfrak{X}$ acting on each constituent space X^n ($n \in \mathbb{Z}_+$) as

$$X^n \ni \bar y = (y_1, \dots, y_n) \mapsto \bar\varphi(\bar y) := \big(\varphi(y_1), \dots, \varphi(y_n)\big) \in X^n. \tag{3.6}$$

For $x \in X$, we also define "shifted" diffeomorphisms

$$\bar\varphi_x(\bar y) := \bar\varphi(\bar y + x) - x, \quad \bar y \in \mathfrak{X} \tag{3.7}$$

(see the shift notation (2.19)). Finally, we introduce a special class of diffeomorphisms $\hat\varphi$ on $\mathcal{Z}$ acting only in the $\bar y$-coordinate as follows,

$$\hat\varphi(z) := \bigl(x, \bar\varphi_x(\bar y)\bigr) \equiv \bigl(x, \bar\varphi(\bar y + x) - x\bigr), \quad z = (x, \bar y) \in \mathcal{Z}. \tag{3.8}$$

Remark 3.1. Note that, even though $K_\varphi := \operatorname{supp}\varphi$ is compact in X, the support of the diffeomorphism $\hat\varphi$ (again defined as the closure of the set $\{z \in \mathcal{Z}\colon \hat\varphi(z) \neq z\}$) is given by $\operatorname{supp}\hat\varphi = \mathcal{Z}_{K_\varphi}$ (see (2.41)) and hence is *not* compact in the topology of $\mathcal{Z}$ (see Section 2.1).

In the standard fashion, the maps φ and $\hat\varphi$ can be lifted to measurable "diagonal" transformations (denoted by the same letters) of the configuration spaces Γ_X and $\Gamma_{\mathcal{Z}}$, respectively:

$$\begin{aligned} &\Gamma_X \ni \gamma \mapsto \varphi(\gamma) := \bigl\{\varphi(x),\ x \in \gamma\bigr\} \in \Gamma_X, \\ &\Gamma_{\mathcal{Z}} \ni \hat\gamma \mapsto \hat\varphi(\hat\gamma) := \bigl\{\hat\varphi(z),\ z \in \hat\gamma\bigr\} \in \Gamma_{\mathcal{Z}}. \end{aligned} \tag{3.9}$$

The following lemma shows that the operator $\mathfrak{q}$ commutes with the action of diffeomorphisms (3.9).[6]

Lemma 3.1. *For any diffeomorphism $\varphi \in \mathrm{Diff}_0(X)$ and the corresponding diffeomorphism $\hat\varphi$, it holds*

$$\varphi \circ \mathfrak{q} = \mathfrak{q} \circ \hat\varphi. \tag{3.10}$$

Proof. Fix an arbitrary $\hat\gamma \in \Gamma_{\mathcal{Z}}$. By definition of the map $\mathfrak{q}$,

$$\mathfrak{q}(\hat\gamma) = \bigl\{y_i + x,\ y_i \in \bar y,\ (x, \bar y) \in \hat\gamma\bigr\},$$

and (3.9) implies that

$$\varphi\bigl(\mathfrak{q}(\hat\gamma)\bigr) = \bigl\{\varphi(y_i + x),\ y_i \in \bar y,\ (x, \bar y) \in \hat\gamma\bigr\}.$$

On the other hand, by (3.7) and (3.8) we have

$$\hat\varphi(\hat\gamma) = \bigl\{\bigl(x, \bar\varphi_x(\bar y)\bigr),\ (x, \bar y) \in \hat\gamma\bigr\},$$

so that, by the structure of the map $\bar\varphi_x$,

$$\begin{aligned} \mathfrak{q}\bigl(\hat\varphi(\hat\gamma)\bigr) &= \bigl\{\xi_i + x,\ \xi_i \in \bar\varphi_x(\bar y),\ (x, \bar y) \in \hat\gamma\bigr\} \\ &= \bigl\{\varphi(y_i + x),\ y_i \in \bar y,\ (x, \bar y) \in \hat\gamma\bigr\}, \end{aligned}$$

and the statement of the lemma follows. □

[6] According to relation (3.10), $\mathfrak{q}$ is an *intertwining operator* between associated diffeomorphisms φ and $\hat\varphi$.

Lemma 3.2. *The interaction potential $\hat{\Phi}$ defined in (2.29) is invariant with respect to the diffeomorphisms (3.8), that is, for any $\varphi \in \mathrm{Diff}_0(X)$ we have*

$$\hat{\Phi}\big(\hat{\varphi}(\hat{\gamma})\big) = \hat{\Phi}(\hat{\gamma}), \quad \hat{\gamma} \in \Gamma_{\mathcal{Z}}.$$

In particular, this implies the $\hat{\varphi}$-invariance of the energy functionals defined in (A.1) and (A.2), that is, for any $\hat{\xi} \in \Gamma^0_{\mathcal{Z}}$ and $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$,

$$\hat{E}\big(\hat{\varphi}(\hat{\xi})\big) = \hat{E}(\hat{\xi}), \qquad \hat{E}\big(\hat{\varphi}(\hat{\xi}), \hat{\varphi}(\hat{\gamma})\big) = \hat{E}(\hat{\xi}, \hat{\gamma}).$$

Proof. The claim readily follows by observing that the diffeomorphism (3.8) acts on the $\bar{y}$-coordinates of points $z = (x, \bar{y})$ in a configuration $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$, while the interaction potential $\hat{\Phi}$ (see (2.29)) only depends on their x-coordinates. □

As already mentioned (see (3.1)), we suppose that the measure η is absolutely continuous with respect to Lebesgue measure $\mathrm{d}\bar{y}$ on $\mathfrak{X}$; moreover, in the rest of Section 3.2 it will be assumed that the corresponding density h is such that for any $n \in \mathbb{Z}_+$ the following dichotomy holds: either $h(\bar{y}) > 0$ (for a.a. $\bar{y} \in X^n$) or $h(\bar{y}) = 0$ (for a.a. $\bar{y} \in X^n$). This implies that the measure η is quasi-invariant with respect to the action of transformations $\bar{\varphi} : \mathfrak{X} \to \mathfrak{X}$ ($\varphi \in \mathrm{Diff}_0(X)$), that is, for any $f \in \mathrm{M}_+(\mathfrak{X})$,

$$\int_{\mathfrak{X}} f(\bar{y}) \bar{\varphi}^* \eta(\mathrm{d}\bar{y}) = \int_{\mathfrak{X}} f(\bar{y}) \rho_\eta^{\bar{\varphi}}(\bar{y}) \, \mathrm{d}\bar{y}, \tag{3.11}$$

with the Radon–Nikodym density

$$\rho_\eta^{\bar{\varphi}}(\bar{y}) := \frac{\mathrm{d}(\bar{\varphi}^*\eta)}{\mathrm{d}\eta}(\bar{y}) = \frac{h(\bar{\varphi}^{-1}(\bar{y}))}{h(\bar{y})} J_{\bar{\varphi}}(\bar{y})^{-1} \tag{3.12}$$

(we set $\rho_\eta^{\bar{\varphi}}(\bar{y}) = 1$ if $h(\bar{y}) = 0$ or $h(\bar{\varphi}^{-1}(\bar{y})) = 0$). Here $J_{\bar{\varphi}}(\bar{y})$ is the Jacobian determinant of the diffeomorphism $\bar{\varphi}$; due to the diagonal structure of $\bar{\varphi}$ (see (3.6)) we have $J_{\bar{\varphi}}(\bar{y}) = \prod_{y_i \in \bar{y}} J_\varphi(y_i)$, where $J_\varphi(y)$ is the Jacobian determinant of φ.

Due to the "shift" form of diffeomorphisms (3.8), formulas (3.11), (3.12) readily imply that the product measure $\hat{\theta}(\mathrm{d}z) = \theta(\mathrm{d}x) \otimes \eta(\mathrm{d}\bar{y})$ on $\mathcal{Z} = X \times \mathfrak{X}$ is quasi-invariant with respect to $\hat{\varphi}$, that is, for each $\varphi \in \mathrm{Diff}_0(X)$ and any $f \in \mathrm{M}_+(\mathcal{Z})$,

$$\int_{\mathcal{Z}} f(z) \, \hat{\varphi}^* \hat{\theta}(\mathrm{d}z) = \int_{\mathcal{Z}} f(z) \rho_\varphi(z) \, \hat{\theta}(\mathrm{d}z), \tag{3.13}$$

where the Radon–Nikodym density $\rho_\varphi := \mathrm{d}(\hat{\varphi}^*\hat{\theta})/\mathrm{d}\hat{\theta}$ is given by (see (3.12))

$$\rho_\varphi(z) = \rho_\eta^{\bar{\varphi}_x}(\bar{y}) \equiv \frac{h(\bar{\varphi}^{-1}(\bar{y} + x) - x)}{h(\bar{y})} J_{\bar{\varphi}}(\bar{y} + x)^{-1}, \quad z = (x, \bar{y}) \in \mathcal{Z}. \tag{3.14}$$

We can now state our result on the quasi-invariance of the measure $\hat{\mathrm{g}}$.

Theorem 3.3. *The Gibbs measure* $\hat{\mathrm{g}}$ *constructed in Section 2.4 is quasi-invariant with respect to the action of diagonal diffeomorphisms* $\hat{\varphi}$ *on* $\Gamma_{\mathcal{Z}}$ $(\varphi \in \mathrm{Diff}_0(X))$ *defined by formula (3.8), with the Radon–Nikodym density* $R_{\hat{\mathrm{g}}}^{\hat{\varphi}} = \mathrm{d}(\hat{\varphi}^*\hat{\mathrm{g}})/\mathrm{d}\hat{\mathrm{g}}$ *given by*

$$R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\gamma}) = \prod_{z\in\hat{\gamma}} \rho_\varphi(z), \quad \hat{\gamma} \in \Gamma_{\mathcal{Z}}, \tag{3.15}$$

where $\rho_\varphi(z)$ *is defined in (3.14). Moreover,* $R_{\hat{\mathrm{g}}}^{\hat{\varphi}} \in L^1(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}})$.

Proof. First of all, note that $\rho_\varphi(z) = 1$ for any $z = (x, \bar{y}) \notin \operatorname{supp}\hat{\varphi} = \mathcal{Z}_{K_\varphi}$, where $K_\varphi = \operatorname{supp}\varphi$ (see Remark 3.1), and $\hat{\theta}(\mathcal{Z}_{K_\varphi}) < \infty$ by Proposition 2.11. On the other hand, Theorem 2.6(b) implies that the correlation function $\kappa_{\hat{\mathrm{g}}}^1$ is bounded. Therefore, by Remark A.5 (see Appendix A) we obtain that $\hat{\gamma}(\mathcal{Z}_{K_\varphi}) < \infty$ for $\hat{\mathrm{g}}$-a.a. configurations $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$, hence the product in (3.15) contains finitely many terms different from 1 and so the function $R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\gamma})$ is well defined. Moreover, it satisfies the "localization" equality

$$R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\gamma}) = R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\gamma} \cap \mathcal{Z}_{K_\varphi}) \quad \text{for } \hat{\mathrm{g}}\text{-a.a. } \hat{\gamma} \in \Gamma_{\mathcal{Z}}. \tag{3.16}$$

Following [25, §2.8, Theorem 2.8.2], the proof of the theorem will be based on the use of Ruelle's equation (see Appendix A, Theorem A.1). Namely, according to (A.4) with $\Lambda = \mathcal{Z}_{K_\varphi}$, for any function $F \in \mathrm{M}_+(\Gamma_{\mathcal{Z}})$ we have

$$\begin{aligned}
\int_{\Gamma_{\mathcal{Z}}} F(\hat{\gamma})\, \hat{\varphi}^*\hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) &= \int_{\Gamma_{\mathcal{Z}}} F\big(\hat{\varphi}(\hat{\gamma})\big)\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_\Lambda} \bigg(\int_{\Gamma_{\mathcal{Z}\setminus\Lambda}} F\big(\hat{\varphi}\big(\hat{\xi} \cup \hat{\gamma}'\big)\big)\, \mathrm{e}^{-\hat{E}(\hat{\varphi}(\hat{\xi})) - \hat{E}(\hat{\varphi}(\hat{\xi}), \hat{\varphi}(\hat{\gamma}'))}\, \hat{\mathrm{g}}\big(\mathrm{d}\hat{\gamma}'\big) \bigg) \lambda_{\hat{\theta}}(\mathrm{d}\hat{\xi}) \\
&= \int_{\Gamma_\Lambda} \bigg(\int_{\Gamma_{\mathcal{Z}\setminus\Lambda}} F\big(\hat{\varphi}(\hat{\xi}) \cup \hat{\gamma}'\big)\, \mathrm{e}^{-\hat{E}(\hat{\varphi}(\hat{\xi})) - \hat{E}(\hat{\varphi}(\hat{\xi}), \hat{\gamma}')}\, \hat{\mathrm{g}}\big(\mathrm{d}\hat{\gamma}'\big) \bigg) \lambda_{\hat{\theta}}(\mathrm{d}\hat{\xi}) \\
&= \int_{\Gamma_\Lambda} \bigg(\int_{\Gamma_{\mathcal{Z}\setminus\Lambda}} F\big(\hat{\xi} \cup \hat{\gamma}'\big)\, \mathrm{e}^{-\hat{E}(\hat{\xi}) - \hat{E}(\hat{\xi}, \hat{\gamma}')}\, \hat{\mathrm{g}}\big(\mathrm{d}\hat{\gamma}'\big) \bigg) \hat{\varphi}^* \lambda_{\hat{\theta}}(\mathrm{d}\hat{\xi}),
\end{aligned} \tag{3.17}$$

where $\lambda_{\hat{\theta}}$ is the Lebesgue–Poisson measure corresponding to the reference measure $\hat{\theta}$ (see (A.5)). Since $\hat{\theta}$ is quasi-invariant with respect to diffeomorphisms $\hat{\varphi}$ (see (3.13)), it readily follows from definition (A.5) that the restriction of the Lebesgue–Poisson measure $\lambda_{\hat{\theta}}$ onto the set Γ_Λ is quasi-invariant with respect to $\hat{\varphi}$, with the density given precisely by expression (3.15). Hence, using property (3.16), the right-hand side of (3.17) is reduced to

$$\int_{\Gamma_\Lambda} \bigg(\int_{\Gamma_{\mathcal{Z}\setminus\Lambda}} F\big(\hat{\xi} \cup \hat{\gamma}'\big)\, \mathrm{e}^{-\hat{E}(\hat{\xi}) - \hat{E}(\hat{\xi}, \hat{\gamma}')}\, \hat{\mathrm{g}}\big(\mathrm{d}\gamma'\big) \bigg) R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\xi})\, \lambda_{\hat{\theta}}(\mathrm{d}\hat{\xi})$$

$$= \int_{\Gamma_\Lambda} \left(\int_{\Gamma_{\mathcal{Z}\setminus\Lambda}} F(\hat{\xi} \cup \hat{\gamma}') R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\xi} \cup \hat{\gamma}')\, \mathrm{e}^{-E(\hat{\xi})-E(\hat{\xi},\hat{\gamma}')}\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}') \right) \lambda_{\hat{\theta}}(\mathrm{d}\hat{\xi})$$

$$= \int_{\Gamma_{\mathcal{Z}}} F(\hat{\gamma}) R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\gamma})\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}), \tag{3.18}$$

where we have again used Ruelle's equation (A.4).

As a result, combining (3.17) and (3.18) we obtain

$$\int_{\Gamma_{\mathcal{Z}}} F(\hat{\gamma})\, \hat{\varphi}^* \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_{\mathcal{Z}}} F(\hat{\gamma}) R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\gamma})\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}), \tag{3.19}$$

which proves the quasi-invariance of $\hat{\mathrm{g}}$. In particular, setting $F \equiv 1$ in Eq. (3.19) yields $\int_{\Gamma_{\mathcal{Z}}} R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\gamma})\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) = 1$, and hence $R_{\hat{\mathrm{g}}}^{\hat{\varphi}} \in L^1(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}})$, as claimed. □

Remark 3.2. Note that the Radon–Nikodym density $R_{\hat{\mathrm{g}}}^{\hat{\varphi}}$ given by (3.15) does not depend on the background interaction potential Φ. As should be evident from the proof above, this is due to (i) the special form of diffeomorphisms $\hat{\varphi}$ on $\mathcal{Z}$ acting only in variable $\bar{y} \in \mathfrak{X}$ (see (3.8)), and (ii) the cylinder structure of the interaction potential $\hat{\Phi}$ that depends only on variable $x \in X$ (see (2.29)). In particular, expression (3.15) applies as well to the "interaction-free" case with $\Phi \equiv 0$ (and hence $\hat{\Phi} \equiv 0$), where the Gibbs measure $\mathrm{g} \in \mathscr{G}(\theta, \Phi = 0)$ is reduced to the Poisson measure π_θ on Γ_X with intensity measure θ (see Appendix A), while the Gibbs measure $\hat{\mathrm{g}} \in \mathscr{G}(\hat{\theta}, \hat{\Phi} = 0)$ amounts to the Poisson measure $\pi_{\hat{\theta}}$ on $\Gamma_{\mathcal{Z}}$ with intensity measure $\hat{\theta}$.

Remark 3.3. As is essentially well known (see, e.g., [2,37]), quasi-invariance of a Poisson measure on the configuration space follows directly from the quasi-invariance of its intensity measure. For a proof adapted to our slightly more general setting (where diffeomorphisms are only assumed to have the support of finite measure), we refer the reader to [9, Proposition A.1]. Incidentally, the expression for the Radon–Nikodym derivative given in [9] (see also [2, Proposition 2.2]) contained a superfluous normalizing constant, which in our context would read

$$C_\varphi := \exp\left(\int_{\mathcal{Z}} \left(1 - \rho_\varphi(z)\right) \hat{\theta}(\mathrm{d}z) \right)$$

(cf. (3.15)). In fact, it is easy to see that $C_\varphi = 1$; indeed, $\rho_\varphi = 1$ outside the set $\operatorname{supp} \hat{\varphi} = \mathcal{Z}_{K_\varphi}$ with $\hat{\theta}(\mathcal{Z}_{K_\varphi}) < \infty$ (see Proposition 2.11), hence

$$\ln C_\varphi = \int_{\mathcal{Z}_{K_\varphi}} \left(1 - \rho_\varphi(z)\right) \hat{\theta}(\mathrm{d}z) = \hat{\theta}(\mathcal{Z}_{K_\varphi}) - \hat{\theta}\left(\hat{\varphi}^{-1}(\mathcal{Z}_{K_\varphi})\right) = 0.$$

Let $\mathcal{I}_{\mathfrak{q}} : L^\infty(\Gamma_X, \mathrm{g}_{\mathrm{cl}}) \to L^\infty(\Gamma_Z, \hat{\mathrm{g}})$ be the isometry defined by the map $\mathfrak{q}$ (see (2.20)),

$$(\mathcal{I}_{\mathfrak{q}} F)(\hat{\gamma}) := F \circ \mathfrak{q}(\hat{\gamma}), \quad \hat{\gamma} \in \Gamma_Z. \tag{3.20}$$

The adjoint operator $\mathcal{I}_{\mathfrak{q}}^*$ is a bounded operator in the corresponding dual spaces,

$$\mathcal{I}_{\mathfrak{q}}^*: L^\infty(\Gamma_Z, \hat{g})' \to L^\infty(\Gamma_X, g_{cl})'. \tag{3.21}$$

Lemma 3.4. *The operator $\mathcal{I}_{\mathfrak{q}}^*$ defined by (3.21) can be restricted, for any $s \geqslant 1$, to the bounded operator*

$$\mathcal{I}_{\mathfrak{q}}^*: L^s(\Gamma_Z, \hat{g}) \to L^s(\Gamma_X, g_{cl}). \tag{3.22}$$

Proof. 1) Let us first consider the case $s = 1$. It is known (see [29]) that, for any σ-finite measure space (M, μ), the corresponding space $L^1(M, \mu)$ can be identified with the subspace V of the dual space $L^\infty(M, \mu)'$ consisting of all linear functionals on $L^\infty(M, \mu)$ continuous with respect to the bounded convergence in $L^\infty(M, \mu)$. That is, $\ell \in V$ if and only if $\ell(\psi_n) \to 0$ for any $\psi_n \in L^\infty(M, \mu)$ such that $|\psi_n| \leqslant 1$ and $\psi_n(x) \to 0$ as $n \to \infty$ for μ-a.a. $x \in M$. Hence, to prove the lemma it suffices to show that, for any $F \in L^1(\Gamma_{\mathcal{Z}}, \hat{g})$, the functional $\mathcal{I}_{\mathfrak{q}}^* F \in L^\infty(\Gamma_{\mathcal{Z}}, \hat{g})'$ is continuous with respect to the bounded convergence in $L^\infty(\Gamma_{\mathcal{Z}}, \hat{g})$. To this end, for any sequence (ψ_n) in $L^\infty(\Gamma_X, g_{cl})$ such that $|\psi_n| \leqslant 1$ and $\psi_n(\gamma) \to 0$ for g_{cl}-a.a. $\gamma \in \Gamma_X$, we have to prove that $\mathcal{I}_{\mathfrak{q}}^* F(\psi_n) \to 0$.

Let us first show that $\mathcal{I}_{\mathfrak{q}} \psi_n(\hat{\gamma}) \equiv \psi_n(\mathfrak{q}(\hat{\gamma})) \to 0$ for $\hat{g}$-a.a. $\hat{\gamma} \in \Gamma_{\mathcal{Z}}$. Set

$$A_\psi := \bigl\{\gamma \in \Gamma_X : \psi_n(\gamma) \to 0\bigr\} \in \mathcal{B}(\Gamma_X),$$
$$\hat{A}_\psi := \bigl\{\hat{\gamma} \in \Gamma_{\mathcal{Z}} : \psi_n\bigl(\mathfrak{q}(\hat{\gamma})\bigr) \to 0\bigr\} \in \mathcal{B}(\Gamma_{\mathcal{Z}}),$$

and note that $\hat{A}_\psi = \mathfrak{q}^{-1}(A_\psi)$; then, recalling relation (2.27), we get

$$\hat{g}(\hat{A}_\psi) = \hat{g}\bigl(\mathfrak{q}^{-1}(A_\psi)\bigr) = g_{cl}(A_\psi) = 1,$$

as claimed. Now, by the dominated convergence theorem this implies

$$\mathcal{I}_{\mathfrak{q}}^* F(\psi_n) = \int_{\Gamma_{\mathcal{Z}}} F(\hat{\gamma}) \mathcal{I}_{\mathfrak{q}} \psi_n(\hat{\gamma})\, \hat{g}(\mathrm{d}\hat{\gamma}) \to 0,$$

and the proof is complete.

2) The case $s > 1$ is in fact much simpler. It is known (see [29]) that, for any σ-finite measure space (M, μ), the dual space $L^s(M, \mu)'$ can be identified with $L^r(M, \mu)$, where $r^{-1} + s^{-1} = 1$. Hence, formula (3.22) follows from the obvious fact that the operator $\mathcal{I}_{\mathfrak{q}}$ defined by (3.20) can also be viewed as an isometry from $L^r(\Gamma_X, g_{cl})$ into $L^r(\Gamma_Z, \hat{g})$. □

Taking advantage of Theorem 3.3 and applying the projection construction, we obtain our main result in this section.

Theorem 3.5. *The Gibbs cluster measure g_{cl} is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$ on Γ_X. The corresponding Radon–Nikodym density is given by $R_{g_{cl}}^{\varphi} = \mathcal{I}_{\mathfrak{q}}^* R_{\hat{g}}^{\hat{\varphi}}$.*

Proof. Note that, due to (2.27) and (3.10),

$$\mathrm{g}_{\mathrm{cl}} \circ \varphi^{-1} = \hat{\mathrm{g}} \circ \mathfrak{q}^{-1} \circ \varphi^{-1} = \hat{\mathrm{g}} \circ \hat{\varphi}^{-1} \circ \mathfrak{q}^{-1}.$$

That is, $\varphi^*\mathrm{g}_{\mathrm{cl}} = \mathrm{g}_{\mathrm{cl}} \circ \varphi^{-1}$ is a push-forward of the measure $\hat{\varphi}^*\hat{\mathrm{g}} = \hat{\mathrm{g}} \circ \hat{\varphi}^{-1}$ under the map $\mathfrak{q}$, that is, $\varphi^*\mathrm{g}_{\mathrm{cl}} = \mathfrak{q}^*\hat{\varphi}^*\hat{\mathrm{g}}$. In particular, if $\hat{\varphi}^*\hat{\mathrm{g}}$ is absolutely continuous with respect to $\hat{\mathrm{g}}$ then so is $\varphi^*\mathrm{g}_{\mathrm{cl}}$ with respect to g_{cl}. Moreover, by the change of measure (2.27) and Theorem 3.3, for any $F \in L^\infty(\Gamma_X, \mathrm{g}_{\mathrm{cl}})$ we have

$$\int_{\Gamma_X} F(\gamma)\, \varphi^*\mathrm{g}_{\mathrm{cl}}(\mathrm{d}\gamma) = \int_{\Gamma_{\mathcal{Z}}} \mathcal{I}_{\mathfrak{q}} F(\hat{\gamma})\, \hat{\varphi}^*\hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_{\mathcal{Z}}} \mathcal{I}_{\mathfrak{q}} F(\hat{\gamma}) R_{\hat{\mathrm{g}}}^{\hat{\varphi}}(\hat{\gamma})\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}). \tag{3.23}$$

By Lemma 3.4, the operator $\mathcal{I}_{\mathfrak{q}}^*$ acts from $L^1(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}})$ to $L^1(\Gamma_X, \mathrm{g}_{\mathrm{cl}})$. Therefore, the right-hand side of (3.23) can be rewritten as

$$\int_{\Gamma_X} F(\gamma) \big(\mathcal{I}_{\mathfrak{q}}^* R_{\hat{\mathrm{g}}}^{\hat{\varphi}}\big)(\gamma)\, \mathrm{g}_{\mathrm{cl}}(\mathrm{d}\gamma),$$

which completes the proof. □

Remark 3.4. The Gibbs cluster measure g_{cl} on the configuration space Γ_X can be used to construct a unitary representation U of the diffeomorphism group $\mathrm{Diff}_0(X)$ by operators in $L^2(\Gamma_X, \mathrm{g}_{\mathrm{cl}})$, given by the formula

$$U_\varphi F(\gamma) = \sqrt{R_{\mathrm{g}_{\mathrm{cl}}}^{\varphi}(\gamma)}\, F\big(\varphi^{-1}(\gamma)\big), \quad F \in L^2(\Gamma_X, \mathrm{g}_{\mathrm{cl}}). \tag{3.24}$$

Such representations, which can be defined for arbitrary quasi-invariant measures on Γ_X, play a significant role in the representation theory of the group $\mathrm{Diff}_0(X)$ [21,38] and quantum field theory [18,19]. An important question is whether representation (3.24) is irreducible. According to [38], this is equivalent to the $\mathrm{Diff}_0(X)$-ergodicity of the measure g_{cl}, which in our case is equivalent to the ergodicity of the measure $\hat{\mathrm{g}}$ with respect to the group of transformations $\hat{\varphi}$ ($\varphi \in \mathrm{Diff}_0(X)$). Adapting the technique developed in [27], it can be shown that the aforementioned ergodicity of $\hat{\mathrm{g}}$ is valid if and only if $\hat{\mathrm{g}} \in \mathrm{ext}\,\mathscr{G}(\hat{\theta}, \hat{\Phi})$. In turn, the latter is equivalent to $\mathrm{g} \in \mathrm{ext}\,\mathscr{G}(\theta, \Phi)$, provided that $\mathrm{g} \in \mathscr{G}_L(\theta, \Phi)$ (see Corollary 2.9).

3.3. Integration-by-parts formula

Let us first state simple sufficient conditions for our measures on configuration spaces to belong to the corresponding moment classes $\mathcal{M}^n$ (see Appendix A, formula (A.10)).

Lemma 3.6. (a) *Let $\mathrm{g} \in \mathscr{G}(\theta, \Phi)$, and suppose that the correlation functions κ_{g}^m are bounded for all $m = 1, \dots, n$. Then $\hat{\mathrm{g}} \in \mathcal{M}^n(\Gamma_{\mathcal{Z}})$, that is,*

$$\int_{\Gamma_{\mathcal{Z}}} \big|\langle f, \hat{\gamma}\rangle\big|^n\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) < \infty, \quad f \in C_0(\mathcal{Z}). \tag{3.25}$$

Moreover, bound (3.25) is valid for any function $f \in \bigcap_{m=1}^n L^m(\mathcal{Z}, \hat{\theta})$.

(b) *If, in addition, the total number of components of a random vector $\bar{y} \in \mathfrak{X}$ has a finite n-th moment*[7] *with respect to the measure η,*

$$\int_{\mathfrak{X}} N_X(\bar{y})^n \,\eta(\mathrm{d}\bar{y}) < \infty, \tag{3.26}$$

then $\mathrm{g}_{\mathrm{cl}} \in \mathcal{M}^n(\Gamma_X)$.

Proof of Lemma 3.6 is deferred to Appendix B. In the rest of this section, we shall assume that the conditions of this lemma are satisfied with $n = 1$. Thus, the measures g, $\hat{\mathrm{g}}$ and g_{cl} belong to the corresponding $\mathcal{M}^1$-classes.

Let $v \in \mathrm{Vect}_0(X)$ (:= the space of compactly supported smooth vector fields on X), and define a vector field $\hat{v}_x$ on $\mathfrak{X}$ by the formula

$$\hat{v}_x(\bar{y}) := \bigl(v(y_1 + x), v(y_2 + x), \ldots\bigr), \qquad \bar{y} = (y_1, y_2, \ldots) \in \mathfrak{X}. \tag{3.27}$$

Observe that if the density $h(\bar{y})$ is differentiable ($\mathrm{d}\bar{y}$-a.e.) then the measure η satisfies the following integration-by-parts formula (see, e.g., [5, §1.3], [6, §5.1.3, p. 207]):

$$\int_{\mathfrak{X}} \nabla^{\hat{v}_x} f(\bar{y})\,\eta(\mathrm{d}\bar{y}) = -\int_{\mathfrak{X}} f(\bar{y})\beta_\eta^{\hat{v}}(x, \bar{y})\,\eta(\mathrm{d}\bar{y}), \qquad f \in C_0^\infty(\mathfrak{X}), \tag{3.28}$$

where $\nabla^{\hat{v}_x}$ is the derivative along the vector field $\hat{v}_x$ and

$$\beta_\eta^{\hat{v}}(x, \bar{y}) := \bigl(\beta_\eta(\bar{y}), \hat{v}_x(\bar{y})\bigr)_{T_{\bar{y}}\mathfrak{X}} + \operatorname{div} \hat{v}_x(\bar{y}) \tag{3.29}$$

is the logarithmic derivative of $\eta(\mathrm{d}\bar{y}) = h(\bar{y})\mathrm{d}\bar{y}$ along $\hat{v}_x$, expressed in terms of the vector logarithmic derivative

$$\beta_\eta(\bar{y}) := \frac{\nabla h(\bar{y})}{h(\bar{y})}, \qquad \bar{y} \in \mathfrak{X}. \tag{3.30}$$

Denote for brevity

$$\|\bar{y}\|_1 := \sum_{y_i \in \bar{y}} |y_i|, \qquad \bar{y} \in \mathfrak{X}, \tag{3.31}$$

and define the space $H^{1,n}(\mathfrak{X})$ ($n \geqslant 1$) as the set of functions $f \in L^n(\mathfrak{X}, \mathrm{d}\bar{y})$ satisfying the condition

$$\int_{\mathfrak{X}} \bigl\|\nabla f(\bar{y})\bigr\|_1^n \,\mathrm{d}\bar{y} \equiv \int_{\mathfrak{X}} \biggl(\sum_{y_i \in \bar{y}} \bigl|\nabla_{y_i} f(\bar{y})\bigr|\biggr)^n \mathrm{d}\bar{y} < \infty. \tag{3.32}$$

[7] Cf. our standard assumption (2.50), where $n = 1$.

Note that $H^{1,n}(\mathfrak{X})$ is a linear space, due to the elementary inequality $(|a|+|b|)^n \leqslant 2^{n-1}(|a|^n + |b|^n)$.

Lemma 3.7. *Assume that $h^{1/n} \in H^{1,n}(\mathfrak{X})$ for some $n \geqslant 1$, and let condition (3.26) hold. Then $\beta_\eta^{\hat{v}} \in L^m(\mathcal{Z}, \hat{\theta})$ for any m such that $1 \leqslant m \leqslant n$.*

Proof. Observe that

$$\begin{aligned}\int_{\mathfrak{X}} \|\beta_\eta(\bar{y})\|_1^n \,\eta(\mathrm{d}\bar{y}) &= \int_{\mathfrak{X}} \bigl(h(\bar{y})^{-1}\|\nabla h(\bar{y})\|_1\bigr)^n h(\bar{y})\,\mathrm{d}\bar{y} \\ &= \int_{\mathfrak{X}} \bigl\|h(\bar{y})^{1/n-1}\nabla h(\bar{y})\bigr\|_1^n \,\mathrm{d}\bar{y} \\ &= n^n \int_{\mathfrak{X}} \bigl\|\nabla\bigl(h(\bar{y})^{1/n}\bigr)\bigr\|_1^n \,\mathrm{d}\bar{y} < \infty, \end{aligned} \tag{3.33}$$

according to the hypothesis $h^{1/n} \in H^{1,n}(\mathfrak{X})$ (see (3.32)). That is, $\|\beta_\eta(\bar{y})\|_1 \in L^n(\mathfrak{X}, \eta)$, which implies (by Lyapunov's inequality) that $\|\beta_\eta(\bar{y})\|_1 \in L^m(\mathfrak{X}, \eta)$ for all $1 \leqslant m \leqslant n$.

Now, to show that $\beta_\eta^{\hat{v}} \in L^m(\mathcal{Z}, \hat{\theta})$ $(1 \leqslant m \leqslant n)$, it suffices to check that each of the two terms on the right-hand side of (3.29) belongs to $L^m(\mathcal{Z}, \hat{\theta})$. To this end, denote $b_v := \sup_{x\in X}|v(x)| < \infty$, $K_v := \operatorname{supp} v \subset X$, and recall that $C_{K_v} := \sup_{y\in X}\theta(K_v - y) < \infty$ by condition (2.49). Then, on using (3.27), we have

$$\begin{aligned}&\int_{\mathcal{Z}} \bigl|\bigl(\beta_\eta(\bar{y}), \hat{v}_x(\bar{y})\bigr)\bigr|^m \,\hat{\theta}(\mathrm{d}z) \\ &\quad\leqslant \int_{\mathcal{Z}} \Bigl(\sum_{y_i\in\bar{y}} \bigl|\beta_\eta(\bar{y})_i\bigr|\cdot\bigl|v(y_i+x)\bigr|\Bigr)^m \theta(\mathrm{d}x)\,\eta(\mathrm{d}\bar{y}) \\ &\quad\leqslant (b_v)^{m-1}\int_{\mathfrak{X}} \Bigl(\sum_{y_i\in\bar{y}}\bigl|\beta_\eta(\bar{y})_i\bigr|\Bigr)^{m-1} \sum_{y_i\in\bar{y}}\bigl|\beta_\eta(\bar{y})_i\bigr|\Bigl(\int_X \bigl|v(y_i+x)\bigr|\,\theta(\mathrm{d}x)\Bigr)\,\eta(\mathrm{d}\bar{y}) \\ &\quad\leqslant (b_v)^m \int_{\mathfrak{X}} \|\beta_\eta(\bar{y})\|_1^{m-1} \sum_{y_i\in\bar{y}}\bigl|\beta_\eta(\bar{y})_i\bigr|\theta(K_v - y_i)\,\eta(\mathrm{d}\bar{y}) \\ &\quad\leqslant (b_v)^m C_{K_v} \int_{\mathfrak{X}} \|\beta_\eta(\bar{y})\|_1^m \,\eta(\mathrm{d}\bar{y}) < \infty, \end{aligned} \tag{3.34}$$

due to (3.33). Similarly, denoting $d_v := \sup_{x\in X}|\operatorname{div} v(x)| < \infty$, we obtain

$$\int_{\mathcal{Z}} \bigl|\operatorname{div}\hat{v}_x(\bar{y})\bigr|^m \,\hat{\theta}(\mathrm{d}x\times\mathrm{d}\bar{y}) = \int_{\mathcal{Z}} \Bigl(\sum_{y_i\in\bar{y}}\bigl|(\operatorname{div} v)(y_i+x)\bigr|\Bigr)^m \theta(\mathrm{d}x)\,\eta(\mathrm{d}\bar{y})$$

$$\leqslant (d_v)^{m-1} \int_{\mathfrak{X}} N_X(\bar{y})^{m-1} \biggl(\sum_{y_i \in \bar{y}} \int_X \bigl| (\operatorname{div} v)(y_i + x) \bigr| \, \theta(\mathrm{d}x) \biggr) \, \eta(\mathrm{d}\bar{y})$$

$$\leqslant (d_v)^m \int_{\mathfrak{X}} N_X(\bar{y})^{m-1} \sum_{y_i \in \bar{y}} \theta(K_v - y_i) \, \eta(\mathrm{d}\bar{y})$$

$$\leqslant (d_v)^m C_{K_v} \int_{\mathfrak{X}} N_X(\bar{y})^m \, \eta(\mathrm{d}\bar{y}) < \infty, \tag{3.35}$$

by assumption (3.26). As a result, combining bounds (3.34) and (3.35) we see that $\beta_\eta^{\hat{v}} \in L^m(\mathcal{Z}, \hat{\theta})$, as claimed. □

The next two theorems are our main results in this section.

Theorem 3.8. *Assume that $h \in H^{1,1}(\mathfrak{X})$. Then, for any function $F \in \mathcal{FC}(\Gamma_X)$, the Gibbs cluster measure g_{cl} satisfies the integration-by-parts formula*

$$\int_{\Gamma_X} \sum_{x \in \gamma} \nabla_x F(\gamma) \cdot v(x) \, \mathrm{g}_{\mathrm{cl}}(\mathrm{d}\gamma) = - \int_{\Gamma_X} F(\gamma) B_{\mathrm{g}_{\mathrm{cl}}}^v(\gamma) \, \mathrm{g}_{\mathrm{cl}}(\mathrm{d}\gamma), \tag{3.36}$$

where $B_{\mathrm{g}_{\mathrm{cl}}}^v(\gamma) := \mathcal{I}_{\mathfrak{q}}^* \langle \beta_\eta^{\hat{v}}, \hat{\gamma} \rangle$ and $\beta_\eta^{\hat{v}}$ is the logarithmic derivative defined in (3.29). Moreover, $B_{\mathrm{g}_{\mathrm{cl}}}^v \in L^1(\Gamma_X, \mathrm{g}_{\mathrm{cl}})$.

Proof. For any function $F \in \mathcal{FC}(\Gamma_X)$ and vector field $v \in \mathrm{Vect}_0(X)$, let us denote for brevity

$$H(x, \gamma) := \nabla_x F(\gamma) \cdot v(x), \quad x \in X, \ \gamma \in \Gamma_X. \tag{3.37}$$

Furthermore, setting $\hat{F} = \mathcal{I}_{\mathfrak{q}} F : \Gamma_{\mathcal{Z}} \to \mathbb{R}$ we introduce the notation

$$\hat{H}(z, \hat{\gamma}) := \nabla_{\bar{y}} \hat{F}(\hat{\gamma}) \cdot \hat{v}_x(\bar{y}), \quad z = (x, \bar{y}) \in \mathcal{Z}, \ \hat{\gamma} \in \Gamma_{\mathcal{Z}}. \tag{3.38}$$

From these definitions, it is clear that

$$\mathcal{I}_{\mathfrak{q}} \biggl(\sum_{x \in \gamma} H(x, \gamma) \biggr)(\hat{\gamma}) = \sum_{z \in \hat{\gamma}} \hat{H}(z, \hat{\gamma}), \quad \hat{\gamma} \in \Gamma_{\mathcal{Z}}. \tag{3.39}$$

Let us show that the Gibbs measure $\hat{\mathrm{g}}$ on $\Gamma_{\mathcal{Z}}$ satisfies the following integration-by-parts formula:

$$\int_{\Gamma_{\mathcal{Z}}} \sum_{z \in \hat{\gamma}} \hat{H}(z, \hat{\gamma}) \, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) = - \int_{\Gamma_{\mathcal{Z}}} \hat{F}(\hat{\gamma}) B_{\hat{\mathrm{g}}}^{\hat{v}}(\hat{\gamma}) \, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}), \tag{3.40}$$

where the logarithmic derivative $B_{\hat{\mathrm{g}}}^{\hat{v}}(\hat{\gamma}) := \langle \beta_\eta^{\hat{v}}, \hat{\gamma} \rangle$ belongs to $L^1(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}})$ (by Lemmas 3.6(b) and 3.7 with $n = 1$). By the change of measure (2.27) and due to relation (3.39), we have

$$\int_{\Gamma_{\mathcal{Z}}} \sum_{z\in\hat{\gamma}} \bigl|\hat{H}(z,\hat{\gamma})\bigr|\,\hat{g}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_X} \sum_{x\in\gamma} \bigl|H(x,\gamma)\bigr|\,g_{\mathrm{cl}}(\mathrm{d}\gamma)$$
$$\leqslant \sup_{(x,\gamma)} \bigl|H(x,\gamma)\bigr| \int_{\Gamma_X} \sum_{x\in\gamma} \mathbf{1}_{K_v}(x)\,g_{\mathrm{cl}}(\mathrm{d}\gamma), \tag{3.41}$$

where $K_v := \operatorname{supp} v$ is a compact set in X. Note that the right-hand side of (3.41) is finite, since the function H is bounded (see (3.37)) and, by Lemma 3.6(b), $g_{\mathrm{cl}} \in \mathcal{M}^1(\Gamma_X)$. Therefore, by Remark A.1 we can apply the GNZ equation (A.3) with the function $\hat{H}$ to obtain

$$\int_{\Gamma_{\mathcal{Z}}} \sum_{z\in\hat{\gamma}} \hat{H}(z,\hat{\gamma})\,\hat{g}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_{\mathcal{Z}}} \biggl(\int_{\mathcal{Z}} \hat{H}\bigl(z,\hat{\gamma}\cup\{z\}\bigr)\mathrm{e}^{-\hat{E}(\{z\},\hat{\gamma})}\,\hat{\theta}(\mathrm{d}z)\biggr)\,\hat{g}(\mathrm{d}\hat{\gamma}). \tag{3.42}$$

Inserting definition (3.37), using Lemma 3.2 and recalling that $\hat{\theta} = \theta \otimes \eta$ (see (2.28)), let us apply the integration-by-parts formula (3.28) for the measure η to rewrite the internal integral in (3.42) as

$$\int_X \mathrm{e}^{-E(\{x\},p_X(\hat{\gamma}))} \biggl(\int_{\mathfrak{X}} \nabla_{\bar{y}} \hat{F}\bigl(\hat{\gamma}\cup\bigl\{(x,\bar{y})\bigr\}\bigr)\cdot \hat{v}_x(\bar{y})\,\eta(\mathrm{d}\bar{y})\biggr)\,\theta(\mathrm{d}x)$$
$$= -\int_X \mathrm{e}^{-E(\{x\},p_X(\hat{\gamma}))} \biggl(\int_{\mathfrak{X}} \hat{F}\bigl(\hat{\gamma}\cup\bigl\{(x,\bar{y})\bigr\}\bigr)\beta_\eta^{\hat{v}}(x,\bar{y})\,\eta(\mathrm{d}\bar{y})\biggr)\,\theta(\mathrm{d}x)$$
$$= -\int_{\mathcal{Z}} \mathrm{e}^{-E(\{p_X(z)\},p_X(\hat{\gamma}))}\,\hat{F}\bigl(\hat{\gamma}\cup\{z\}\bigr)\beta_\eta^{\hat{v}}(z)\,\hat{\theta}(\mathrm{d}z).$$

Returning to (3.42) and again using the GNZ equation (A.3), we see that the right-hand side of (3.42) is reduced to

$$-\int_{\Gamma_{\mathcal{Z}}} \sum_{z\in\hat{\gamma}} \hat{F}(\hat{\gamma})\beta_\eta^{\hat{v}}(z)\,\hat{g}(\mathrm{d}\hat{\gamma}) = -\int_{\Gamma_{\mathcal{Z}}} \hat{F}(\hat{\gamma})B_{\hat{g}}^{\hat{v}}(\hat{\gamma})\,\hat{g}(\mathrm{d}\hat{\gamma}),$$

which proves formula (3.40).

Now, using equality (3.39), we obtain

$$\int_{\Gamma_X} \sum_{x\in\gamma} H(x,\gamma)\,g_{\mathrm{cl}}(\mathrm{d}\gamma) = \int_{\Gamma_{\mathcal{Z}}} \biggl(\sum_{(x,\bar{y})\in\hat{\gamma}} \nabla_{\bar{y}}\mathcal{I}_{\mathrm{q}}F(\hat{\gamma})\cdot\hat{v}_x(\bar{y})\biggr)\,\hat{g}(\mathrm{d}\hat{\gamma})$$
$$= -\int_{\Gamma_{\mathcal{Z}}} \mathcal{I}_{\mathrm{q}}F(\hat{\gamma})B_{\hat{g}}^{\hat{v}}(\hat{\gamma})\,\hat{g}(\mathrm{d}\hat{\gamma})$$
$$= -\int_{\Gamma_X} F(\gamma)\mathcal{I}_{\mathrm{q}}^* B_{\hat{g}}^{\hat{v}}(\gamma)\,g_{\mathrm{cl}}(\mathrm{d}\gamma),$$

where $\mathcal{I}_{\mathrm{q}}^* B_{\hat{g}}^{\hat{v}} \in L^1(\Gamma_X, g_{\mathrm{cl}})$ by Lemma 3.4. Thus, formula (3.36) is proved. □

Remark 3.5. Observe that the logarithmic derivative $B_{\hat{g}}^{\hat{v}}(\hat{\gamma}) = \langle \beta_\eta^{\hat{v}}, \hat{\gamma}\rangle$ (see (3.40)) does not depend on the interaction potential Φ, and in particular coincides with that in the case $\Phi \equiv 0$, where the Gibbs measure g is reduced to the Poisson measure π_θ. Nevertheless, the logarithmic derivative $B_{g_{cl}}^v$ does depend on Φ via the map $\mathcal{I}_q^*$.

According to Theorem 3.8, $B_{g_{cl}}^v \in L^1(\Gamma_{\mathcal{Z}}, g_{cl})$. Under some additional conditions this statement can be enhanced.

Lemma 3.9. *Assume that the hypotheses of Lemmas 3.6 and 3.7 are satisfied with some integer $n \geqslant 2$. Then $B_{g_{cl}}^v \in L^n(\Gamma_{\mathcal{Z}}, g_{cl})$.*

Proof. By Lemmas 3.6(a) and 3.7, it follows that $\langle \beta_\eta^{\hat{v}}, \hat{\gamma}\rangle \in L^n(\Gamma_{\mathcal{Z}}, \hat{g})$. By Lemma 3.4, $\mathcal{I}_q^*$ is a bounded operator from $L^n(\Gamma_{\mathcal{Z}}, \hat{g})$ to $L^n(\Gamma_X, g_{cl})$, which implies that $B_{g_{cl}}^v = \mathcal{I}_q^* \langle \beta_\eta^{\hat{v}}, \hat{\gamma}\rangle \in L^n(\Gamma_{\mathcal{Z}}, g_{cl})$. □

Formula (3.36) can be extended to more general vector fields on Γ_X. Let $\mathcal{FV}(\Gamma_X)$ be the class of vector fields V of the form $V(\gamma) = (V(\gamma)_x)_{x\in\gamma}$,

$$V(\gamma)_x = \sum_{j=1}^k G_j(\gamma) v_j(x) \in T_x X,$$

where $G_j \in \mathcal{FC}(\Gamma_X)$ and $v_j \in \mathrm{Vect}_0(X)$, $j = 1, \dots, k$ ($k \in \mathbb{N}$). For any such V we set

$$B_{g_{cl}}^V(\gamma) := \bigl(\mathcal{I}_q^* B_{\hat{g}}^{\mathcal{I}_q V}\bigr)(\gamma),$$

where $B_{\hat{g}}^{\mathcal{I}_q V}(\hat{\gamma})$ is the logarithmic derivative of $\hat{g}$ along $\mathcal{I}_q V(\hat{\gamma}) := V(q(\hat{\gamma}))$ (see [2]). Note that $\mathcal{I}_q V$ is a vector field on $\Gamma_{\mathcal{Z}}$ owing to the obvious equality

$$T_{\hat{\gamma}} \Gamma_{\mathcal{Z}} = T_{q(\hat{\gamma})} \Gamma_X.$$

Clearly,

$$B_{g_{cl}}^V(\gamma) = \sum_{j=1}^k \biggl(G_j(\gamma) B_{g_{cl}}^{v_j}(\gamma) + \sum_{x\in\gamma} \nabla_x G_j(\gamma) \cdot v_j(x) \biggr).$$

Theorem 3.10. *For any $F_1, F_2 \in \mathcal{FC}(\Gamma_X)$ and $V \in \mathcal{FV}(\Gamma_X)$, we have*

$$\int_{\Gamma_X} \sum_{x\in\gamma} \nabla_x F_1(\gamma) \cdot V(\gamma)_x F_2(\gamma)\, g_{cl}(\mathrm{d}\gamma) = -\int_{\Gamma_X} F_1(\gamma) \sum_{x\in\gamma} \nabla_x F_2(\gamma) \cdot V(\gamma)_x\, g_{cl}(\mathrm{d}\gamma)$$
$$- \int_{\Gamma_X} F_1(\gamma) F_2(\gamma) B_{g_{cl}}^V(\gamma)\, g_{cl}(\mathrm{d}\gamma).$$

Proof. The proof can be obtained by a straightforward generalization of the arguments used in the proof of Theorem 3.8. □

We define the *vector logarithmic derivative* of g_{cl} as a linear operator

$$B_{\mathrm{g}_{\mathrm{cl}}} : \mathcal{FV}(\Gamma_X) \to L^1(\Gamma_X, \mathrm{g}_{\mathrm{cl}})$$

via the formula

$$B_{\mathrm{g}_{\mathrm{cl}}} V(\gamma) := B^V_{\mathrm{g}_{\mathrm{cl}}}(\gamma).$$

This notation will be used in the next section.

4. The Dirichlet form and equilibrium stochastic dynamics

Throughout this section, we assume that the conditions of Lemma 3.6 are satisfied with $n = 2$. Thus, the measures g, $\hat{\mathrm{g}}$ and g_{cl} belong to the corresponding $\mathcal{M}^2$-classes. Our considerations will involve the Γ-gradients (see Section 3.1) on different configuration spaces, such as Γ_X, $\Gamma_{\mathfrak{X}}$ and $\Gamma_{\mathcal{Z}}$; to avoid confusion, we shall denote them by ∇^Γ_X, $\nabla^\Gamma_{\mathfrak{X}}$ and $\nabla^\Gamma_{\mathcal{Z}}$, respectively.

4.1. *The Dirichlet form associated with* g_{cl}

Let us introduce a pre-Dirichlet form $\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$ associated with the Gibbs cluster measure g_{cl}, defined on functions $F_1, F_2 \in \mathcal{FC}(\Gamma_X) \subset L^2(\Gamma_X, \mathrm{g}_{\mathrm{cl}})$ by

$$\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}(F_1, F_2) := \int_{\Gamma_X} \left\langle \nabla^\Gamma_X F_1(\gamma), \nabla^\Gamma_X F_2(\gamma) \right\rangle_\gamma \mathrm{g}_{\mathrm{cl}}(\mathrm{d}\gamma). \tag{4.1}$$

Let us also consider the operator $H_{\mathrm{g}_{\mathrm{cl}}}$ defined by

$$H_{\mathrm{g}_{\mathrm{cl}}} F := -\Delta^\Gamma F + B_{\mathrm{g}_{\mathrm{cl}}} \nabla^\Gamma_X F, \quad F \in \mathcal{FC}(\Gamma_X), \tag{4.2}$$

where $\Delta^\Gamma F(\gamma) := \sum_{x\in\gamma} \Delta_x F(\gamma)$ and Δ_x denotes the Laplacian on X acting with respect to $x \in \gamma$.

The next theorem readily follows from general theory of (pre-)Dirichlet forms associated with measures from the class $\mathcal{M}^2(\Gamma_X)$ which satisfy the integration-by-parts formula (see [3,31]).

Theorem 4.1. (a) *Pre-Dirichlet form* (4.1) *is well defined, i.e.,* $\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}(F_1, F_2) < \infty$ *for all* $F_1, F_2 \in \mathcal{FC}(\Gamma_X)$.

(b) *Expression* (4.2) *defines a symmetric operator* $H_{\mathrm{g}_{\mathrm{cl}}}$ *in* $L^2(\Gamma_X, \mathrm{g}_{\mathrm{cl}})$ *whose domain includes* $\mathcal{FC}(\Gamma_X)$.

(c) *The operator* $H_{\mathrm{g}_{\mathrm{cl}}}$ *is the generator of the pre-Dirichlet form* $\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$, *i.e.,*

$$\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}(F_1, F_2) = \int_{\Gamma_X} F_1(\gamma) H_{\mathrm{g}_{\mathrm{cl}}} F_2(\gamma)\, \mathrm{g}_{\mathrm{cl}}(\mathrm{d}\gamma), \quad F_1, F_2 \in \mathcal{FC}(\Gamma_X). \tag{4.3}$$

Formula (4.3) implies that the form $\mathcal{E}_{g_{cl}}$ is closable. We keep the notation $\mathcal{E}_{g_{cl}}$ and $H_{g_{cl}}$ for the corresponding closure and its generator, respectively; note that the latter is the Friedrichs extension of $(H_{g_{cl}}, \mathcal{FC}(\Gamma_X))$. It follows from the properties of the *carré du champ* $\sum_{x\in\gamma} \nabla_x F_1(\gamma)\cdot\nabla_x F_2(\gamma)$ that $\mathcal{E}_{g_{cl}}$ is a quasi-regular local Dirichlet form on a bigger state space $\ddot{\Gamma}_X$ consisting of all integer-valued Radon measures on X (see [31, condition (Q), p. 298, and §4.5.1]). By general theory of Dirichlet forms (see [30]), this implies the following result (cf. [2,3,9]).

Theorem 4.2. *There exists a conservative diffusion process* $\mathbf{X} = (\mathbf{X}_t, t \geqslant 0)$ *on* $\ddot{\Gamma}_X$*, properly associated with the Dirichlet form* $\mathcal{E}_{g_{cl}}$*, that is, for any function* $F \in L^2(\ddot{\Gamma}_X, g_{cl})$ *and all* $t \geqslant 0$*, the map*

$$\ddot{\Gamma}_X \ni \gamma \mapsto p_t F(\gamma) := \int_\Omega F(\mathbf{X}_t)\,\mathrm{d}P_\gamma$$

is an $\mathcal{E}_{g_{cl}}$*-quasi-continuous version of* $\exp(-tH_{g_{cl}})F$*. Here* Ω *is the canonical sample space (of* $\ddot{\Gamma}_X$*-valued continuous functions on* $\mathbb{R}_+$*) and* $(P_\gamma, \gamma \in \ddot{\Gamma}_X)$ *is the family of probability distributions of the process* $\mathbf{X}$ *conditioned on the initial value* $\gamma = \mathbf{X}_0$ (g_{cl}*-a.s.). The process* $\mathbf{X}$ *is unique up to* g_{cl}*-equivalence. In particular,* $\mathbf{X}$ *is* g_{cl}*-symmetric (i.e.,* $\int F_1 p_t F_2\,\mathrm{d}g_{cl} = \int F_2 p_t F_1\,\mathrm{d}g_{cl}$ *for all measurable functions* $F_1, F_2 : \ddot{\Gamma}_X \to \mathbb{R}_+$*) and* g_{cl} *is its invariant measure.*

4.2. Irreducibility of the Dirichlet form

In this section, we assume that any cluster contains a.s. no more than N points, where N is fixed; that is, the measure η (see Section 2.2) is supported on the reduced space $\mathfrak{X} = \bigsqcup_{k=0}^N X^k$ (cf. (2.1); for convenience and with some abuse of notation, we preserve the same symbol $\mathfrak{X}$ for this space).

Similarly to (4.1), let $\mathcal{E}_{\hat{g}}$ be the pre-Dirichlet form associated with the Gibbs measure $\hat{g}$, defined on functions $F_1, F_2 \in \mathcal{FC}(\Gamma_{\mathcal{Z}}) \subset L^2(\Gamma_{\mathcal{Z}}, \hat{g})$ by

$$\mathcal{E}_{\hat{g}}(F_1, F_2) := \int_{\Gamma_{\mathcal{Z}}} \big\langle \nabla^\Gamma_{\mathcal{Z}} F_1(\hat{\gamma}), \nabla^\Gamma_{\mathcal{Z}} F_2(\hat{\gamma})\big\rangle_{\hat{\gamma}}\,\hat{g}(\mathrm{d}\hat{\gamma}). \tag{4.4}$$

The integral in (4.4) is finite because $\hat{g} \in \mathcal{M}^2 \subset \mathcal{M}^1$. This also implies that the Γ-gradient in (4.4) can be considered as an (unbounded) operator $\nabla^\Gamma_{\mathcal{Z}} : L^2(\Gamma_{\mathcal{Z}}, \hat{g}) \to L^2\mathcal{V}(\Gamma_{\mathcal{Z}}, \hat{g})$ with domain $\mathcal{FC}(\Gamma_{\mathcal{Z}})$, where $L^2\mathcal{V}(\Gamma_{\mathcal{Z}}, \hat{g})$ is the space of square-integrable vector fields on $\Gamma_{\mathcal{Z}}$.

So far, not much was required from the underlying Gibbs measure $g \in \mathscr{G}(\theta, \Phi)$, apart from suitable bounds on its correlation functions. But in this section we want to establish and explore a relationship between the Dirichlet forms of the measures g_{cl} and $\hat{g}$. In order to guarantee the existence of the latter Dirichlet form (that is, the closability of the form (4.4)), we need certain regularity of $\hat{g}$, and in particular certain smoothness of the interaction potential Φ. More precisely, let us assume that g is a *Ruelle measure*, that is, its interaction potential Φ is a *translation-invariant pair potential* (see Appendix A, Example A.1), $\Phi(\{x, x'\}) = \phi_0(x - x') = \phi_0(x' - x)$ $(x, x' \in X)$, with function ϕ_0 satisfying the standard conditions of *superstability* (SS), *lower regularity* (LR) and *integrability* (I) (see [35,36]). In addition, we assume that the corresponding reference measure is absolutely continuous, $\theta(\mathrm{d}x) = \mathrm{e}^{-\varphi(x)}\mathrm{d}x$ ($x \in X$), and the following *differentiability* condition (D) holds: the function $\mathrm{e}^{-\psi(x)}$, where $\psi := \phi_0 + \varphi$, is weakly differentiable and $\nabla\psi \in L^2 \cap L^1(X, \mathrm{e}^{-\psi(x)}\mathrm{d}x)$.

According to [3], conditions (SS), (LR), (I) and (D) imply an integration-by-parts formula for the measure g. From the construction of the measure $\hat{\mathrm{g}}$ (see Section 2.4), it easily follows that all these conditions also hold for $\hat{\mathrm{g}}$ (on the space $\mathcal{Z}$) and hence, again by [3], $\hat{\mathrm{g}}$ satisfies the integration-by-parts formula

$$\int_{\Gamma_{\mathcal{Z}}} \nabla_w^{\Gamma} F_1(\hat{\gamma}) F_2(\hat{\gamma})\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) = -\int_{\Gamma_{\mathcal{Z}}} F_1(\hat{\gamma})\big(\nabla_w^{\Gamma} F_2(\hat{\gamma}) + B_{\hat{\mathrm{g}}}^{w}(\hat{\gamma}) F_2(\hat{\gamma})\big)\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}), \tag{4.5}$$

for any $F_1, F_2 \in \mathcal{FC}(\Gamma_{\mathcal{Z}})$ and $w \in \mathrm{Vect}_0(\mathcal{Z})$, where $B_{\hat{\mathrm{g}}}^{w} \in L^2(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}})$ is the logarithmic derivative of the measure $\hat{\mathrm{g}}$ along the vector field w.

Remark 4.1. Adaptation of a general formula from [3] shows that the logarithmic derivative $B_{\hat{\mathrm{g}}}^{w}$ is given by

$$B_{\hat{\mathrm{g}}}^{w}(\hat{\gamma}) = \big\langle \beta_{\hat{\theta}}^{w}, \hat{\gamma}\big\rangle + \sum_{\{z_1, z_2\} \in \hat{\gamma}} \nabla\widetilde{\phi}_0(z_1 - z_2) \cdot \big(w(z_1) - w(z_2)\big),$$

where $\beta_{\hat{\theta}}^{w}$ is the logarithmic derivative of the measure $\hat{\theta}$ in the direction w and $\widetilde{\phi}_0(z) := \phi_0(p_X(z))$. However, the explicit form of $B_{\hat{\mathrm{g}}}^{w}$ is not needed for our purposes.

It follows from the integration-by-parts formula (4.5) that the pre-Dirichlet form $\mathcal{E}_{\hat{\mathrm{g}}}$ defined in (4.4) is closable (we keep the same notation for the closure and denote by $\mathcal{D}(\mathcal{E}_{\hat{\mathrm{g}}})$ its domain).

Following [3], we introduce an extension $\mathcal{E}_{\hat{\mathrm{g}}}^{\max}$ of the form $\mathcal{E}_{\hat{\mathrm{g}}}$ as follows. Let $\mathrm{div}_{\mathcal{Z}}^{\Gamma} : L^2\mathcal{V}(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}}) \to L^2(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}})$ be the operator adjoint to $(\nabla_{\mathcal{Z}}^{\Gamma}, \mathcal{FC}(\Gamma_{\mathcal{Z}}))$, which is well defined on $\mathcal{FV}(\Gamma_{\mathcal{Z}})$. Then the adjoint $(d_{\mathcal{Z}}^{\Gamma}, \mathcal{D}(d_{\mathcal{Z}}^{\Gamma}))$ of the operator $(\mathrm{div}_{\mathcal{Z}}^{\Gamma}, \mathcal{FV}(\Gamma_{\mathcal{Z}}))$ is a closed extension of the operator $(\nabla_{\mathcal{Z}}^{\Gamma}, \mathcal{FC}(\Gamma_{\mathcal{Z}}))$. Now, consider the bilinear form

$$\mathcal{E}_{\hat{\mathrm{g}}}^{\max}(F_1, F_2) := \int_{\Gamma_{\mathcal{Z}}} \big\langle d_{\mathcal{Z}}^{\Gamma} F_1(\hat{\gamma}), d_{\mathcal{Z}}^{\Gamma} F_2(\hat{\gamma})\big\rangle_{\hat{\gamma}}\, \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) \tag{4.6}$$

with domain $\mathcal{D}(\mathcal{E}_{\hat{\mathrm{g}}}^{\max}) = \mathcal{D}(d_{\mathcal{Z}}^{\Gamma})$; clearly, $(\mathcal{E}_{\hat{\mathrm{g}}}^{\max}, \mathcal{D}(d_{\mathcal{Z}}^{\Gamma}))$ is an extension of the form $(\mathcal{E}_{\hat{\mathrm{g}}}, \mathcal{D}(\mathcal{E}_{\hat{\mathrm{g}}}))$.

Our aim is to study a relationship between the forms $\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$ and $\mathcal{E}_{\hat{\mathrm{g}}}^{\max}$ and to characterize in this way the kernel of $\mathcal{E}_{\mathrm{g}_{\mathrm{cl}}}$. We need some preparations. Let us recall that the projection map $\mathfrak{q} : \mathcal{Z} \to \Gamma_X^{\sharp}$ was defined in (2.18) as $\mathfrak{q} := \mathfrak{p} \circ s$, where the map $s : \mathcal{Z} \to \mathfrak{X}$ is given by

$$\mathcal{Z} \ni (x, \bar{y}) \mapsto s(x, \bar{y}) := \bar{y} + x \in \mathfrak{X}.$$

As usual, we preserve the same notation for the induced maps of the corresponding configuration spaces. It follows directly from the definition of the map $\mathfrak{p}$ (see (2.3)) that

$$\big(\nabla_X^{\Gamma} F\big) \circ \mathfrak{p} = \nabla_{\mathfrak{X}}^{\Gamma}(F \circ \mathfrak{p}), \qquad F \in \mathcal{FC}(\Gamma_X), \tag{4.7}$$

where we use the identification of the tangent spaces

$$T_{\bar\gamma}\Gamma_{\mathfrak{X}} = \bigoplus_{\bar y\in\bar\gamma} T_{\bar y}\mathfrak{X} = \bigoplus_{\bar y\in\bar\gamma}\bigoplus_{y_i\in\bar y} T_{y_i}X = \bigoplus_{y_i\in\mathfrak{p}(\bar\gamma)} T_{y_i}X = T_{\mathfrak{p}(\bar\gamma)}X. \tag{4.8}$$

Lemma 4.3. *Let $F\in\mathcal{FC}(\Gamma_X)$. Then the function $\hat F := F\circ\mathfrak{q}$ belongs to the domain $\mathcal{D}(\mathcal{E}^{\max}_{\hat{\mathrm{g}}})$.*

Proof. By definition of $\mathcal{D}(\mathcal{E}^{\max}_{\hat{\mathrm{g}}})$, we need to show that there exists a vector field $W\in L^2\mathcal{V}(\Gamma_{\mathcal{Z}},\hat{\mathrm{g}})$ such that, for any $V\in\mathcal{FV}(\Gamma_{\mathcal{Z}})$,

$$\int_{\Gamma_{\mathcal{Z}}} \hat F(\hat\gamma)\,\mathrm{div}^{\Gamma}_{\mathcal{Z}} V(\hat\gamma)\,\hat{\mathrm{g}}(\mathrm{d}\hat\gamma) = -\int_{\Gamma_{\mathcal{Z}}} \big\langle W(\hat\gamma), V(\hat\gamma)\big\rangle_{\hat\gamma}\,\hat{\mathrm{g}}(\mathrm{d}\hat\gamma). \tag{4.9}$$

Without loss of generality we can assume that $V = Gv$, where $G\in\mathcal{FC}(\Gamma_{\mathcal{Z}})$ and $v\in\mathrm{Vect}_0(\mathcal{Z})$. Then (see [3])

$$\mathrm{div}^{\Gamma}_{\mathcal{Z}} V = \nabla^{\Gamma}_v G + B^v_{\hat{\mathrm{g}}} G.$$

Note that, even though the function $\hat F$ does not belong to the class $\mathcal{FC}(\Gamma_{\mathcal{Z}})$, it has the form

$$\hat F(\hat\gamma) = f\big(\langle\phi_1,\hat\gamma\rangle,\dots,\langle\phi_k,\hat\gamma\rangle\big),\quad \hat\gamma\in\Gamma_{\mathcal{Z}},$$

where $k\in\mathbb{N}$, $f\in C^\infty_b(\mathbb{R}^k)$ and $\phi_1,\dots,\phi_k\in C^\infty_{\hat\theta}(X)$ (:= the set of C^∞-functions on X with $\hat\theta$-bounded support and bounded derivatives). Integration-by-parts formula (4.5) can be extended to such functions in a straightforward manner. Applying this formula, we obtain

$$\begin{aligned}\int_{\Gamma_{\mathcal{Z}}} \hat F(\hat\gamma)\big(\nabla^{\Gamma}_v G(\hat\gamma) + B^v_{\hat{\mathrm{g}}}(\hat\gamma)G(\hat\gamma)\big)\,\hat{\mathrm{g}}(\mathrm{d}\hat\gamma) &= -\int_{\Gamma_{\mathcal{Z}}} \nabla^{\Gamma}_v \hat F(\hat\gamma)G(\hat\gamma)\,\hat{\mathrm{g}}(\mathrm{d}\hat\gamma)\\ &= -\int_{\Gamma_{\mathcal{Z}}} \big\langle\nabla^{\Gamma}_{\mathcal{Z}}\hat F(\hat\gamma), v\big\rangle_{\hat\gamma} G(\hat\gamma)\,\hat{\mathrm{g}}(\mathrm{d}\hat\gamma)\\ &= -\int_{\Gamma_{\mathcal{Z}}} \big\langle\nabla^{\Gamma}_{\mathcal{Z}}\hat F(\hat\gamma), G(\hat\gamma)v\big\rangle_{\hat\gamma}\,\hat{\mathrm{g}}(\mathrm{d}\hat\gamma),\end{aligned}$$

and Eq. (4.9) holds with $W = \nabla^{\Gamma}_{\mathcal{Z}}\hat F$.

It remains to show that $W\in L^2(\Gamma_{\mathcal{Z}},\hat{\mathrm{g}})$. Let us introduce the linear operator $\mathrm{d}s^*:\mathfrak{X}\to\mathcal{Z}$ by the formula

$$\mathrm{d}s^*(\bar y) := \big(\|\bar y\|_1, \bar y\big),\quad \bar y\in\mathfrak{X} = \bigsqcup_{k=0}^{N} X^k, \tag{4.10}$$

where $\|\bar y_1\| = \sum_{y_i\in\bar y} y_i$ (see (3.31)). Note that $\|\mathrm{d}s^*\| = \sqrt{N+1}$. As suggested by the notation, $\mathrm{d}s^*$ coincides with the adjoint of the derivative

$$\mathrm{d}s(z) : T_z\mathcal{Z} \to T_{s(z)}\mathfrak{X}$$

under the identification $T_{\bar{y}}\mathfrak{X} = \mathfrak{X}$, $T_z\mathcal{Z} = \mathcal{Z}$. A direct calculation shows that for any differentiable function f on $\mathfrak{X}$ the following commutation relation holds

$$\big(\mathrm{d}s^*\nabla f\big) \circ s = \nabla(f \circ s). \tag{4.11}$$

Here the symbol ∇ denotes the gradient on the corresponding space (i.e., $\mathfrak{X}$ on the left and $\mathcal{Z}$ on the right).

Let

$$\mathrm{d}s^*(\hat{\gamma}) : T_{s(\hat{\gamma})}\Gamma_{\mathfrak{X}} = \bigoplus_{\bar{y}\in s(\hat{\gamma})} T_{\bar{y}}\mathfrak{X} \to \bigoplus_{z\in\hat{\gamma}} T_z\mathcal{Z} = T_{\hat{\gamma}}\Gamma_{\mathcal{Z}}$$

be the natural lifting of the operator $\mathrm{d}s^*$; in view of (4.8) this can be interpreted as a bounded operator

$$\mathrm{d}s^*(\hat{\gamma}) : T_{\mathrm{q}(\hat{\gamma})}\Gamma_X \to T_{\hat{\gamma}}\Gamma_{\mathcal{Z}}$$

with the norm (cf. (4.10))

$$\big\|\mathrm{d}s^*(\hat{\gamma})\big\| = \sqrt{N+1}, \tag{4.12}$$

which induces the (bounded) operator

$$\mathcal{I}_{\mathrm{q}}\mathrm{d}s^* : L^2\mathcal{V}(\Gamma_X, \mathrm{g}_{\mathrm{cl}}) \to L^2\mathcal{V}(\Gamma_{\mathcal{Z}}, \hat{\mathrm{g}}) \tag{4.13}$$

acting as follows

$$\big(\mathcal{I}_{\mathrm{q}}\mathrm{d}s^* V\big)(\hat{\gamma}) = \mathrm{d}s^*(\hat{\gamma})V\big(\mathrm{q}(\hat{\gamma})\big), \quad V \in L^2\mathcal{V}(\Gamma_X, \mathrm{g}_{\mathrm{cl}}).$$

Note that formula (4.11) together with (4.7) implies

$$\big(\mathrm{d}s^*\nabla_X^\Gamma F\big) \circ \mathrm{q} = \nabla_{\mathcal{Z}}^\Gamma(F \circ \mathrm{q}), \quad F \in \mathcal{FC}(\Gamma_X). \tag{4.14}$$

Hence, using the change of variables (B.6), relation (4.12) and Theorem 4.1(a), for any $F \in \mathcal{FC}(\Gamma_X)$ we have

$$\begin{aligned}\int_{\Gamma_{\mathcal{Z}}} \big|\nabla_{\mathcal{Z}}^\Gamma(F\circ \mathrm{q})\big|^2\,\hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) &= \int_{\Gamma_{\mathcal{Z}}} \big|\mathrm{d}s^*\nabla_X^\Gamma F\big(\mathrm{q}(\hat{\gamma})\big)\big|_{\hat{\gamma}}^2\,\hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) \\ &= \int_{\Gamma_X} \big|\mathrm{d}s^*\nabla_X^\Gamma F(\gamma)\big|_\gamma^2\,\mathrm{g}_{\mathrm{cl}}(\mathrm{d}\gamma) \\ &\leqslant (N+1)\int_{\Gamma_X} \big|\nabla_X^\Gamma F(\gamma)\big|_\gamma^2\,\mathrm{g}_{\mathrm{cl}}(\mathrm{d}\gamma) < \infty, \end{aligned} \tag{4.15}$$

which implies that $\hat{F} = F\circ \mathrm{q} \in \mathcal{D}(\mathcal{E}_{\hat{\mathrm{g}}}^{\max})$, as claimed. This completes the proof of the lemma. □

Theorem 4.4. *For the Dirichlet forms $\mathcal{E}_{g_{cl}}$ and $\mathcal{E}_{\hat{g}}^{\max}$ defined in* (4.3) *and* (4.6)*, respectively, their domains satisfy the relation $\mathcal{I}_q(\mathcal{D}(\mathcal{E}_{g_{cl}})) \subset \mathcal{D}(\mathcal{E}_{\hat{g}}^{\max})$. Furthermore, $F \in \operatorname{Ker}\mathcal{E}_{g_{cl}}$ if and only if $\mathcal{I}_q F \in \operatorname{Ker}\mathcal{E}_{\hat{g}}^{\max}$.*

Proof. Formula (4.14) can be rewritten in terms of operators acting in the corresponding L^2-spaces,

$$\mathcal{I}_q \mathrm{d}s^* \nabla_X^\Gamma F = d_{\mathcal{Z}}^\Gamma \mathcal{I}_q F, \quad F \in \mathcal{FC}(\Gamma_X). \tag{4.16}$$

Using the bound (4.15) we have, for any $F \in \mathcal{FC}(\Gamma_X)$,

$$\begin{aligned}
\|F\|_{\mathcal{E}_{g_{cl}}}^2 &:= \mathcal{E}_{g_{cl}}(F,F) + \int_{\Gamma_X} F(\gamma)^2 \, g_{cl}(\mathrm{d}\gamma) \\
&\geqslant (N+1)^{-1} \mathcal{E}_{\hat{g}}^{\max}(\mathcal{I}_q F, \mathcal{I}_q F) + \int_{\Gamma_{\mathcal{Z}}} \big(\mathcal{I}_q F(\hat{\gamma})\big)^2 \, \hat{g}(\mathrm{d}\hat{\gamma}) \\
&\geqslant (N+1)^{-1} \|\mathcal{I}_q F\|_{\mathcal{E}_{\hat{g}}^{\max}}^2,
\end{aligned}$$

which implies that $\mathcal{I}_q(\mathcal{D}(\mathcal{E}_{g_{cl}})) \subset \mathcal{D}(\mathcal{E}_{\hat{g}}^{\max})$, thus proving the first part of the theorem.

Further, using approximation arguments and the continuity of the operator (4.13), one can show that equality (4.16) extends to the domain $\mathcal{D}(\mathcal{E}_{g_{cl}})$ and

$$\mathcal{E}_{\hat{g}}^{\max}(\mathcal{I}_q F, \mathcal{I}_q F) = \int_{\Gamma_{\mathcal{Z}}} \big|\big(\mathcal{I}_q \mathrm{d}s^* \overline{\nabla}_X^\Gamma F\big)(\hat{\gamma})\big|_{\hat{\gamma}}^2 \, \hat{g}(\mathrm{d}\hat{\gamma}), \quad F \in \mathcal{D}(\mathcal{E}_{g_{cl}}), \tag{4.17}$$

where $\overline{\nabla}_X^\Gamma$ is the closure of the operator $(\nabla_X^\Gamma, \mathcal{FC}(\Gamma_X))$. Since $\operatorname{Ker}(\mathcal{I}_q \mathrm{d}s^*) = \{0\}$, formula (4.17) readily implies that $\mathcal{I}_q F \in \operatorname{Ker}\mathcal{E}_{\hat{g}}^{\max}$ if and only if $\overline{\nabla}_X^\Gamma F = 0$, which is equivalent to $F \in \operatorname{Ker}\mathcal{E}_{g_{cl}}$. □

Let us recall that a bilinear form $\mathcal{E}$ is termed *irreducible* if the condition $\mathcal{E}(F,F) = 0$ implies that $F = \text{const}$ for any bounded $F \in \mathcal{D}(\mathcal{E})$. If, in addition, $\mathcal{E}$ is a Dirichlet form, the condition of boundedness of F can be dropped, see [3]. Observe that $\mathcal{E}_{\hat{g}}^{\max}$ is not in general a Dirichlet form.

Corollary 4.5. *The Dirichlet form $\mathcal{E}_{g_{cl}}$ is irreducible if $\mathcal{E}_{\hat{g}}^{\max}$ is such.*

Proof. The result follows immediately from Theorem 4.4 and the obvious fact that if $\mathcal{I}_q F = \text{const}$ ($\hat{g}$-a.s.) then $F = \text{const}$ (g_{cl}-a.s.). □

Remark 4.2. It has been proved in [3] that the form $\mathcal{E}_{\hat{g}}^{\max}$ is irreducible if and only if $\hat{g} \in \operatorname{ext}\mathscr{G}(\hat{\theta}, \hat{\Phi})$; in turn, the latter is equivalent to $g \in \operatorname{ext}\mathscr{G}(\theta, \Phi)$ provided that $g \in \mathscr{G}_L(\theta, \Phi)$ (see Corollary 2.9).

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Appendix A. Gibbs measures on configuration spaces

Let us briefly recall the definition and some properties of (grand canonical) Gibbs measures on the configuration space $\varGamma_X$. For a more systematic exposition and further details, see the classical books [17,33,35]; more recent useful references include [3,25,26].

Denote by $\varGamma_X^0 := \{\gamma \in \varGamma_X : \gamma(X) < \infty\}$ the subspace of finite configurations in X. Let $\varPhi : \varGamma_X^0 \to \mathbb{R} \cup \{+\infty\}$ be a measurable function (called the *interaction potential*) such that $\varPhi(\emptyset) = 0$. A simple, most common example is that of the *pair interaction potential*, i.e., such that $\varPhi(\gamma) = 0$ unless configuration γ consists of exactly two points.

Definition A.1. The *energy* $E : \varGamma_X^0 \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E(\xi) := \sum_{\zeta \subset \xi} \varPhi(\zeta) \quad \big(\xi \in \varGamma_X^0\big), \qquad E(\emptyset) := 0. \tag{A.1}$$

The *interaction energy* between configurations $\xi \in \varGamma_X^0$ and $\gamma \in \varGamma_X$ is given by

$$E(\xi, \gamma) := \begin{cases} \sum_{\gamma \supset \gamma' \in \varGamma_X^0} \varPhi(\xi \cup \gamma') & \text{if } \sum_{\gamma \supset \gamma' \in \varGamma_X^0} |\varPhi(\xi \cup \gamma')| < \infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{A.2}$$

Definition A.2. Let $\mathscr{G}(\theta, \varPhi)$ denote the class of all *grand canonical Gibbs measures* corresponding to the reference measure θ and the interaction potential $\varPhi$, that is, the probability measures on $\varGamma_X$ that satisfy the *Dobrushin–Lanford–Ruelle* (*DLR*) *equation* (see, e.g., [3, Eq. (2.17), p. 251]).

In the present paper, we use an equivalent characterization of the Gibbs measures via the *Georgii–Nguyen–Zessin* (*GNZ*) *equation* (see [16, Theorem 3.5], [32, Theorem 2]).

Theorem A.1. *A measure* g *on the configuration space* $\varGamma_X$ *belongs to the Gibbs class* $\mathscr{G}(\theta, \varPhi)$ *if and only if either of the following conditions holds*:

(i) (GNZ equation) *For any function* $H \in \mathrm{M}_+(X \times \varGamma_X)$,

$$\int_{\varGamma_X} \sum_{x \in \gamma} H(x, \gamma)\, \mathrm{g}(\mathrm{d}\gamma) = \int_{\varGamma_X} \bigg(\int_X H\big(x, \gamma \cup \{x\}\big) \mathrm{e}^{-E(\{x\}, \gamma)} \theta(\mathrm{d}x) \bigg)\, \mathrm{g}(\mathrm{d}\gamma). \tag{A.3}$$

(ii) (Ruelle's equation) *For any bounded function* $F \in \mathrm{M}_+(\varGamma_X)$ *and any compact set* $\varLambda \subset X$,

$$\int_{\varGamma_X} F(\gamma)\, \mathrm{g}(\mathrm{d}\gamma) = \int_{\varGamma_\varLambda} \mathrm{e}^{-E(\xi)} \bigg(\int_{\varGamma_{X \setminus \varLambda}} F(\xi \cup \gamma)\, \mathrm{e}^{-E(\xi, \gamma)}\, \mathrm{g}(\mathrm{d}\gamma) \bigg)\, \lambda_\theta(\mathrm{d}\xi), \tag{A.4}$$

where λ_θ *is the* Lebesgue–Poisson measure *on* Γ_X^0 *defined by the formula*

$$\lambda_\theta(\mathrm{d}\xi) = \sum_{n=0}^{\infty} \mathbf{1}\{\xi(\Lambda) = n\} \frac{1}{n!} \bigotimes_{x\in\xi} \theta(\mathrm{d}x), \quad \xi \in \Gamma_\Lambda^0. \tag{A.5}$$

Remark A.1. Using a standard argument based on the decomposition $H = H^+ - H^-$, $|H| = H^+ + H^-$ with $H^+ := \max\{H, 0\}$, $H^- := \max\{-H, 0\}$, one can see that Eq. (A.3) is also valid for an arbitrary measurable function $H : X \times \Gamma_X \to \mathbb{R}$ provided that

$$\int_{\Gamma_X} \sum_{x\in\gamma} \big|H(x,\gamma)\big|\, \mathrm{g}(\mathrm{d}\gamma) < \infty. \tag{A.6}$$

Remark A.2. In paper [32], the result of Theorem A.1 is proved under the additional assumptions of *stability* of the interaction potential Φ and *temperedness* of the measure g. In a subsequent work by Kuna [25, Theorems 2.2.4 and A.1.1], these assumptions have been lifted.

Remark A.3. Inspection of [32, Theorem 2] or [25, Theorem A.1.1] reveals that the proof of the implication (A.3) $\Rightarrow$ (A.4) is valid for *any* set $\Lambda \in \mathcal{B}(X)$ satisfying the conditions $\theta(\Lambda) < \infty$ and $\gamma(\Lambda) < \infty$ (g-a.s.). Hence, Ruelle's equation (A.4) is valid for such sets as well.

In the "interaction-free" case where $\Phi \equiv 0$, the unique grand canonical Gibbs measure coincides with the Poisson measure π_θ (with intensity measure θ). In the general situation, there are various types of conditions to ensure that the class $\mathscr{G}(\theta, \Phi)$ is non-empty (see [14,17,33,35,36] and also [25,24,26]).

Example A.1. The following are four classical examples of translation-invariant pair interaction potentials (i.e., such that $\Phi(\{x, y\}) = \phi_0(x - y) \equiv \phi_0(y - x)$), for which $\mathscr{G}(\theta, \Phi) \neq \emptyset$.

(1) (*Hard core potential*) $\phi_0(x) = +\infty$ for $|x| \leqslant r_0$, otherwise $\phi_0(x) = 0$ ($r_0 > 0$).
(2) (*Purely repulsive potential*) $\phi_0 \in C_0^2(\mathbb{R}^d)$, $\phi_0 \geqslant 0$ on $\mathbb{R}^d$, and $\phi_0(0) > 0$.
(3) (*Lennard–Jones type potential*) $\phi_0 \in C^2(\mathbb{R}^d \setminus \{0\})$, $\phi_0 \geqslant -a > -\infty$ on $\mathbb{R}^d$, $\phi_0(x) := c|x|^{-\alpha}$ for $|x| \leqslant r_1$ ($c > 0$, $\alpha > d$), and $\phi_0(x) = 0$ for $|x| > r_2$ ($0 < r_1 < r_2 < \infty$).
(4) (*Lennard–Jones* "6–12" *potential*) $d = 3$, $\phi_0(x) = c(|x|^{-12} - |x|^{-6})$ for $x \neq 0$ ($c > 0$) and $\phi_0(0) = +\infty$.

Definition A.3. For a Gibbs measure $\mathrm{g} \in \mathscr{G}(\theta, \Phi)$ on Γ_X, its *correlation function* $\kappa_\mathrm{g}^n : X^n \to \mathbb{R}_+$ *of the n-th order* ($n \in \mathbb{N}$) is defined by the following property: for any function $\phi \in \mathrm{M}_+(X^n)$, symmetric with respect to permutations of its arguments, it holds

$$\int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} \phi(x_1,\dots,x_n)\, \mathrm{g}(\mathrm{d}\gamma) = \frac{1}{n!} \int_{X^n} \phi(x_1,\dots,x_n) \kappa_\mathrm{g}^n(x_1,\dots,x_n)\, \theta(\mathrm{d}x_1)\cdots\theta(\mathrm{d}x_n). \tag{A.7}$$

For $n=1$ and $\phi(x)=\mathbf{1}_B(x)$, formula (A.7) specializes to

$$\int_{\Gamma_X} \gamma(B)\,\mathrm{g}(\mathrm{d}\gamma)=\int_B \kappa_{\mathrm{g}}^1(x)\,\theta(\mathrm{d}x). \tag{A.8}$$

Example A.2. In the Poisson case (i.e., $\Phi\equiv 0$), we have $\kappa_{\pi_\theta}^n(x)\equiv n!$ ($n\in\mathbb{N}$).

Remark A.4. Using the GNZ equation (A.3) with $H(x,\gamma)=\phi(x)$, from definition (A.8) it follows that

$$\kappa_{\mathrm{g}}^1(x)=\int_{\Gamma_X} \mathrm{e}^{-E(\{x\},\gamma)}\,\mathrm{g}(\mathrm{d}\gamma),\qquad x\in X. \tag{A.9}$$

In particular, representation (A.9) implies that if $\Phi\geqslant 0$ (non-attractive interaction potential) then $\kappa_{\mathrm{g}}^1(x)\leqslant 1$ for all $x\in X$, so that κ_{g}^1 is bounded.

Remark A.5. If the first-order correlation function $\kappa_{\mathrm{g}}^1(x)$ is integrable on any set $B\in\mathcal{B}(X)$ of finite θ-measure (e.g., if κ_{g}^1 is bounded on X, cf. Remark A.4), then, according to (A.8), the mean number of points in $\gamma\cap B$ is finite, also implying that $\gamma(B)<\infty$ for g-a.a. configurations $\gamma\in\Gamma_X$ (cf. Remark A.3). Conversely, if $\kappa_{\mathrm{g}}^1(x)\geqslant c>0$ for all $x\in X$ and the mean number of points in $\gamma\cap B$ is finite, then it follows from (A.8) that $\theta(B)<\infty$.

Definition A.4. For a probability measure μ on Γ_X, the notation $\mu\in\mathcal{M}^n(\Gamma_X)$ signifies that

$$\int_{\Gamma_X} \bigl|\langle\phi,\gamma\rangle\bigr|^n\,\mu(\mathrm{d}\gamma)<\infty,\qquad \phi\in C_0(X). \tag{A.10}$$

Definition A.5. We denote by $\mathscr{G}_L(\theta,\Phi)$ the set of all Gibbs measures $\mathrm{g}\in\mathscr{G}(\theta,\Phi)$ such that all its correlation functions κ_{g}^n are well defined and satisfy, for some constant $c>0$ and all $n\in\mathbb{N}$, the estimate

$$\bigl|\kappa_{\mathrm{g}}^n(x_1,\dots,x_n)\bigr|\leqslant\bigl(cn^2\bigr)^n,\qquad (x_1,\dots,x_n)\in X^n. \tag{A.11}$$

The following useful criterion was proved by Lenard [28].

Proposition A.2. *Let* $\mathrm{g}_1,\mathrm{g}_2\in\mathscr{G}_L(\theta,\Phi)$ *and* $\kappa_{\mathrm{g}_1}^n=\kappa_{\mathrm{g}_2}^n$ *for all* $n\in\mathbb{N}$. *Then* $\mathrm{g}_1=\mathrm{g}_2$.

Definition A.6. We denote by $\operatorname{ext}\mathscr{G}(\theta,\Phi)$ the set of extreme elements of the (convex) set $\mathscr{G}(\theta,\Phi)$, that is, those measures $\mathrm{g}\in\mathscr{G}(\theta,\Phi)$ that cannot be written as $\mathrm{g}=\frac{1}{2}(\mathrm{g}_1+\mathrm{g}_2)$ with $\mathrm{g}_1,\mathrm{g}_2\in\mathscr{G}(\theta,\Phi)$ and $\mathrm{g}_1\neq\mathrm{g}_2$.

Using Ruelle's equation (A.4), it is easy to obtain the following result (cf. [25, Corollary 2.2.6]).

Proposition A.3. *For a Gibbs measure* $\mathrm{g} \in \mathscr{G}(\theta, \Phi)$ *and any compact set* $\Lambda \in \mathcal{B}(X)$, *define the measure* g_Λ *on the configuration space* Γ_Λ *as the corresponding marginal of* g,

$$\mathrm{g}_\Lambda(A) := \mathrm{g}\big(\{\gamma \in \Gamma_X:\ \gamma \cap \Lambda \in A\}\big), \quad A \in \mathcal{B}(\Gamma_\Lambda).$$

Then g_Λ *is absolutely continuous with respect to the Lebesgue–Poisson measure* λ_θ, *and its Radon–Nikodym density* $S_\Lambda := \mathrm{d}\mathrm{g}_\Lambda/\mathrm{d}\lambda_\theta \in L^1(\Gamma_\Lambda, \lambda_\theta)$ *is given by*

$$S_\Lambda(\xi) = \mathrm{e}^{-E(\xi)} \int\limits_{\Gamma_{X\setminus\Lambda}} \mathrm{e}^{-E(\xi,\gamma)}\, \mathrm{g}(\mathrm{d}\gamma), \quad \xi \in \Gamma_\Lambda. \tag{A.12}$$

Appendix B. Proof of Lemma 3.6

(a) Using the multinomial expansion, for any $f \in C_0(\mathcal{Z})$ we have

$$\begin{aligned}\int\limits_{\Gamma_{\mathcal{Z}}} \big|\langle f, \hat\gamma\rangle\big|^n\, \hat{\mathrm{g}}(\mathrm{d}\hat\gamma) &\leqslant \int\limits_{\Gamma_{\mathcal{Z}}} \Big(\sum_{z\in\hat\gamma} \big|f(z)\big|\Big)^n\, \hat{\mathrm{g}}(\mathrm{d}\hat\gamma) \\ &= \sum_{m=1}^n \int\limits_{\Gamma_{\mathcal{Z}}} \sum_{\{z_1,\dots,z_m\}\subset\hat\gamma} \phi_n(z_1,\dots,z_m)\, \hat{\mathrm{g}}(\mathrm{d}\hat\gamma),\end{aligned} \tag{B.1}$$

where $\phi_n(z_1,\dots,z_m)$ is a symmetric function given by

$$\phi_n(z_1,\dots,z_m) := \sum_{\substack{i_1,\dots,i_m\geqslant 1\\ i_1+\cdots+i_m=n}} \frac{n!}{i_1!\cdots i_m!} \big|f(z_1)\big|^{i_1}\cdots\big|f(z_m)\big|^{i_m}. \tag{B.2}$$

By formula (A.7), the integral on the right-hand side of (B.1) is reduced to

$$\frac{1}{m!} \int\limits_{\mathcal{Z}^m} \phi_n(z_1,\dots,z_m)\, \kappa_{\hat{\mathrm{g}}}^m(z_1,\dots,z_m)\, \hat\theta(\mathrm{d}z_1)\cdots\hat\theta(\mathrm{d}z_m). \tag{B.3}$$

By Theorem 2.6(b), the hypotheses of the lemma imply that $0 \leqslant \kappa_{\hat{\mathrm{g}}}^m \leqslant c_m$ with some constant $c_m < \infty$ $(m = 1,\dots,n)$. Hence, substituting (B.2) we obtain that the integral in (B.3) is bounded by

$$\sum_{\substack{i_1,\dots,i_m\geqslant 1\\ i_1+\cdots+i_m=n}} \frac{c_m n!}{i_1!\cdots i_m!} \prod_{j=1}^m \int\limits_{\mathcal{Z}} \big|f(z_j)\big|^{i_j}\, \hat\theta(\mathrm{d}z_j) < \infty, \tag{B.4}$$

since each integral in (B.4) is finite owing to the assumption $f \in C_0(\mathcal{Z})$. Moreover, bound (B.4) is valid for any function $f \in \bigcap_{m=1}^n L^m(\mathcal{Z}, \hat\theta)$. Returning to (B.1), this yields (3.25).

(b) Using the change of measure (2.27), for any $\phi \in C_0(X)$ we obtain

$$\int_{\Gamma_X} \left|\langle \phi, \gamma\rangle\right|^n \mathrm{g}_{\mathrm{cl}}(\mathrm{d}\gamma) = \int_{\Gamma_{\mathcal{Z}}} \left|\langle \phi, \mathfrak{q}(\hat{\gamma})\rangle\right|^n \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_{\mathcal{Z}}} \left|\langle \mathfrak{q}^*\phi, \hat{\gamma}\rangle\right|^n \hat{\mathrm{g}}(\mathrm{d}\hat{\gamma}), \tag{B.5}$$

where

$$\mathfrak{q}^*\phi(x, \bar{y}) := \sum_{y_i \in \bar{y}} \phi(y_i + x), \quad (x, \bar{y}) \in \mathcal{Z}. \tag{B.6}$$

Due to part (a) of the lemma, it suffices to show that $\mathfrak{q}^*\phi \in L^m(\mathcal{Z}, \hat{\theta})$ for any $m = 1, \ldots, n$. By the elementary inequality $(a_1 + \cdots + a_k)^m \leqslant k^{m-1}(a_1^m + \cdots + a_k^m)$, from (B.6) we have

$$\int_{\mathcal{Z}} \left|\mathfrak{q}^*\phi(z)\right|^m \hat{\theta}(\mathrm{d}z) \leqslant \int_{\mathcal{Z}} N_X(\bar{y})^{m-1} \sum_{y_i \in \bar{y}} \left|\phi(y_i + x)\right|^m \hat{\theta}(\mathrm{d}x \times \mathrm{d}\bar{y}), \tag{B.7}$$

where $N_X(\bar{y})$ is the "dimension" of the vector $\bar{y}$ (see (2.38)). Recalling that $\hat{\theta} = \theta \otimes \eta$ and denoting $b_\phi := \sup_{x \in X} |\phi(x)| < \infty$ and $K_\phi := \operatorname{supp} \phi \subset X$, the right-hand side of (B.7) is dominated by

$$\begin{aligned}
&\int_{\mathfrak{X}} N_X(\bar{y})^{m-1} \left((b_\phi)^m \sum_{y_i \in \bar{y}} \int_X \mathbf{1}_{K_\phi - y_i}(x)\, \theta(\mathrm{d}x) \right) \eta(\mathrm{d}\bar{y}) \\
&\quad = (b_\phi)^m \int_{\mathfrak{X}} N_X(\bar{y})^{m-1} \sum_{y_i \in \bar{y}} \theta(K_\phi - y_i)\, \eta(\mathrm{d}\bar{y}) \\
&\quad \leqslant (b_\phi)^m \sup_{y \in X} \theta(K_\phi - y) \int_{\mathfrak{X}} N_X(\bar{y})^m\, \eta(\mathrm{d}\bar{y}) < \infty,
\end{aligned}$$

according to assumptions (2.49) and (3.26). The proof is complete.

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