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An Independence Result on Cotorsion Theories over Valuation Domains

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It is shown that, over suitable valuation domains R with field of quotients Q, the cotorsion theory $\mathcal{G}_K$ generated by $K = Q/R$ coincides with the cotorsion theory $\mathcal{C}_\partial$ cogenerated by the Fuchs' divisible module ∂, provided that Gödel's Axiom of Constructibility $V = L$ is assumed. On the other hand, assuming Martin's Axiom and the negation of the Continuum Hypothesis, it is proved that the cotorsion theory $\mathcal{G}_K$ is strictly smaller than $\mathcal{C}_\partial$ by exhibiting a strongly $(\aleph_1 - K)$-free divisible module M of projective dimension 2 such that $\mathrm{Ext}^1_R(M, K) = 0$. Applications to Whitehead modules are derived. © 2001 Academic Press

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1. INTRODUCTION

Cotorsion theories have been introduced in 1979 by the second author [S] in the context of abelian groups. Recently, they played a crucial role in the solution of the Flat Cover Conjecture (see [BEE]). This important achievement is based on a preliminary result by Eklof and Trlifaj [ET], which gives a method for constructing from any module M a module N "controlled" by M such that $\mathrm{Ext}^1_R(M, N) = 0$. The paper [ET] was inspired by results by Göbel and Shelah [GS], which solved long-standing problems in abelian group theory about "splitters" and homological properties of cotorsion theories.

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Recall that a *cotorsion theory* consists of a pair $(\mathcal{F}, \mathcal{C})$ of classes of modules over a fixed ring R (in this paper, R is always a commutative integral domain with 1), such that $\mathcal{F} = ^{\perp}\mathcal{C}$ and $\mathcal{C} = \mathcal{F}^{\perp}$, where for a class $\mathcal{M}$ of R-modules,

$$^{\perp}\mathcal{M} = \{N \in \mathrm{Mod}(R) \mid \mathrm{Ext}^1_R(N, M) = 0 \text{ for all } M \in \mathcal{M}\},$$

$$\mathcal{M}^{\perp} = \{N \in \mathrm{Mod}(R) \mid \mathrm{Ext}^1_R(M, N) = 0 \text{ for all } M \in \mathcal{M}\}.$$

If the class $\mathcal{M}$ consists of a single module M, we will simply write $^{\perp}M$ and $M^{\perp}$. Given two cotorsion theories $(\mathcal{F}, \mathcal{C})$ and $(\mathcal{F}', \mathcal{C}')$, we set $(\mathcal{F}, \mathcal{C}) \leq (\mathcal{F}', \mathcal{C}')$ if $\mathcal{C} \subseteq \mathcal{C}'$ or, equivalently, if $\mathcal{F}' \subseteq \mathcal{F}$.

Given a class $\mathcal{M}$ of modules, the pairs

$$\mathcal{G}_{\mathcal{M}} = (^{\perp}\mathcal{M}, (^{\perp}\mathcal{M})^{\perp}) \qquad \text{and} \qquad \mathcal{C}_{\mathcal{M}} = (^{\perp}(\mathcal{M}^{\perp}), \mathcal{M}^{\perp})$$

are cotorsion theories, called the cotorsion theories *generated* and *cogenerated* by $\mathcal{M}$, respectively. For more facts on cotorsion theories we refer the reader to the papers [S, GS, and ET].

The increasing interest in cotorsion theories demonstrated by the quoted papers led Trlifaj [T] to investigate more systematically the cotorsion theories induced by tilting and cotilting modules (these are generalizations of progenerators and injective cogenerators, respectively). In particular, he characterized the cotorsion theory $\mathcal{C}_{\partial}$ cogenerated by Fuchs' module ∂, which is a Tilting module (as proved by Facchini [Fa]) and the generator of the category of divisible modules. He showed that if R is a valuation domain (or, more generally, a Prüfer domain), then $\mathcal{C}_{\partial} = (\mathcal{P}_1, \mathcal{D})$, where $\mathcal{P}_1$ is the class of modules of projective dimension ≤ 1, and $\mathcal{D}$ is the class of divisible modules.

Cotorsion theories generated by a single module are generally more difficult to handle; just think of the Whitehead problem that led Shelah [Sh1] in 1974 to the first independence result in abelian group theory. Recall that a Whitehead module is an R-module M such that $\mathrm{Ext}^1_R(M, R) = 0$, i.e., $M \in ^{\perp} R$ (for an extensive treatment of Whitehead abelian groups we refer the reader to the monograph by Eklof and Mekler [EM], and for Whitehead modules over domains to the paper by Becker *et al*. [BFS] or to the recent monograph [FS2]).

In this paper we investigate the cotorsion theory $\mathcal{G}_K$ generated by the divisible module $K = Q/R$, where R is a suitable valuation domain and Q is its field of quotients. The choice of K is motivated by at least two reasons. The first reason is that, for every module M, $\mathrm{Ext}^1_R(M, Q/R) \cong \mathrm{Ext}^2_R(M, R)$. Hence we can look at modules in $^{\perp}K$ as a "shifted version" from Ext^1 to Ext^2 of Whitehead modules. Moreover, if $0 \to H \to F \to M \to 0$ is a free presentation of M, then $\mathrm{Ext}^2_R(M, R) \cong \mathrm{Ext}^1_R(H, R)$; whence M belongs to $^{\perp}K$ exactly if H is a Whitehead module. The second reason is the crucial role played by K with respect to the functors Hom and Ext in the covariant situation. Actually, using the classical Matlis category equivalence induced by the functors $\mathrm{Hom}_R(K, \bullet)$ and $K \otimes_R \bullet$ (see [M3]), we will show that, over complete valuation domains such that $\mathrm{p.d.}\, Q = 1$, for an h-divisible torsion module D and a complete torsion-free module C corresponding to each other in the Matlis equivalence, $D \in^{\perp} K$ if and only if $C \in^{\perp} R$; i.e., C is a Whitehead module. These facts, considered in Section 5, show that there exists a strong connection between the cotorsion theory $\mathscr{G}_K$ and the Whitehead modules.

We will see in Section 4 that, under suitable conditions on R, we can establish an independence result. This is slightly surprising in view of the remarks made in the preceding paragraph. However, the valuation domains for which this happens must be carefully chosen: first of all, they must satisfy the homological conditions $\mathrm{gl.d.}\, R = 2$ and $\mathrm{p.d.}\, Q = 1$. They must also be, roughly speaking, as far as possible from being almost maximal. Valuation domains satisfying these conditions will be called IC-domains. We will show that, assuming the Gödel Axiom of Constructibility (V = L), over an IC-domain the cotorsion theory $\mathscr{G}_K$ coincides with the cotorsion theory $\mathscr{C}_\partial$ investigated by Trlifaj. In order to get this result, we state at the end of Section 3 a version of the Shelah Singular Compactness Theorem [Sh2] stating that, for singular cardinals λ, every λ-generated torsion divisible module whose $<\lambda$-generated divisible submodules are K-free is also K-free.

In order to prove a result in the opposite direction, we must impose the additional condition that K is countable. An IC-domain satisfying also this condition will be called an ICC-domain. We will prove that, assuming Martin's Axiom and the negation of the Continuum Hypothesis, over an ICC-domain the cotorsion theory $\mathscr{G}_K$ is strictly smaller than $\mathscr{C}_\partial$. The way to reach this result resembles the way to prove that there exist non-free Whitehead abelian groups (see [EM]). The role of strongly $\aleph_1$-free groups is played by the strongly $(\aleph_1 - K)$-free modules (for their definition see Section 3), and free resolutions are replaced by h-exact sequences of divisible modules, introduced by de la Rosa and Fuchs [DF].

Many results in this paper hold over valuation domains R satisfying some (and not all) of the above conditions, and we will specify the precise hypotheses on R in each statement. But we will need the full force of ICC-domains in order to obtain the main independence result.

2. IC-DOMAINS AND THE CLASS $^{\perp}K$

From now on, R will always denote a valuation domain, Q its field of quotients, K the divisible module Q/R, P the maximal ideal of R, and $\mathrm{Spec}(R)$ its prime spectrum. We denote by $\mathscr{P}_n$ the class of R-modules of projective dimension $\leq n$, by $\mathcal{I}_n$ the class of modules of injective dimension $\leq n$, and by $\mathcal{D}$ the class of divisible R-modules. The projective (resp., injective) dimension of a module M is denoted by p.d. M (resp., i.d. M). The global dimension of the ring R is denoted by gl.d. R. For basic facts on these invariants we refer to [FS1] and [FS2].

It is well known that the Fuchs' divisible module ∂ (see [FS1, p. 123]) generates (in the usual sense) the class $\mathcal{D}$ of divisible modules. Denote by Gen(M) the class of modules generated by the module M. In [Fa] Facchini proves that ∂ is a tilting module, i.e., Gen$(\partial) = \partial^{\perp}$. For a valuation domain R (or, more generally, for a Prüfer domain), Trlifaj [T] proves that the cotorsion theory cogenerated by ∂ is $\mathcal{C}_\partial = (\mathcal{P}_1, \mathcal{D})$. Note that the module ∂ is a splitter (in the sense of [GS]); i.e., $\text{Ext}^1_R(\partial, \partial) = 0$.

In this section we introduce a class of valuation domains that gives rise to an independence result for the cotorsion theory $\mathcal{G}_K = ({}^{\perp}K, ({}^{\perp}K)^{\perp})$ generated by K. For these domains p.d. $Q = 1$ holds, hence $\text{Ext}^1_R(K, K) = 0$; i.e., K is a splitter. Note that for general valuation domains K is not a splitter. An example of valuation domain R with p.d. $Q = 2$ such that $\text{Ext}^1_R(K, K) \neq 0$ is furnished in [F2] under the assumption of V = L (see also [FS2, p. 267]).

The next lemma shows that $\mathcal{G}_K \leq \mathcal{C}_\partial$ over any valuation domain.

LEMMA 2.1. *For every valuation domain R and every R-module M, the following facts hold*:

(a) $\mathcal{P}_1 \subseteq^{\perp} K$;

(b) $\text{Ext}^1_R(M, K) \cong \text{Ext}^2_R(M, R)$.

Proof. From the short exact sequence $0 \to R \to Q \to K \to 0$ we get for every R-module M the long exact sequence

$$0 = \text{Ext}^1_R(M, Q) \to \text{Ext}^1_R(M, K) \to \text{Ext}^2_R(M, R) \to \text{Ext}^2_R(M, Q) = 0$$

from which (b) follows; if p.d. $M \leq 1$, then $\text{Ext}^2_R(M, R) = 0$, so $M \in^{\perp} K$. ∎

The valuation domains that we will introduce in this section satisfy certain homological and topological conditions. The homological conditions are

$$\text{p.d.}\, Q = 1 \qquad \text{and} \qquad \text{gl.d.}\, R = 2$$

(domains satisfying p.d. $Q = 1$ are called *Matlis domains* in [FS2]). The two combined conditions can be stated by saying that every submodule of Q is at most countably generated, by well-known results by Kaplansky [K] and Osofsky [O]; thus they are essentially conditions on the value group of R.

We fix for the moment our attention on valuation domains satisfying these homological conditions. Remark that, given any domain R, p.d. $Q = 1$

and gl.d. $R = 2$ if and only if all torsion-free R-modules have projective dimension ≤ 1.

The following facts are easily established for modules over these valuation domains (one can find the proofs, for instance, in [FS2]; recall that a module is h-divisible if it is an epic image of an injective module):

(i) all divisible modules are h-divisible, and the torsion part of a divisible module splits off (see [M2]);

(ii) the class $\mathcal{P}_1$ is closed under submodules;

(iii) if D is a divisible module, then i.d. $D \leq 1$; in fact, for h-divisible modules D the inequality i.d. $(D) + 1 \leq$ gl.d. R holds (see [M2]);

(iv) K is a cotorsion module in the sense of Enochs or Warfield, i.e., $\mathrm{Ext}^1_R(F, K) = 0$ for all torsion-free modules F; equivalently, $\mathrm{Ext}^1_R(Q, K) = 0$ and i.d. $K = 1$ (see [W2 or FS1, p. 243]);

(v) the class ${}^{\perp}K$ is closed under submodules, since i.d. $K = 1$.

Recall that a short exact sequence is h-exact if K is projective with respect to it; by a result of de la Rosa and Fuchs [DF] (see also [FS2, VII.2.10]), given any torsion h-divisible module D, there exists an h-exact sequence

$$0 \to A \to B \to D \to 0 \tag{1}$$

with A h-divisible module and B isomorphic to a direct sum of copies of K.

PROPOSITION 2.2. *Assume that R is a valuation domain such that* p.d. $Q = 1$. *If T is a torsion divisible module, then*:

(a) $\mathrm{Ext}^1_R(D, T) = 0$ *for all modules D which are isomorphic to direct sums of copies of K*;

(b) $T \cong K^{(\alpha)}$ *for some cardinal α if and only if* p.d. $T = 1$;

(c) *if* gl.d. $R = 2$, *then there exists an h-exact sequence*

$$0 \to K^{(\alpha)} \to K^{(\beta)} \to T \to 0,$$

where α and β are suitable cardinals.

Proof. (a) We already mentioned that (over Prüfer domains) $\mathcal{P}_1 = {}^{\perp}\mathcal{D}$ and $\mathcal{P}_1^{\perp} = \mathcal{D}$. As direct sums of copies of K obviously belong to $\mathcal{P}_1$, the claim follows.

(b) The proof can be found in [F1] (see also [FS2, VII.3.5]).

(c) In the exact sequence (1) the module A has projective dimension 1, by fact (ii) above, hence it is isomorphic to a direct sum of copies of K by (b) (see also [FS1, VI.2.5]). ∎

If R is an almost maximal valuation domain (i.e., all proper quotients of R are complete in the ideal topology), then the cotorsion theory $\mathcal{G}_K$ is the minimal one; in fact, K is injective, thus ${}^{\perp}K = \text{Mod}(R)$ and $({}^{\perp}K)^{\perp}$ coincides with $\mathcal{I}_0$, the class of injective modules. Furthermore, R is characterized by the fact that $\text{Ext}^1_R(R/J, K) = 0$ for all its ideals J: note that this equality holds over any valuation domain R if J is a principal ideal.

If we want $\mathcal{G}_K$ to be as large as possible, it is natural to look for valuation domains which are, roughly speaking, as far as possible from being almost maximal. Thus we are led to introduce the following class of valuation domains.

DEFINITION. A valuation domain R is said to be *antimaximal* if, given any nonprincipal ideal J, $\text{Ext}^1_R(R/J, K) \neq 0$.

The antimaximality of the valuation domain R can be restated by saying that a cyclic module belongs to ${}^{\perp}K$ exactly if it is cyclically presented; i.e., it has projective dimension ≤ 1. Thus valuation domains R such that ${}^{\perp}K = \mathcal{P}_1$ must be antimaximal. The next result describes in topological terms the antimaximal valuation domains.

PROPOSITION 2.3. *For a valuation domain R the following conditions are equivalent*:

(1) *if I is a nonzero ideal of R such that R/I is Hausdorff in the topology of the nonzero submodules*, *then R/I is not complete*;

(2) *R is antimaximal*;

(3) *if J is a nonzero ideal of R such that* $\text{Ext}^1_R(J, R) = 0$, *the* $J \cong R$.

Proof. (1) $\Rightarrow$ (2). We briefly sketch the proof, referring to the paper [SZ2] for facts on 2-generated modules and their associated units in a maximal immediate extension S of R. Assume that J is a nonprincipal ideal of R. Pick $0 \neq a \in J$ and let $I = aR : J$. Then R/I is Hausdorff, hence it is not complete. Thus there exists a unit $u \in S \backslash R$ such that its breadth ideal $B(u)$ equals I (see [SZ2]). Furthermore, there exists a 2-generated indecomposable module M that fits in a pure-exact sequence

$$0 \to R/aR \to M \to R/J \to 0.$$

The inclusion $R/aR \leq Q/aR$ gives rise to the push-out

$$\begin{array}{ccccccccc} 0 \longrightarrow & R/aR & \longrightarrow & M & \longrightarrow & R/J & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \| & & \\ 0 \longrightarrow & Q/aR \cong K & \longrightarrow & N & \longrightarrow & R/J & \longrightarrow & 0, \end{array}$$

where the bottom exact sequence is nonsplitting; otherwise M, as a finitely generated submodule of a direct sum of two uniserial modules, would be decomposable, by a result by Matlis [M1] (see also [FS1]). Therefore $\mathrm{Ext}^1_R(R/J, K) \neq 0$, as desired.

(2) $\Rightarrow$ (1). Let I be a nonzero ideal of R such that R/I is Hausdorff. Pick $0 \neq a \in I$ and let $J = \{r \in R | rI < aR\}$ (in [SZ2] such an ideal J is denoted by $aR :: I$). J is nonprincipal, hence there exists a nonsplitting exact sequence

$$0 \to Q/R \to Y \to R/J \to 0.$$

Let $y \in Y$ be a preimage of a generator of R/J. There exists an element $x \in Q/R$ such that $\mathrm{Ann}\, x = \mathrm{Ann}\, y$. Then it is easily checked that $M = yR + xR$ is an indecomposable 2-generated submodule of Y whose associated unit in S has breadth ideal $B(u) = I$, whence R/I is not complete (see [SZ2]).

(2) $\Leftrightarrow$ (3). $\mathrm{Ext}^1_R(R/J, K)$ is isomorphic, by Lemma 2.1, to $\mathrm{Ext}^2_R(R/J, R)$, which is isomorphic to $\mathrm{Ext}^1_R(J, R)$ in view of the exact sequence

$$0 = \mathrm{Ext}^1_R(R, R) \to \mathrm{Ext}^1_R(J, R) \to \mathrm{Ext}^2_R(R/J, R) \to \mathrm{Ext}^2_R(R, R) = 0,$$

whence the equivalence follows trivially. ∎

We introduce now the classes of valuation domains over which our independence results will be proved in Section 4.

DEFINITION. An antimaximal valuation domain R such that $\mathrm{p.d.}\, Q = 1$ and $\mathrm{gl.d.}\, R = 2$ is called *IC-domain*.

A stronger condition will be needed on the valuation domain R when we deal with Martin's Axiom.

DEFINITION. A valuation domain R is said to be an *ICC-domain* if K is countable.

Note that an ICC-domain is certainly an IC-domain, since K countable implies that all the submodules of Q are countably generated, and, given a nonzero ideal I of R such that the quotient R/I is Hausdorff in the topology of the nonzero submodules, R/I is not complete, since $|R/I| < 2^{\aleph_0}$; so R is antimaximal by Proposition 2.3.

The terms "IC" and "ICC" remind one that the considered domains satisfy incompleteness conditions on the quotients, modulo nonzero ideals, and countability conditions.

Note also that a valuation domain R is a IC- or an ICC-domain exactly if its completion $\widetilde{R}$ has the same property; in fact, it is well known that $\widetilde{R}$ is an immediate extension of R, so they have the same value groups, and $Q/R \cong \widetilde{Q}/\widetilde{R}$ as R-modules.

We provide some examples of IC- and ICC-domains; the first one is our prototype.

EXAMPLE 2.4. Denote as usual by $\mathbb{Q}$ the ring of rational numbers, by $\mathbb{Z}$ the ring of the integers, and by $\mathbb{Z}_p$ its localization at the prime p. The discrete rank-2 valuation domain $R = \mathbb{Z}_p + X\mathbb{Q}[[X]]$, consisting of the power series with rational coefficients and constant term in $\mathbb{Z}_p$, is an ICC-domain. In fact, $Q/R = \bigcup_{n\in\omega}(X^{-n}R/R)$, where $X^{-n}R/R \cong R/X^nR$ is countable for every $n \in \omega$, so $|Q/R| = \aleph_0$. Note that R is complete, hence $|R| = 2^{\aleph_0}$.

The subring $\mathbb{Z}_p + X\mathbb{Q}[X]_{(X)}$ of R is also an ICC-domain, which is not complete.

A generalization of the preceding example to valuation domains of countable Krull dimension is given by the family of valuation domains considered in the next example.

EXAMPLE 2.5. Let R be a strongly discrete valuation domain (i.e., $L > L^2$ for all $0 \neq L \in \operatorname{Spec}(R)$), with $\operatorname{Spec}(R)$ countable. Then the prime ideals of R form a well-ordered decreasing chain $P = P_0 > P_1 > P_2 > \cdots > P_\sigma > \cdots$, where $\sigma < \alpha$ and α is a countable ordinal. If $|R/P| < 2^{\aleph_0}$, then it is easily seen that $|R/I| < 2^{\aleph_0}$ for every nonzero ideal I of R. Hence R is an IC-domain. The same conclusion holds if we assume that R/P_σ is not complete for all $\sigma > 0$.

If R/P is countable, then K is also countable, so R is an ICC-domain.

EXAMPLE 2.6. Let F be a field of cardinality κ. Consider the field of quotients Q of the ring of formal polynomials with coefficients in F and exponents in Δ, a dense subgroup of the additive group $\mathbb{R}$ of the real numbers. The valuation domain R of the canonical valuation $v\colon Q \to \mathbb{R} \cup \{\infty\}$ has Krull dimension 1, is not a principal ideal domain, and is an IC-domain.

If F and Δ are countable, then R is an ICC-domain.

We start now to investigate the class ${}^\perp K$ in the case that R is an IC-domain. We observe that the fact that K is cotorsion obviously implies that a module M belongs to ${}^\perp K$ if and only if its torsion part tM belongs to it.

The next Proposition 2.8 is of most importance for the structure of torsion modules in the class ${}^\perp K$. It extends to modules in the class ${}^\perp K$ a well-known property of modules of projective dimension ≤ 1 (see [FS2, VI.6.4]). Recall that a module is coherent if every finitely generated submodule is finitely presented; for valuation domains, this amounts to saying that every finitely generated submodule is a direct sum of cyclically presented modules (see [W3 or FS1]). Recall also that the *length* of a finitely generated module coincides with the number of a minimal set of generators of M and also with the length of a pure-composition series.

We need the following preliminary crucial lemma.

LEMMA 2.7. *Let R be an antimaximal valuation domain and J a nonprincipal ideal of R. Then* $\operatorname{Ext}^1_R(R/J, K)$ *is not finitely generated.*

Proof. The exact sequence $0 \to J \to R \to R/J \to 0$ induces the exact sequence

$$\mathrm{Hom}_R(R, K) \xrightarrow{\eta} \mathrm{Hom}_R(J, K) \to \mathrm{Ext}^1_R(R/J, K) \to 0.$$

We claim that

if $f \in \mathrm{Hom}_R(J, K)$ and $r \notin \mathrm{Ann}_R f$, then $f \in r\,\mathrm{Hom}_R(J, K) + \mathrm{Im}\,\eta$.

If $f \in \mathrm{Im}\,\eta$ the claim is trivial; thus, assume that $f \notin \mathrm{Im}\,\eta$. Let $aR = \mathrm{Ker}\, f$, where $0 \neq a \in J$, so that $\mathrm{Ann}_R f = R : a^{-1}J$. We have the commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{f} & Q/R \\ \downarrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \epsilon} \\ J/aR & \xrightarrow{\alpha} & J/aR, \end{array}$$

where π is the canonical surjection, α is an automorphism of J/aR, and ϵ is the embedding induced by the multiplication by a^{-1}. The endomorphism ring $\mathrm{End}_R(J/aR)$ is isomorphic to the completion $\widetilde{S}$ of the ring $S = R/(R : a^{-1}J)$ (see [SL] or [FS1]), and $S < \widetilde{S}$ by Proposition 2.3. It is immediate to check that α corresponds, under this isomorphism, to a unit of $\widetilde{S} \setminus S$. Since $r \notin R : a^{-1}J$, it follows that $rS \neq 0$, hence $\alpha - u \in r\widetilde{S}$ for some $u \in \mathrm{End}(J/aR)$ corresponding to a unit of S. Clearly $f - \epsilon u \pi \in r\,\mathrm{Hom}_R(J, Q/R)$, and since $\epsilon u \pi \in \mathrm{Im}\,\eta$ the claim is proved.

Assume now, by way of contradiction, that $\mathrm{Ext}^1_R(R/J, K)$ is finitely generated. Choose preimages $f_1, \ldots, f_n \in \mathrm{Hom}_R(J, K)$ of a finite set of generators of $\mathrm{Ext}^1_R(R/J, K)$. For all $1 \leq i \leq n$, $\mathrm{Ann}_R f_i$ is strictly contained in P, since J is nonprincipal. Pick an element $r \in P \setminus \bigcup_i \mathrm{Ann}_R f_i$. Then the claim ensures that $f_i \in r\,\mathrm{Hom}_R(J, K) + \mathrm{Im}\,\eta$ for all i, hence

$$\begin{aligned} \mathrm{Hom}_R(J, K)/\mathrm{Im}\,\eta &= (r\,\mathrm{Hom}_R(J, K) + \mathrm{Im}\,\eta)/\mathrm{Im}\,\eta \\ &= r(\mathrm{Hom}_R(J, K)/\mathrm{Im}\,\eta). \end{aligned}$$

Therefore $\mathrm{Ext}^1_R(R/J, K) \cong r\,\mathrm{Ext}^1_R(R/J, K)$; this fact contradicts the assumption that $\mathrm{Ext}^1_R(R/J, K)$ is finitely generated. ∎

With the aid of the previous lemma, we can prove the following.

PROPOSITION 2.8. *Let R be an IC-domain and M an R-module such that $\mathrm{Ext}^1_R(M, K)$ is finitely generated. Then M is coherent. In particular, this holds if $M \in {}^{\perp} K$.*

Proof. Since the module M is coherent if and only if its torsion part tM is coherent, and $\mathrm{Ext}^1_R(N, K)$ is a quotient of $\mathrm{Ext}^1_R(M, K)$ for every submodule N of M, we can assume that M is a torsion module. We must show that a finitely generated submodule N of M is a direct sum of cyclically presented modules. We induct on the length n of N. If $n = 1$, then $N \cong R/I$ and I must be principal, by Lemma 2.7. Assume $n > 1$ and the claim is true for $n - 1$. Let H be a pure submodule of N of length $n - 1$ (see [SZ1]). By induction, $H = \bigoplus_{1 \leq i \leq n-1} R/r_i R$. Let $N = H + zR$; we claim that $J = \mathrm{Ann}(z + H)$ is principal. Assume, by way of contradiction, that $J \ncong R$, so $\mathrm{Ext}^1_R(R/J, K) \neq 0$. Then we have the exact sequence

$$\mathrm{Hom}_R(H, K) \to \mathrm{Ext}^1_R(R/J, K) \to \mathrm{Ext}^1_R(N, K) \to 0.$$

The last Ext is finitely generated, since it is a quotient of $\mathrm{Ext}^1_R(M, K)$, and the first Hom is also finitely generated, since it is isomorphic to H. Therefore $\mathrm{Ext}^1_R(R/J, K)$ is finitely generated, contradicting Lemma 2.7. Thus J is principal and consequently H is a summand of N, so N is finitely presented. ∎

It is well known (see [FS1, IV.4.4]) that countably generated coherent modules have projective dimension ≤ 1, whence we get the following immediate consequence of the preceding proposition.

COROLLARY 2.9. *Let R be an IC-domain. Then every countably generated torsion R-module in ${}^{\perp}K$ has projective dimension ≤ 1.*

Examples of torsion modules not in ${}^{\perp}K$ are abundant. The next two results furnish two different kinds of these modules.

COROLLARY 2.10. *Let R be an IC-domain and T an injective torsion R-module. Then T is not coherent, hence $T \notin {}^{\perp} K$.*

Proof. Every injective torsion module is the injective hull of a direct sum of modules isomorphic to $E(R/I)$, for suitable non-zero ideals I of R. $E(R/I)$ is obviously not coherent for I nonprincipal. If I is principal, then $E(R/I) \cong E(K)$ is not coherent either, since it contains finitely generated noncyclic modules which are uniform, as it is not uniserial; hence the conclusion follows from Proposition 2.8. ∎

The next result will be used later on in Section 5. Recall that a *v-ideal* of R is an ideal which is the intersection of the principal ideals containing it.

PROPOSITION 2.11. *Let R be an IC-domain. Then $\prod_\alpha K$ is not coherent for all infinite cardinals α, whence $\prod_\alpha K \notin {}^{\perp} K$; equivalently, $t(\prod_\alpha K) \notin {}^{\perp} K$.*

Proof. We can assume that R is complete, since all the involved modules are canonically modules over the completion of R. Furthermore, it is enough to consider the case $\alpha = \aleph_0$. Assume first that there exits a nonprincipal v-ideal J. Clearly there is an embedding of R/J into $\prod_{\aleph_0} K$; by Proposition 2.3, $\mathrm{Ext}^1_R(R/J, K) \neq 0$, so the conclusion follows from (v).

Assume now that all v-ideals are principal; as R is not a principal ideal domain, this happens exactly if the value group of R is the additive group $\mathbb{R}$ of the reals. P is countably generated and $\mathrm{Ext}^1_R(R/P, K) \neq 0$. There exists a nonsplitting exact sequence $0 \to K \xrightarrow{\eta} M \to R/P \to 0$, which gives the long exact sequence

$$0 \to \mathrm{Hom}_R(M,K) \xrightarrow{\phi} \mathrm{Hom}_R(K,K) \to \mathrm{Ext}^1_R(R/P,K) \to \mathrm{Ext}^1_R(M,K) \to 0.$$

Since $\mathrm{Ext}^1_R(R/P, K)$ is semisimple and $\mathrm{Hom}_R(K, K) \cong \widetilde{R} = R$, there follows that $\mathrm{Coker}\phi \cong R/P$. If $\mathrm{Ext}^1_R(M, K) = 0$, then $\mathrm{Ext}^1_R(R/P, K) \cong R/P$ is finitely generated, contradicting Lemma 2.7. So $\mathrm{Ext}^1_R(M, K) \neq 0$.

To conclude the proof, in view of (v) it is enough to prove that there is an embedding of M into $\prod_{\aleph_0} K$. For every $r \in P$, $rM \leq \eta K$, hence we can define a homomorphism ϕ_r: $M \to K$ by setting $\phi_r(x) = rx$, for all $x \in M$. If P is the union of the ascending chain of cyclic ideals $r_1 R < r_2 R < \cdots < r_n R < \cdots$, let ϕ: $M \to \prod_{\aleph_0} K$ be the diagonal map of the homomorphisms $\phi_{r_n} (n \in \omega)$. Then $\mathrm{Ker}\,\phi = \bigcap_{n\in\omega} \mathrm{Ker}\,\phi_n = \bigcap_{n\in\omega} M[r_n] = M[P]$; but $M[P] = 0$, as M is uniform and $K[P] = 0$. ∎

The preceding results show that, given an IC-domain R, the following implications hold for a R-module M:

$$\text{p.d.}\, M \leq 1 \Rightarrow \mathrm{Ext}^1_R(M, K) = 0 \Rightarrow M \text{ is coherent.}$$

(The first implication holds for every valuation domain, by Lemma 2.1.) If M is countably generated, then both the implications can be reversed, by Corollary 2.9. On the other side, the next Theorem 3.6 will show that there are uncountably generated coherent R-modules of projective dimension 2, and the discussion in Section 4 will show that reversing the first implication is undecidable in ZFC. The following is an open question.

Question. Can one prove in ZFC that there are coherent modules over IC-domains that do not belong to the class ${}^{\perp}K$?

3. DIVISIBLE MODULES AND K-FREE MODULES

In this section we study divisible modules over valuation domains R with $\text{p.d.}\, Q = 1$ and $\text{gl.d.}\, R = 2$. The main result is Theorem 3.6, which provides the construction of a strongly $(\aleph_1 - K)$-free torsion divisible module (for its definition see below, after Proposition 3.4) which is not K-free.

In the following definition we consider the simplest divisible torsion modules.

DEFINITION. A divisible module D over a valuation domain R is said to be *K-free* if it is isomorphic to $K^{(\alpha)}$ for some cardinal α. This α is called the *K-rank* of D.

The K-rank of a K-free module M is an invariant of M, since it coincides with the dimension of the R/P-vector space $M[r]/PM[r]$ for all $0 \neq r \in R$.

Obviously, if p.d. $Q = 1$, then K-free modules have projective dimension 1, so they are coherent; the converse holds for countably generated torsion divisible modules, since they have projective dimension ≤ 1.

The next result gives a useful characterization of those coherent torsion divisible modules which are $\aleph_1$-generated. Recall that an $\aleph_1$-filtration of a module M is a smooth ascending chain of countably generated submodules of M whose union is M itself.

PROPOSITION 3.1. *Let R be a valuation domain such that* p.d. $Q = 1$. *An $\aleph_1$-generated torsion divisible module is coherent if and only if it has an $\aleph_1$-filtration consisting of countably generated K-free submodules.*

Proof. Let D be an $\aleph_1$-generated torsion divisible coherent module. Using a K-free resolution of D by means of an h-exact sequence, ensured by Proposition 2.2(c), it follows easily that D has an $\aleph_1$-filtration of countably generated divisible submodules, which are coherent and consequently K-free, by the preceding remark. The converse is obvious. ∎

The next result improves on Proposition 2.3 for valuation domains R with p.d. $Q = 1$, showing that K can be replaced by arbitrary torsion divisible coherent module.

PROPOSITION 3.2. *For a valuation domain R such that* p.d. $Q = 1$, *the following conditions are equivalent*:

(1) *R is antimaximal*;

(2) *if J is a nonzero ideal of R and D is a nonzero torsion divisible coherent module such that* $\mathrm{Ext}_R^1(R/J, D) = 0$, *then* $J \cong R$.

Proof. (1) $\Rightarrow$ (2). Arguing as in the proof of Proposition 2.3, condition (1) is equivalent to the existence of a 2-generated indecomposable module M that fits in a pure-exact sequence $0 \to R/aR \to M \to R/J \to 0$. The inclusion $R/aR \leq D$ gives rise to the push-out

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/aR & \longrightarrow & M & \longrightarrow & R/J & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & X & \longrightarrow & R/J & \longrightarrow & 0. \end{array}$$

Assume by contradiction that the bottom exact sequence is splitting. Then, up to isomorphisms, M is contained in $D_0 \oplus R/J$ where D_0 is a countably generated divisible submodule of D (making use of the fact that D is an epimorphic image of a K-free module). Now D_0 is K-free, since it is countably generated and coherent; thus the finitely generated module M is contained in the direct sum of a K-free module of finite K-rank and of the module R/J. Thus M is decomposable by Matlis [M1] (see also [FS1]). Therefore $\mathrm{Ext}^1_R(R/J, D) \neq 0$, as desired.

(2) ⇒ (1). Trivial. ∎

The connection of divisible and K-free modules with the class $^{\perp}K$, and so with the cotorsion theory $\mathcal{G}_K$, is clarified by the next result.

PROPOSITION 3.3. *Let R be a valuation domain with* p.d. $Q = 1$ *and* gl.d. $R = 2$. *Then the following facts are equivalent*:

(a) $\mathcal{P}_1 =^{\perp} K$;

(b) *every divisible torsion module $D \in^{\perp} K$ has projective dimension* ≤ 1, *i.e., D is K-free*;

(c) $K^{(\lambda)} \in (^{\perp}K)^{\perp}$ *for all cardinals* λ.

Proof. That $(a) \Rightarrow (b)$ is trivial.

(b) ⇒ (a) We must show that, if T is an arbitrary torsion module in $^{\perp}K$, then p.d. $T = 1$. From Theorem 3.4 in [T] we know that the cotorsion theory $\mathcal{C}_\delta = (\mathcal{P}_1, \mathcal{D})$ is complete; i.e., there exists an exact sequence $0 \to T \to D \to D/T \to 0$ with D divisible and p.d. $(D/T) = 1$. We deduce the exact sequence

$$0 = \mathrm{Ext}^1_R(D/T, K) \to \mathrm{Ext}^1_R(D, K) \to \mathrm{Ext}^1_R(T, K) = 0,$$

whence $\mathrm{Ext}^1_R(D, K) = 0$, so the hypothesis ensures that p.d. $D = 1$. Therefore p.d. $T \leq 1$, as desired.

(a) ⇒ (c) is obvious, since $K^{(\lambda)}$ is divisible.

(c) ⇒ (b) Let D be a divisible torsion module in $^{\perp}K$. By Proposition 2.2(c), there exists an h-exact sequence $0 \to K^{(\alpha)} \to K^{(\beta)} \to D \to 0$, which splits by hypothesis. Therefore, D is K-free. ∎

At this point it is natural to ask whether the three equivalent conditions in Proposition 3.3 are always satisfied for an IC-domain. We will focus on condition (b), dealing with divisible modules. For these modules we introduce two notions which resemble that of λ-free and strongly λ-free abelian groups, for λ an infinite cardinal (see [EM, p. 87]). Recall that, given a cardinal λ, a module is *λ-generated* (resp., *$<\lambda$-generated*) if the minimal cardinality of generating systems of M equals λ (resp., is strictly smaller than λ).

DEFINITION. A torsion divisible module D over a valuation domain R is said to be $(\lambda - K)$-*free*, where λ is an infinite cardinal, if any divisible submodule M of D, which is $<\lambda$-generated, is K-free.

If p.d. $Q = 1$, then in view of the existence of the exact sequence (1) for a torsion divisible module D, D is $(\lambda - K)$-free exactly if every submodule N which is $<\lambda$-generated is contained in a K-free submodule M of D which is still $<\lambda$-generated. Note that a torsion divisible module is $(\aleph_1 - K)$-free exactly if it is coherent.

DEFINITION. A torsion divisible module D over a valuation domain R is said to be *strongly* $(\lambda - K)$-*free*, where λ is an infinite cardinal, if it is $(\lambda - K)$-free and any submodule M of D, which is $<\lambda$-generated, is contained in a $<\lambda$-generated K-free submodule D' of D such that D/D' is $(\lambda - K)$-free.

Remark. A remark on the preceding definitions is in order. One can define, for arbitrary modules, the notions of $(\lambda - \mathcal{P}_1)$-modules and strongly $(\lambda - \mathcal{P}_1)$-modules, instead of those of $(\lambda - K)$-free and strongly $(\lambda - K)$-free modules, respectively, which are given only for torsion divisible modules. However, since we use these concepts only for divisible modules, we prefer to use the preceding definitions and leave these more general notions to subsequent investigation.

Obviously, K-free modules are strongly $(\lambda - K)$-free for every infinite cardinal λ. Examples of torsion divisible modules which fail to be $(\aleph_1 - K)$-free are the torsion submodules of the infinite direct products of copies of K; in fact, they are not coherent.

Our next goal is to exhibit an $\aleph_1$-generated strongly $(\aleph_1 - K)$-free torsion divisible module of projective dimension 2 over valuation domains R with gl.d. $R = 2$ and p.d. $Q = 1$. We need two lemmas.

LEMMA 3.4. *Let R be a valuation domain with* p.d. $Q = 1$ *and* gl.d. $R = 2$. *Let $B = \bigoplus_{n\in\omega} K_n$, with $K_n \cong K$ for all n. Then there exists a K-free module T of countable K-rank containing B, such that* p.d. $(T/B) = 2$ *and $T/\bigoplus_{n\leq h} K_n$ is K-free for all $h \in \omega$.*

Proof. Let J be a nonprincipal ideal of R and $0 \neq r \in J$. Let $rR = r_0R < r_1R < r_2R < \cdots < r_nR < \cdots$ be a countable ascending chain of ideals of R such that $\bigcup_{n\in\omega} r_nR = J$, and let $a_{n+1}r_{n+1} = r_n$ for suitable $a_{n+1} \in R$ and for all $n \in \omega$. Consider the submodule $r^{-1}J$ of Q; note that p.d.$(Q/r^{-1}J) = 2$. For every $n \in \omega$, let $t_n = r^{-1}r_n + R \in K$ and denote by $(t_n)_m$ the element t_n when considered in the copy K_m of K in B. Let now K_ω be another copy of K and consider the module T generated by B and K_ω, subject to the relations

$$(t_n)_\omega = (t_n)_0 + (a_1t_n)_1 + (a_1a_2t_n)_2 + \cdots + (a_1a_2\cdots a_{n-1}t_n)_{n-1}$$

for every $n \in \omega$. It is easy to verify that B embeds in T and that $T/B \cong Q/r^{-1}J$. T is clearly countably generated; every finite set $\{y_1, \ldots, y_n\}$ of elements of T generates a finitely presented submodule; in fact, the elements y_i belong to $\bigoplus_{n \le h} K_n + K_\omega$ for some $h \in \omega$, thus there is only a finite number of relations between them. Thus T is a coherent, whence it is K-free, being countably generated. It is pretty clear that, for every $h \in \omega$, $T/(\bigoplus_{n \le h} K_n)$ is isomorphic to T, hence it is K-free. ∎

By a slight elaboration of the preceding result, we obtain the second lemma.

LEMMA 3.5. *If R is a valuation domain with* p.d. $Q = 1$ *and* gl.d. $R = 2$, *then there exist K-free modules $B' < T'$ of countable K-rank, such that*

(a) *B' is the union of a countable ascending chain of submodules $\{C_n\}_{n \in \omega}$, each one of countable K-rank;*

(b) *C_{n+1}/C_n and T'/C_n are K-free of countable K-rank for all $n \in \omega$;*

(c) p.d.$(T'/B') = 2$.

Proof. For every $n \in \omega$, let A_n be a K-free module of K-rank $\aleph_0$, and let $A = \bigoplus_{n \in \omega} A_n$. Consider $B_n = \bigoplus_{m \le n} K_m$, and define $C_n = (\bigoplus_{m \le n} A_m) \oplus B_n$. Set $B = \bigcup_{n \in \omega} B_n$ and define T to be the module constructed in Lemma 3.4. Then $T' = A \oplus T$ and $B' = \bigcup_{n \in \omega} C_n$ obviously satisfy the required conditions. ∎

We are now ready for the main construction of this section, which imitates the proof of Theorem 1.3 in [EM, p. 183]. Since we make use of homological results typical for our context, for the sake of completeness we give all the details of the construction.

THEOREM 3.6. *Let R be a valuation domain with* p.d. $Q = 1$ *and* gl.d. $R = 2$. *There exists an $\aleph_1$-generated strongly $(\aleph_1 - K)$-free divisible torsion module of projective dimension* 2.

Proof. We define by induction a continuous well-ordered ascending chain of divisible modules $\{D_\nu\}_{\nu < \aleph_1}$ satisfying the following conditions for every $\nu < \aleph_1$:

(1) D_ν is a countably generated K-free module;

(2) if ν is a successor ordinal, D_μ/D_ν is K-free for every $\nu < \mu < \aleph_1$;

(3) if ν is a limit ordinal, p.d. $(D_{\nu+1}/D_\nu) = 2$.

Assume that we have done the wanted construction. Let $D = \bigcup_{\nu < \aleph_1} D_\nu$. Then D is clearly an $\aleph_1$-generated divisible torsion module. Property (2) guarantees that D is strongly $(\aleph_1 - K)$-free. By a well-known result by Eklof [E2] (see also [FS1, p. 75]), property (3) ensures that p.d. $D = 2$. Thus it remains to show that the wanted construction is possible.

Assume D_μ has been defined for every $\mu < \delta$, for some $\delta < \aleph_1$, in such a way that the properties (1)–(3) hold for every $\nu < \mu < \delta$. If δ is a limit ordinal, we obviously define $D_\delta = \bigcup_{\mu<\delta} D_\mu$. We have to show that $\{D_\mu\}_{\mu\leq\delta}$ satisfies the properties (1)–(3). Property (3) is clearly satisfied. To check Property (2) we have to show that D_δ/D_ν is K-free for every successor ordinal $\nu < \delta$. Let τ_n be an increasing sequence of successor ordinals such that $\sup \tau_n = \delta$. Then $D_\delta = \bigcup_n D_{\tau_n}$ and $D_\delta/D_\nu = \bigcup_n D_{\tau_n}/D_\nu$. By the inductive hypothesis, D_{τ_n}/D_ν and $D_{\tau_{n+1}}/D_{\tau_n}$ are K-free for every n, hence by Auslander's Lemma (see [FS1, p. 73]) D_δ/D_ν has projective dimension 1; i.e., it is K-free. By the same argument, D_δ is K-free too, thus (1) is satisfied.

If $\delta = \nu + 1$ and ν is nonlimit, let $D_\delta = D_\nu \oplus K^{(\aleph_0)}$. In this case the properties (1)–(3) are easily checked.

It remains to consider the case $\delta = \nu + 1$ for a limit ordinal ν. Let again τ_n be an increasing sequence of successor ordinals such that $\sup \tau_n = \nu$. By the inductive hypothesis, D_{τ_n} and $D_{\tau_{n+1}}/D_{\tau_n}$ are K-free modules of K-rank $\aleph_0$. Consider B' and C_n as given in Lemma 3.5. For every n, there is an isomorphism ϕ_n: $D_{\tau_n} \to C_n$. Since $D_{\tau_{n+1}}/D_{\tau_n}$ is K-free, it is possible, by Proposition 2.2(a), to define an isomorphism ϕ: $D_\nu = \bigcup_n D_{\tau_n} \to B'$ inducing ϕ_n on D_{τ_n} for all n. Let T' be as in Lemma 3.5 and define D_δ as the pushout of the diagram

$$\begin{array}{ccc} B' & \xrightarrow{\phi^{-1}} & D_\nu \\ \downarrow & & \downarrow \\ T' & \longrightarrow & D_\delta. \end{array}$$

Properties (1) and (3) are clearly satisfied. To check (2), we have to show that D_δ/D_μ is K-free for every successor ordinal $\mu < \nu$. Choose $\tau_n > \mu$; then D_{τ_n}/D_μ is K-free by induction and D_δ/D_{τ_n} is K-free, since it is isomorphic to T'/C_n; thus D_δ/D_μ is K-free too, by Proposition 2.2(a). ∎

The remaining part of this section is devoted to illustrating a version of Shelah's Singular Compactness Theorem [Sh2] adapted to our setting. Since this theorem has been applied to different situations, we only outline the proof.

THEOREM 3.7 (Singular Compactness). *Let λ be a singular cardinal and R a valuation domain with* p.d. $Q = 1$ *and* gl.d. $R = 2$. *Then a λ-generated torsion divisible module is K-free, provided that its $<\lambda$-generated divisible submodules are K-free.*

Proof. We refer the reader to Paragraph 3.6 in Chapter IV of [EM] for the general version of the Singular Compactness Theorem adapted to modules that we use here. We must specify the class of "free" modules $\mathcal{F}$ considered there and their "bases." Fix the cardinal $\mu = \aleph_0$. It remains to check that the properties (a)–(e) are satisfied.

Define $\mathcal{F}$, the class of "free" modules, as the class of K-free modules. Fix a strictly ascending chain $\{x_n R\}_{n\in\omega}$ of submodules of K whose union is K, so that $S = \{x_n\}_{n\in\omega}$ is a system of generators for K. For every $M \in \mathcal{F}$, we define a "basis" X of M in the following way.

Pick a decomposition of $M \cong \bigoplus_{i\in I} K_i$, with $K_i \cong K$ for all $i \in I$; pick a generating system S_i in each copy K_i corresponding to the system S of K via the isomorphism $K_i \cong K$; and then set

$$X = \left\{ Y | Y = \bigcup_{i\in J} S_i \text{ for some } J \subseteq I \right\}.$$

The five properties (a)–(e) required in Paragraph 3.6 of Chapter IV in [EM] are evidently satisfied in our context, so the theorem is proved once we have observed that λ-"free" modules (obtained by substituting "free" for free in Definition 1.1 in [EM, p. 83]) agree with the $(\lambda - K)$-free modules defined above. ∎

4. THE INDEPENDENCE RESULTS ON $\mathcal{G}_K$

In order to determine the cotorsion theory $\mathcal{G}_K$, in view of Proposition 3.3 we are led to consider the projective dimensions of the torsion divisible modules in ${}^{\perp}K$. We need two preliminary results, which are adapted from analogous results for abelian groups (for the next one see Lemma 1.4 in [EM, p. 346]).

LEMMA 4.1. *Let R be a valuation domain with* p.d. $Q = 1$ *and* gl.d. $R = 2$. *Let λ be an uncountable regular cardinal, and T a divisible torsion R-module with a λ-filtration $\{T_\alpha\}_{\alpha<\lambda}$ of divisible submodules. There exists an h-exact sequence*

$$0 \to H \to F \xrightarrow{\phi} T \to 0,$$

where H and F are K-free, such that $F = \bigoplus_{\alpha<\lambda} F_\alpha$, $H = \bigoplus_{\alpha<\lambda} H_\alpha$; furthermore, for all $\alpha < \lambda$, F_α and H_α are $<\lambda$-generated, and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{\beta<\alpha} H_\beta & \longrightarrow & \bigoplus_{\beta<\alpha} F_\beta & \xrightarrow{\phi_\alpha} & T_\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H & \longrightarrow & F & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

with h-exact rows, where the vertical maps are inclusions.

Proof. We define the modules F and H by transfinite induction. By Proposition 2.2(c), there exists an h-exact sequence

$$0 \to H_0 \to F_0 \to T_1 \to 0,$$

where H_0 and F_0 are K-free. Assume that F_β and H_β have been defined for all $\beta < \alpha$ in such a way that, if $\beta < \gamma < \alpha$, ϕ_γ is an extension of ϕ_β and the exact sequence

$$0 \to \bigoplus_{\beta<\gamma} H_\beta \to \bigoplus_{\beta<\gamma} F_\beta \xrightarrow{\phi_\gamma} T_\gamma \to 0 \tag{2}$$

is h-exact. If α is a limit ordinal, ϕ_α is the union of the maps ϕ_β for $\beta < \alpha$. The exact sequence

$$0 \to \bigoplus_{\beta<\alpha} H_\beta \to \bigoplus_{\beta<\alpha} F_\beta \xrightarrow{\phi_\alpha} T_\alpha \to 0$$

is h-exact, since in the exact sequence

$$\mathrm{Hom}_R\Bigl(K, \bigoplus_{\beta<\alpha} F_\beta\Bigr) \to \mathrm{Hom}_R(K, T_\alpha) \to \mathrm{Ext}^1_R\Bigl(K, \bigoplus_{\beta<\alpha} H_\beta\Bigr)$$

the last term vanishes, by Proposition 2.2(a).

Suppose now $\alpha = \gamma + 1$. Fix a h-exact sequence

$$0 \to D_\gamma \to F_\gamma \xrightarrow{\psi_\gamma} T_{\gamma+1} \to 0,$$

where D_γ and F_γ are K-free $<\lambda$-generated. Then we define

$$\phi_{\gamma+1} : \Bigl(\bigoplus_{\beta<\gamma} F_\beta\Bigr) \oplus F_\gamma \to T_{\gamma+1}$$

by setting $\phi_{\gamma+1} = \phi_\gamma \oplus \psi_\gamma$. We have

$$\mathrm{Ker}(\phi_{\gamma+1}) = \Bigl\{x - y \Bigm| x \in \bigoplus_{\beta<\gamma} F_\beta, y \in F_\gamma : \phi_\gamma(x) = \psi_\gamma(y)\Bigr\}.$$

If $x - y \in \mathrm{Ker}(\phi_{\gamma+1})$, then obviously $y \in G_\gamma = \psi_\gamma^{-1} T_\gamma$. But G_γ is divisible and contained in F_γ, consequently, by Proposition 2.2(b) and fact (ii) in Section 2, G_γ is K-free. The h-exactness of the sequence (2) implies that there exists a homomorphism η: $G_\gamma \to \bigoplus_{\beta<\gamma} F_\beta$ such that $\psi_\gamma | G_\gamma = \phi_\gamma \eta$. Now define $H_\gamma = \{\eta(y) - y | y \in G_\gamma\}$. Clearly $H_\gamma \le \mathrm{Ker}(\phi_{\gamma+1})$. A straightforward computation shows that $\mathrm{Ker}(\phi_{\gamma+1}) = (\bigoplus_{\beta<\gamma} H_\beta) \oplus H_\gamma$; the new exact sequence

$$0 \to \bigoplus_{\beta\le\gamma} H_\beta \to \bigoplus_{\beta\le\gamma} F_\beta \xrightarrow{\phi_{\gamma+1}} T_{\gamma+1} \to 0$$

is obviously h-exact. ∎

Remark. It is possible to give an alternative proof of Lemma 4.1 for arbitrary torsion modules, using the torsion-free covers of the submodules in the filtration instead of the h-exact sequences for divisible modules.

The next lemma reproduces Lemma 1.5 in [EM, p. 347], stated here with slightly different hypotheses; but the proof is the same, so it is omitted.

LEMMA 4.2. *Let F be a module containing two submodules H_0 and H_1 such that F/H_0 is K-free and $\mathrm{Ext}^1_R(F/(H_0+H_1), K) \neq 0$. Then there exist two homomorphisms $\phi_0, \phi_1\colon H_1 \to K$ such that no homomorphism $\phi\colon H_0 \to K$ extends to a homomorphism $\psi_0\colon F \to K$ such that $\psi_0|H_0 = \phi_0$ and to a homomorphism $\psi_1\colon F \to K$ such that $\psi_1|H_1 = \phi_1$.*

We can now prove the first consistency result.

THEOREM 4.3. *Let R be an IC-domain. Assuming $V = L$, every R-module in the class ${}^{\perp}K$ has projective dimension ≤ 1. Equivalently, the cotorsion theory $\mathcal{G}_K$ coincides with the cotorsion theory $\mathcal{C}_\delta$.*

Proof. In view of Proposition 3.3, it is enough to prove that if T is a divisible torsion module satisfying $\mathrm{Ext}^1_R(T, K) = 0$, then p.d. $T = 1$. Assume, by way of contradiction, that p.d. $T = 2$ and T has minimal cardinality λ among the divisible torsion modules in ${}^{\perp}K$ with this property. λ is uncountable, otherwise T, which is a coherent module by Proposition 2.8, would have projective dimension 1; furthermore, it is regular, by Theorem 3.7. Choose a λ-filtration $\{T_\alpha\}_{\alpha<\lambda}$ of divisible submodules and an h-exact sequence $0 \to H \to F \to T \to 0$ as in Lemma 4.1. The subset of λ

$$E = \left\{\alpha < \lambda \mid T_{\alpha+1}/T_\alpha \text{ is not } K\text{-free}\right\}$$

is stationary in λ, and, by the minimality of λ, $\mathrm{Ext}^1_R(T_{\alpha+1}/T_\alpha, K) \neq 0$. Since $T_{\alpha+1}/T_\alpha$ is isomorphic to $F/(H_0 + H_1)$, where

$$F = \bigoplus_{\beta\leq\alpha} F_\beta, \qquad H_0 = \bigoplus_{\beta<\alpha} F_\beta, \qquad H_1 = H_\alpha,$$

we can apply Lemma 4.2 to obtain two homomorphisms $\phi_{0\alpha}, \phi_{1\alpha} : H_1 \to K$ satisfying the conditions in that lemma. Define a partition $\mathcal{P}$ such that, for each $\alpha \in E$ and each homomorphism $h : \bigoplus_{\beta<\alpha} F_\beta \to K$, $\mathcal{P}(h) = 0$ exactly if h does not extend to a homomorphism $\bigoplus_{\beta\leq\alpha} F_\beta \to K$ extending $\phi_{0\alpha}$. Note that $\mathcal{P}(h) = 1$ implies that h does not extend to a homomorphism extending $\phi_{1\alpha}$. V = L implies the weak diamond principle $\Phi_\lambda(E)$, which ensures the existence of a weak diamond function ρ for the partition $\mathcal{P}$ (see [EM, pp. 142–143]). Consider the homomorphism $\phi\colon H = \bigoplus_{\alpha<\lambda} H_\alpha \to K$ defined as $\phi = \bigoplus_{\alpha<\lambda} \phi_{\rho(\alpha)\alpha}$. Then ϕ does not extend to a homomorphism from $F = \bigoplus_{\alpha<\lambda} F_\alpha$ to K (the proof is as in [EM, XII.1.6]). From the exact sequence

$$\mathrm{Hom}_R(F, K) \to \mathrm{Hom}_R(H, K) \to \mathrm{Ext}^1_R(T, K) \to \mathrm{Ext}^1_R(F, K) = 0$$

we deduce that $\mathrm{Ext}^1_R(T, K) \neq 0$, a contradiction. ∎

An immediate consequence of the preceding theorem is the following.

COROLLARY 4.4. *Let R be an IC-domain. Assuming $V = L$, the cotorsion theory $\mathcal{G}_\partial$ generated by Fuchs' module ∂ coincides with the cotorsion theory $\mathcal{C}_\partial$ cogenerated by ∂.*

Proof. Since $\mathcal{G}_K = \mathcal{C}_\partial$, by Theorem 4.3 it is enough to prove that, given a module M, $\mathrm{Ext}^1_R(M, \partial) = 0$ if and only if $\mathrm{Ext}^1_R(M, K) = 0$. But for an IC-domain we have that $\partial \cong (\oplus Q) \oplus (\oplus K)$, hence $\mathrm{Ext}^1_R(M, \partial) = 0$ if and only if $\mathrm{Ext}^1_R(M, \oplus K) = 0$. Thus the conclusion follows from Proposition 3.3. ∎

Theorem 4.3 shows that, under V = L, the cotorsion theory $\mathcal{G}_K$ is complete, since it is cogenerated by a single module (see [ET]: for the definition of the complete cotorsion theory we refer the reader to [T]). So the following question naturally arises.

QUESTION. Is the cotorsion theory $\mathcal{G}_K$ always complete?

We now address ourselves to the opposite consistency result. Following Eklof, we use a special case of Martin's Axiom in the form of a theorem, as illustrated in Theorem 7.1 in [E1], which we recall here.

THEOREM MA. *Assume Martin's Axiom. Let A and B be sets of cardinality $<2^{\aleph_0}$, and let $\mathcal{P}$ be a family of functions with the following properties*:

[P1] *for every $f \in \mathcal{P}$, f is a function from a subset of A into B*;

[P2] *for every $a \in A$ and every $f \in \mathcal{P}$, there exists a $g \in \mathcal{P}$ such that the domain of g contains both a and the domain of f, and g extends f*;

[P3] *for every uncountable subset $\mathcal{P}'$ of $\mathcal{P}$, there exist two different elements f_1, f_2 in $\mathcal{P}'$ and an element $f_3 \in \mathcal{P}$ extending both f_1 and f_2.*

Then there exists a function $g\colon A \to B$ such that for any finite subset F of A there is $f \in \mathcal{P}$ defined on F such that g extends f. ∎

We need also the following technical lemma.

LEMMA 4.5. *Let R be a valuation domain with* gl.d. $R = 2$ *and* p.d. $Q = 1$. *Let D be a strongly $(\aleph_1 - K)$-free divisible torsion module. If A is an $\aleph_1$-generated submodule of D of projective dimension ≤ 1, there exists a divisible submodule C of D containing A such that* p.d. $C = 1$.

Proof. Let $\{A_\nu\}_{\nu<\aleph_1}$ be an $\aleph_1$-filtration of countably generated submodules of A. We define, by induction, a strictly increasing chain of divisible submodules $\{C_\nu\}_{\nu<\aleph_1}$ of D whose union C is the required module. Let C_0 be a countably generated divisible submodule of D containing A_0 such that D/C_0 is $(\aleph_1 - K)$-free. If ν is a limit ordinal, define $C_\nu = \bigcup_{\mu<\nu} C_\mu$; if $\nu = \delta + 1$, let C_ν be a countably generated divisible submodule of D such that $A_\delta + C_\delta \subseteq C_\nu$ and such that D/C_δ is $(\aleph_1 - K)$-free. Then, for every $\nu < \aleph_1$, $C_{\nu+1}/C_\nu$ has projective dimension 1, since it is a countably generated submodule of the $(\aleph_1 - K)$-free module D/C_ν. Thus pd.$C = 1$, by Auslander's Lemma. ∎

Using Theorem MA and the full force of the hypotheses of ICC-domains (here IC-domains are not enough), we can prove now the second consistency result.

THEOREM 4.6. *Let R be an ICC-domain. Assuming the Martin Axiom and the negation of the Continuum Hypothesis, every $\aleph_1$-generated strongly $(\aleph_1 - K)$-free R-module belongs to ${}^\perp K$. Therefore $\mathcal{G}_K$ is strictly smaller than $\mathcal{C}_\partial$.*

Proof. We follow the pattern of the proof of Theorem 7.2 in [E1]. Let D be a $\aleph_1$-generated strongly $(\aleph_1 - K)$-free R-module, and let $0 \to K \to B \xrightarrow{\pi} D \to 0$ be an exact sequence. Recall that D, as an $(\aleph_1 - K)$-free module, is coherent. We will prove that the sequence is splitting, by applying Theorem MA to a set $\mathcal{P}$ of partial splittings of π. In fact, if S is a finitely generated submodule of D, then the sequence $0 \to K \to \pi^{-1}(S) \xrightarrow{\pi} S \to 0$ splits, as S is finitely presented by the coherency of D. Let $\mathcal{P}$ be the set of all the homomorphisms $\phi: S \to B$ such that $\pi\phi = 1_S$, ranging S over the set of the finitely generated submodules of D. If we prove that $\mathcal{P}$ satisfies conditions P2 and P3 listed in Theorem MA, we will obtain a splitting homomorphism $\psi : D \to B$ for π.

We first prove that $\mathcal{P}$ satisfies P2. Let S be a finitely generated submodule of D and let $\phi: S \to B$ be such that $\pi\phi = 1_S$. It is useful, for the next purposes, to consider, instead of a single element, a finite subset F of D. Let S' be the submodule of D generated by S and F; clearly S'/S is finitely presented. By a straightforward argument one can extend ϕ to a homomorphism $\phi': S' \to B$ such that $\pi\phi' = 1_{S'}$, by using the purity of K in B and the divisibility of B. So P2 follows.

We prove now that $\mathcal{P}$ satisfies condition P3 under a particular assumption; namely, we assume that

(•) $\mathcal{P}'$ is an uncountable subset of $\mathcal{P}$ such that there exists a K-free submodule D' of D which contains the domain of every $\phi \in \mathcal{P}'$.

We remark that the K-rank of D' is necessarily $\aleph_1$, in view of the countability of K. Choose a representation of D' in the form $\bigoplus_{\nu<\omega_1} K_\nu$, with

$K_\nu \cong K$ for all ν. If S is the domain of some $\phi \in \mathcal{P}'$, let $A_S = \{\nu | \pi_\nu(S) \neq 0\}$, where $\pi_\nu : D' \to K_\nu$ is the canonical projection. A_S is finite and $\nu(S)$ is a finitely generated submodule of K. Let $\{x_n\}_{n\in\omega}$ be a set of generators of K such that $\{x_\nu R\}_{n\in\omega}$ is a strictly ascending chain of submodules of K; there exists a minimum natural number n_S such that $S \leq \bigoplus_{\nu\in A_S}(x_{n_S}R)_\nu$ (we adopt here the notation used in Lemma 4.3). Let $F_S = \{(x_{n_S})_\nu \in D' | \nu \in A_S\}$. Since we have proved that $\mathcal{P}$ satisfies P2, we may assume that $\mathcal{P}'$ is such that the domain of each $\phi' \in \mathcal{P}'$ is of the form $\bigoplus_{\nu\in A_S}(x_{n_S}R)_\nu$. Since $\mathcal{P}'$ is uncountable and A_S is finite for every S, we may assume that there exist two natural numbers n and m such that the domain of every $\phi' \in \mathcal{P}'$ is of the form $\bigoplus_{\nu\in G}(x_nR)_\nu$ for some finite subset G of ν consisting of exactly m elements. So let $\mathcal{P}' = \{\phi_\nu | \nu < \omega_1\}$, where the domain of ϕ_ν is $\bigoplus_{\mu\in G_\nu}(x_nR)_\mu$, where G_ν is a finite subset of ν such that $|G_\nu| = m$ for all ν. Let $Y_\nu = \{(x_n)_\mu | \mu \in G_\nu\}$. The set

$$\mathcal{T} = \{T \mid T \subseteq Y_\nu \text{ for uncountably many } \nu\}$$

is inductive, since Y_ν has cardinality m for every ν. Let T_0 be a maximal element of $\mathcal{T}$. Now the proof goes on exactly as the proof of Theorem 7.2 in [E1]. Denote by T_1 the submodule of D' generated by T_0. Then T_1 is a finite direct sum of copies of x_nR, hence $\operatorname{Hom}_R(T_1, K)$ is countable. If T_1 is contained in the domain of ϕ_ν and in the domain of ϕ_μ for distinct ν and μ, the restriction of $\phi_\nu - \phi_\mu$ to T_1 is different form zero only for at most countably many choices. Thus we may assume that ϕ_ν agrees with ϕ_μ for uncountably many ν and μ, such that T_0 is contained in $Y_\nu \cap Y_\mu$. Renumbering, we may assume that $T_0 \subseteq Y_0$; for every $y \in Y_0 \backslash T_0$ the set $\{\nu : y \in Y_\nu \text{ and } T_0 \subseteq Y_\nu\}$ is countable by the maximality of T_0. Hence there exists $\nu \neq 0$ such that $y \notin Y_\nu$ for every $y \in Y_0 \backslash T_0$; i.e., $Y_\nu \cap Y_0 = T_0$. Let S be the submodule of D' generated by $Y_0 \cup Y_\nu$; since ϕ_0 and ϕ_ν agree on T_0, we may define a homomorphism ψ: $S \to B$ by setting $\psi(y) = \phi_\nu(y)$ if y is an element of Y_ν and $\psi(y) = \phi_0(y)$ if y is an element of Y_0. Clearly, ψ satisfies $\pi(\psi(y)) = y$ for every $y \in S$, hence $\psi \in \mathcal{P}$. Thus condition P3 is proved under the assumption (•).

Now we prove that

(••) for any uncountable subset $\mathcal{P}'$ of $\mathcal{P}$ there exists a K-free submodule $D' \subseteq D$ of K-rank $\aleph_1$ and an uncountable subset $\mathcal{P}'' \subseteq \mathcal{P}$ such that D' contains the domain of every $\phi \in \mathcal{P}''$.

Suppose $P' = \{\phi_\nu | S_\nu \to B\}$, where S_ν is a finitely generated submodule of D and $\pi\phi_\nu = 1_S$. We may assume that $\mathcal{P}'$ is such that every $S_\nu \in \mathcal{P}'$ has length (minimal number of generators) at most m for some natural number m. Let $\mathcal{T}$ be the set of all the finitely generated submodules T of D such that $T \subseteq S_\nu$ for uncountably many ν. $\mathcal{T}$ is inductive, since the length of any $T \in \mathcal{T}$ is not greater than the length of any S_ν in which it is contained (see [FS2, XII.1.7]). So let T_0 be a maximal element of $\mathcal{T}$; we may assume that T_0 is contained in every $S_\nu \in \mathcal{P}'$. We construct now, by induction, a submodule A of D of projective dimension 1 which contains the domain of every $\phi \in \mathcal{P}''$, for some uncountable subset $\mathcal{P}'' \subseteq \mathcal{P}'$. A will be the union of a smooth ascending chain of countably generated submodules $\{A_\nu | \nu < \omega_1\}$ such that p.d.$(A_{\nu+1}/A_\nu) \leq 1$, for every ν. So p.d.$A \leq 1$ by Auslander's Lemma.

Let $A_0 = T_0$; suppose we have defined $\{A_\mu | \mu < \nu\}$ and a strictly increasing sequence of ordinals $\{\sigma_{\mu+1} | \mu < \nu\}$ such that $A_{\mu+1} \supseteq S_{\sigma_{\mu+1}}$. If ν is a limit ordinal, define $A_\nu = \bigcup_{\mu<\nu} A_\mu$. If $\nu = \delta + 1$, consider a countably generated divisible submodule C_δ of D such that $A_\delta \subseteq C_\delta$ and D/C_δ is $(\aleph_1 - K)$-free (here we use the hypothesis that D is strongly $(\aleph_1 - K)$-free). There must exist an ordinal $\lambda > \sigma_{\mu+1}$, for every $\mu < \nu$, such that $C_\delta \cap S_\lambda = T_0$; otherwise, for uncountably many λ there would exist an element $c_\lambda \in C_\delta \cap S_\lambda$ such that $c_\lambda \notin T_0$. But since C_δ is countable (because C_δ is K-free of countable K-rank) there would exist an element $c \in C_\delta \cap S_\lambda$, $c \notin T_0$, for uncountably many λ. Thus the module generated by T_0 and c would be in $\mathcal{T}$, contradicting the maximality of T_0. So there exists $\sigma_\nu > \sigma_{\mu+1}$, for every $\mu < \nu$, such that $C_\delta \cap S_{\sigma_\nu} = T_0$. Define $A_{\delta+1} = A_\delta + S_{\sigma_\nu}$. Then it is immediate to check that $A_{\delta+1} \cap C_\delta = A_\delta$. Thus $A_{\delta+1}/A_\delta$ is countably generated and isomorphic to $(A_{\delta+1} + C_\delta)/C_\delta$, which is contained in D/C_δ; hence p.d.$(A_{\delta+1}/A_\delta) \leq 1$ and p.d. $A \leq 1$. To conclude the proof, it remains to show that A is contained in a divisible submodule $D' \leq D$ with p.d. $D' \leq 1$. This is guaranteed by Lemma 4.5. ∎

Recently, Eklof and Shelah proved that the cotorsion theory generated by $\mathbb{Z}$: $\mathcal{G}_\mathbb{Z} = ({}^\perp\mathbb{Z}, ({}^\perp\mathbb{Z})^\perp)$ cannot be cogenerated by a set of groups, provided that the uniformization principle UP (see [EM]) is assumed. Analogously, we ask

QUESTION. Does the uniformization principle UP imply that the cotorsion theory $\mathcal{G}_K$ cannot be cogenerated by a set of modules?

5. APPLICATIONS TO $\mathcal{W}^1$- AND $\mathcal{W}^2$-MODULES

We explain now why it is conceivable that an independence result appears in the investigation of the cotorsion theory $\mathcal{G}_K$. Given any module M, Lemma 2.1 shows that $\mathrm{Ext}^1_R(M, K) \cong \mathrm{Ext}^2_R(M, R)$. Call the module M a *$\mathcal{W}^2$-module* (where $\mathcal{W}$ stays for Whitehead) if $\mathrm{Ext}^2_R(M, R) = 0$. Then the equality ${}^\perp K = \mathcal{P}_1$ amounts to saying that every $\mathcal{W}^2$-module has projective dimension ≤ 1. In this form, our problem of determining the class ${}^\perp K$ can be viewed as a *shifted version* of the Whitehead problem, which asks whether every $\mathcal{W}^1$-module has projective dimension 0 (where for a $\mathcal{W}^1$-module we mean a Whitehead module). Thus the results of the preceding section can be reformulated in the previous terminology as follows.

THEOREM 5.1. (a) *Let R be an IC-domain. Assuming* $V = L$, *an R module is a* $\mathscr{W}^2$*-module if and only if it has projective dimension* ≤ 1.

(b) *Let R be an ICC-domain. Assuming Martin's Axiom and the negation of the Continuum Hypothesis, there exists a* $\mathscr{W}^2$*-module of projective dimension* 2.

From the results on Whitehead modules in [BFS] (see also Theorem 10.10 in Chapter XVI of [FS2]), we derive immediately the following

COROLLARY 5.2. *Let R be a countable valuation domain. Assuming* $V = L$, *an R-module is both a* $\mathscr{W}^1$*- and a* $\mathscr{W}^2$*-module if and only if it is free.*

Consider now a module M over an arbitrary valuation domain R, and pick an exact sequence

$$O \to H \to F \to M \to 0$$

with F free. Then $\mathrm{Ext}^1_R(H, R) \cong \mathrm{Ext}^2_R(M, R)$, whence M is a $\mathscr{W}^2$-module if and only if H is a $\mathscr{W}^1$-module. There follows that every Whitehead R-module which is a submodule of a free module is projective if and only if every $\mathscr{W}^2$-module has projective dimension ≤ 1.

Consequently, every independence result on Whitehead modules which are submodules of free modules translates into an independence result on the class ${}^{\perp}K$ (hence on the cotorsion theory $\mathscr{G}_K$), and vice versa. In particular, from Theorems 4.3 and 4.6 we get the following

THEOREM 5.3. (a) *Let R be an IC-domain. Assuming* $V = L$, *every Whitehead R-module which is a submodule of a free module is free.*

(b) *Let R be an ICC-domain. Assuming Martin's Axiom and the negation of the Continuum Hypothesis, there are nonfree Whitehead R-modules which are submodules of free modules.*

In connection with Theorem 5.3, we recall that Theorem 6.7 in [BFS] states that, assuming V = L, a torsion-free module over a countable valuation domain is Whitehead exactly if it is free. Note, however, that an ICC-domain can have cardinality $2^{\aleph_0}$ (see Example 2.4).

Furthermore, Proposition 4.2 in [ES] states that, over any domain $R \neq Q$, it is consistent with ZFC + GCH that, given any R-module A, there exists a nonprojective module M such that $\mathrm{Ext}^1_R(M, A) = 0$. Setting $A = R$, one obtains a nonprojective Whitehead module M. Note that the module M is not a submodule of a free module, since submodules of free modules are projective whenever gl.d. $R = 1$.

There is another close connection, besides that of $\mathscr{W}^2$-modules, between Whitehead modules and the class ${}^\perp K$ of the cotorsion theory $\mathscr{G}_K$. The key of this connection is the classical Matlis category equivalence between torsion h-divisible modules and complete torsion-free modules, induced by the two functors $\mathrm{Hom}_R(K, \bullet)$ and $K \otimes_R \bullet$.

PROPOSITION 5.4. *Let R be a complete valuation domain with* p.d. $Q = 1$. *Let D be a torsion divisible module and M a torsion-free complete module corresponding to each other in the Matlis equivalence. Then D belongs to ${}^\perp K$ if and only if M is a $\mathscr{W}^1$-module.*

Proof. Assume that D belongs to ${}^\perp K$. Consider an exact sequence

$$0 \to R \to X \to \mathrm{Hom}_R(K, D) = M \to 0.$$

The middle term X is clearly torsion-free complete; applying the functor $K \oplus_R \bullet$ we obtain the splitting exact sequence

$$0 \to K \to K \oplus_R X \to D \to 0.$$

Applying to this sequence the functor $\mathrm{Hom}_R(K, \bullet)$ we obtain the original sequence, which must split, too. The converse is similar. ∎

Proposition 5.4 enables us to translate results on the class ${}^\perp K$ to results on $\mathscr{W}^1$-modules. We provide in the following some examples of this application.

PROPOSITION 5.5. *Let R be a complete IC-domain. Then*

(a) *$\prod_\alpha R$ is not a $\mathscr{W}^1$-module for all infinite cardinals α*; *and*
(b) *torsion-free pure-injective modules are not $\mathscr{W}^1$-modules.*

Proof. (a) We have that $\prod_\alpha K = D \oplus T$, where D is divisible torsion-free and T is torsion divisible. Note that $\prod_\alpha R \cong \mathrm{Hom}_R(K, \prod_\alpha K) \cong \mathrm{Hom}_R(K, T)$, so the conclusion follows from Proposition 5.4 and Proposition 2.11.

(b) It is well known that in the Matlis equivalence torsion-free pure-injective modules correspond to injective torsion modules; whence the statement follows from Corollary 2.10 and Proposition 5.4. ∎

The independence results on the cotorsion theory $\mathscr{G}_K$ obtained in Section 4 are translated in the next two theorems.

THEOREM 5.6. *Let R be a complete IC-domain. Assuming $V = L$, a torsion-free complete R-module is a $\mathscr{W}^1$-module if and only if it is isomorphic to the completion of a free R-module.*

Proof. By Theorem 4.3 and Proposition 5.4, the torsion-free complete $\mathcal{W}^1$-modules are exactly the Matlis equivalent of the K-free modules, hence of the form $\mathrm{Hom}_R(K, \oplus_\alpha K)$; from the exact sequence

$$0 \to \oplus_\alpha R \to \oplus_\alpha Q \to \oplus_\alpha K \to O$$

we derive the isomorphism $\mathrm{Hom}_R(K, \oplus_\alpha K) \cong \mathrm{Ext}^1_R(K, \oplus_\alpha R)$, which is the completion of the free module $\oplus_\alpha R$ in the R-topology. ∎

THEOREM 5.7. *Let R be a complete ICC-domain. Assuming Martin's Axiom and the negation of the Continuum Hypothesis, there exists a torsion-free complete $\mathcal{W}^1$-module which is not isomorphic to the completion of a free R-module.*

Proof. An immediate consequence of Theorem 4.6 and Proposition 5.4. ∎

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