An Application of Hadamard Multiplication to Operators on Weighted Hardy Spaces

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ABSTRACT

The monomials $z^n$ form an orthogonal basis for the weighted Hardy Hilbert spaces. It is shown that versions of some natural operators defined on different weighted Hardy spaces are related (in the corresponding orthonormal bases) by a Hadamard product map. Cases in which the Hadamard multiplier is positive are established, and the results used to derive new norm and spectral information for operators on the weighted spaces from well-known information about operators on the usual (unweighted) Hardy space.

Many operators associated with analytic functions have versions on each member of a family of Hilbert spaces. In this article, we show how the different versions of some of these operators are related by a positive Hadamard multiplier, and we show how to use known properties of an operator on one of these spaces and standard inequalities about Hadamard multipliers to establish similar properties of versions on other spaces.

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Let $l^2(\mathbb{N})$ denote the Hilbert space of sequences of complex numbers that are square summable; for sequences $x = (x_0, x_1, x_2, \ldots)$ and $y$ in $l^2(\mathbb{N})$, the inner product is given by

$$\langle x, y \rangle = \sum_{k=0}^{\infty} x_k \overline{y_k},$$

and the norm is given by $\|x\|^2 = \langle x, x \rangle$. The vectors $e_0 = (1, 0, 0, \ldots)$, $e_1 = (0, 1, 0, \ldots)$, ... are a convenient orthonormal basis for $l^2(\mathbb{N})$, and every bounded operator $A$ (some unbounded ones, too) on $l^2(\mathbb{N})$ has an associated infinite matrix $A_{ij} = \langle Ae_j, e_j \rangle$. [The Hilbert space $l^2(\mathbb{N})$ is in the background in all the discussions of this paper, and matrices for all operators will be with respect to this orthonormal basis.] We let $U$ denote the unweighted unilateral shift on $l^2(\mathbb{N})$ (that is, the operator $U$ is given by $Ue_k = e_{k+1}$), and for a given bounded sequence of positive numbers $w_k$, we let $W$ denote the weighted unilateral shift given by $We_k = w_ke_{k+1}$. We consider bounded sequences because $\|W\| = \sup w_k$, and we consider positive sequences because a weighted shift with arbitrary weights $w_k$ is unitarily equivalent to the shift with weights $|w_k|$. An easy calculation shows that $W*e_0 = 0$ and $W*e_k = w_{k-1} e_{k-1}$ for $k \geq 1$. Weighted shifts are well-studied, important operators for both theory and applications.

Since all Hilbert spaces with countably infinite (orthogonal) dimension are “the same” as $l^2(\mathbb{N})$ from the point of view of Hilbert space theory, it might seem superfluous to discuss any other spaces. However, many interesting and important operators are motivated and better understood if we consider other presentations of Hilbert space. In an important class of these Hilbert spaces, the weighted Hardy spaces, the vectors are functions analytic on the open unit disk, $D$. A function analytic on the unit disk has a power series expansion $f(z) = \sum_{k=0}^{\infty} f(k) z^k$ that converges in the unit disk. A suitable sequence $\beta$ of positive numbers can be used to define an inner product on analytic functions by

$$\langle f, g \rangle = \sum f(k) \overline{g(k)} \beta(k)^2,$$

and the weighted Hardy space $H_\beta^2$ is

$$H_\beta^2 = \{ f : \|f\|_\beta^2 < \infty \}.$$

None of the spaces $H_\beta^2$ contains all functions holomorphic in the unit disk, but all contain the analytic polynomials as a dense linear subspace. More-
over, the monomials are an orthogonal set and we have $\|z^n\| = \beta(n)$. It follows that the correspondence $e_n \leftrightarrow z^n/\beta(n)$ establishes an isometric isomorphism between $l^2(\mathbb{N})$ and $H^2_{\beta}$.

The classical Hardy, Bergman, and Dirichlet spaces on the unit disk, usually defined by integrals, may be thought of in this way. The usual Hardy space $H^2$ has norm defined by

$$
\|f\|^2 = \lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta / 2\pi
$$

and is the unweighted case $\beta(n) = 1$. It follows from this definition and convergence theorems for integrals that $H^2$ may be thought of as the closure of the analytic polynomials in $L^2$ of the circle (with measure $d\theta / 2\pi$), or as the $L^2$ functions whose negative Fourier coefficients are all zero. From this point of view, $H^2$ is obtained by projecting square integrable functions onto their analytic parts. The Bergman space has norm defined by

$$
\|f\|^2 = \int_D |f(x + iy)|^2 \, dx \, dy / \pi
$$

and is the case $\beta(n) = (n + 1)^{-1/2}$. The Bergman space may be thought of as the closure of the analytic polynomials in the space $L^2(D, dA)$ of functions square integrable with respect to normalized area measure $dA = dx \, dy / \pi$ and may be obtained by projecting these functions onto their analytic parts. The Dirichlet space has norm defined by

$$
\|f\|^2 = \sum_{k=0}^{\infty} |\hat{f}(k)|^2 + \int_D |f'(x + iy)|^2 \, dx \, dy / \pi
$$

and is the case $\beta(n) = (n + 1)^{1/2}$.

The usual Hardy space is associated with the shift $U$, and in his article on weighted shifts [10], Shields shows how a weighted shift $W$ is associated with the weighted Hardy space $H^2_{\beta}$. Indeed, the weighted shift $W$ is the operator of multiplication by $z$ on $H^2_{\beta}$, that is, $f(z) \mapsto zf(z)$, where $\beta(n) = w_0 w_1 \cdots w_{n-1}$. Operations defined in terms of functions, by composition with a fixed function (composition operators) or by multiplication by a function and projection back to an analytic function (Toeplitz operators), give natural operators that we expect would have closely related versions on each of these weighted spaces. For example, if $\varphi$ is in $L^\infty(d\theta / 2\pi)$, that is, $\varphi$ is a bounded Lebesgue measurable function on the unit circle, and has Fourier
series $\varphi \sim \sum_{-\infty}^{\infty} a_k e^{ik\theta}$, the Toeplitz operator $T_\varphi$ is the operator on $l^2(\mathbb{N})$ defined by $T_\varphi e_j = \sum_{i=0}^{\infty} a_{i-j} e_i$. For Toeplitz [12], these were matrices arising as quadratic forms coming from differential equations, but Toeplitz operators are best understood on $H^2$ considered as the subspace of $L^2$ of the circle consisting of the functions whose negative Fourier coefficients vanish. There we think of them arising from multiplication and projection: $T_\varphi f(n) = \hat{\varphi} f(n)$ for $n$ nonnegative, $T_\varphi f(n) = 0$ for $n$ negative. Less precisely, but more clearly, we have a Fourier-like series

$$T_\varphi \sim a_0 I + \sum_{n=1}^{\infty} \left[ a_n I^n + a_{-n} (I^*)^n \right].$$

The main result is that if $\|W\| < 1$, the analogous weighted Toeplitz operators

$$V_\varphi \sim a_0 I + \sum_{n=1}^{\infty} \left[ a_n W^n + a_{-n} (W^*)^n \right]$$

are well defined and $\|V_\varphi\| \leq \|\varphi\|_\infty$. From this result, we derive corollaries analogous to standard results for the classical Toeplitz operators. These theorems give new proofs for some well-known results for Bergman Toeplitz operators with harmonic symbol.

The technique for changing spaces developed here allows one to obtain new norm and spectral inequalities for Toeplitz and composition operators on weighted spaces from known norm and spectral information on other spaces. Although norm and spectral information is known in special cases for these operators (see, for example, [1,2,6,9,13]), little is known that applies to a wide class of examples. Shields and Wallen [11] used a related technique to prove that the commutant of a weighted shift $W$ is the strong operator closed algebra generated by $W$.

To begin, we recall the definition and some theorems about Hadamard multipliers.

**Definition.** If $A$ and $B$ are complex matrices of the same size, the Hadamard product of $A$ and $B$, denoted $A \cdot B$, is the matrix $C$ whose entries are $c_{ij} = a_{ij} b_{ij}$.

This product is called the Schur product by some authors. The most basic result about the Hadamard product is the following theorem, usually attributed to Schur.
Theorem 1 (Schur [8,4]). If $A$ and $B$ are positive (definite) matrices, then $A \cdot B$ is also positive (definite).

Hadamard multiplication by a fixed matrix defines a linear transformation on the vector space of matrices, and Schur's theorem says that this transformation preserves positivity when the multiplier is a positive matrix. When an orthonormal basis is chosen for a Hilbert space, this Hadamard multiplication gives a linear transformation on the algebra of bounded operators on the Hilbert space that is completely positive for positive multipliers (see V. I. Paulsen's Completely Bounded Maps and Dilations [7] for a survey).

Theorem 2 [7, p. 31]. If $B$ represents a bounded operator and $A$ is a positive matrix with $\sup(a_{ij}) = \alpha < \infty$, then $A \cdot B$ represents a bounded operator and

$$\|A \cdot B\| \leq \alpha \|B\|.$$ 

Let $R$ be a linear transformation of the space of analytic polynomials into the space of formal power series:

$$Rz^j = \sum_{i=0}^{\infty} r_{ij}z^i,$$

where the sum on the right is regarded as a formal power series. Let $H$ be a Hilbert space for which $\{z^n\}_{n=0}^{\infty}$ is an orthogonal basis, and let

$$e_n = \frac{1}{\|z^n\|} z^n$$

be the corresponding orthonormal basis. The linear transformation $R$ induces an operator (possibly unbounded) on $H$ and with respect to this basis, its matrix is

$$(R)_{ij} = (Re_j)_i = \left(R \left( \frac{1}{\|z^j\|} z^i \right) \right)_i = r_{ij} \left( \frac{1}{\|z^j\|} \right) z^i = r_{ij} \frac{\|z^i\|}{\|z^j\|}.$$ 

Stating this formally, recalling that $\|z^j\| = \beta(j)$ in $H_\beta^2$, we obtain the following theorem.
Theorem 3. Suppose $R$ is a linear transformation of the space of analytic polynomials into the space of formal power series that induces the operator $A$ on the weighted Hardy space $H^2_\alpha$ and the operator $B$ on the weighted Hardy space $H^2_\beta$. Then the matrices for the operators $A$ and $B$ satisfy

$$B = M \cdot A,$$

where the entries of $M$ are given by

$$m_{ij} = \frac{\beta(i)\alpha(j)}{\beta(j)\alpha(i)}.$$

To use this observation with Theorem 1 or 2, we need to show the positivity of the multiplier matrix. The positivity proof depends on the following easy lemma.

Lemma 4. Let $G$ be the $n \times n$ L-shaped matrix

$$G = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_n \\ \gamma_2 & \gamma_2 & \gamma_3 & \cdots & \gamma_n \\ \gamma_3 & \gamma_3 & \gamma_3 & \cdots & \gamma_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_n & \gamma_n & \gamma_n & \cdots & \gamma_n \end{pmatrix}.$$

Then $G$ is positive if and only if $\gamma_1 \geq \gamma_2 \geq \gamma_3 \geq \cdots \geq \gamma_n \geq 0$.

Proof. The diagonal entries of a positive matrix are all nonnegative. If the monotonicity condition is not met, say $\gamma_j < \gamma_{j+1}$, it is easily checked that the $2 \times 2$ principal submatrix

$$\begin{pmatrix} \gamma_j & \gamma_{j+1} \\ \gamma_{j+1} & \gamma_{j+1} \end{pmatrix}$$

has a strictly negative eigenvalue.

We prove the converse by induction. Clearly, the result is true for $n = 1$. Suppose the result has been proved for $n = k$. Given a $(k+1) \times (k+1)$ L-shaped matrix as above, let $G' = G - \gamma_{k+1}J$ where $J$ is the matrix all of whose entries are 1's. Now $G'$ is the direct sum of a zero matrix and an L-shaped matrix that is positive by the induction hypothesis. Since $\gamma_{k+1}J$ is a multiple of an orthogonal projection, it follows that $G = G' + \gamma_{k+1}J$ is positive. 

\qed
THEOREM 5. Let $M$ be the $n \times n$ matrix, for $n$ a positive integer or infinity, with

$$m_{ij} = \begin{cases} \frac{p_i}{p_j} & \text{for } i \geq j, \\ \frac{p_j}{p_i} & \text{for } i < j \end{cases}$$

where the $p_i$ are positive. Then $M$ is positive if and only if $p_1 \geq p_2 \geq \cdots > 0$.

Proof. If the monotonicity condition is not met, say $p_j < p_{j+1}$, it is easily checked that the $2 \times 2$ principal submatrix

$$\begin{pmatrix} 1 & \frac{p_{j+1}}{p_j} \\ \frac{p_j}{p_{j+1}} & 1 \end{pmatrix}$$

has a strictly negative eigenvalue.

On the other hand, suppose that $p_1 \geq p_2 \geq \cdots > 0$. For $n$ a positive integer, it is easy to see that $M = G \cdot H$, where

$$G = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_n \\ p_2 & p_2 & p_3 & \cdots & p_n \\ p_3 & p_3 & p_3 & \cdots & p_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_n & p_n & p_n & \cdots & p_n \end{pmatrix}$$

and

$$H = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{p_1} & \frac{1}{p_1} & \frac{1}{p_1} & \cdots & \frac{1}{p_1} \\ \frac{1}{p_2} & \frac{1}{p_2} & \frac{1}{p_2} & \cdots & \frac{1}{p_2} \\ \frac{1}{p_3} & \frac{1}{p_3} & \frac{1}{p_3} & \cdots & \frac{1}{p_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_n} & \frac{1}{p_n} & \frac{1}{p_n} & \cdots & \frac{1}{p_n} \end{pmatrix}.$$
The lemma shows that $G$ and $H$ are both positive, and Schur's theorem implies that $M$ is positive.

Since an infinite matrix is positive if and only if its finite corners are positive, the result follows.

We are now ready to prove the theorem on weighted Toeplitz operators mentioned above. Precisely, $V_\varphi$ is the operator defined by $V_\varphi e_j = \sum_{i=0}^\infty b_{ij} e_i$ where $b_{ij} = a_{i-j} \beta(i)/\beta(j)$ for $i \geq j$ and $b_{ij} = a_{i-j} \beta(j)/\beta(i)$ for $i < j$, that is,

$$V_\varphi = M \cdot T_\varphi$$

for $M$ as in Theorem 5 with $p_i = \beta(i-1)$. The best-known examples are the harmonic Toeplitz operators on the Bergman space [where $\beta(n) = (n+1)^{-1/2}$]. In this case,

$$V_\varphi f = P\tilde{\varphi}f,$$

where $P$ is the orthogonal projection of $L^2(D, dA)$ onto the Bergman space and $\tilde{\varphi}$ is the harmonic function on $D$ with boundary values $\varphi$, that is, $V_\varphi$ is the Bergman Toeplitz operator with symbol $\tilde{\varphi}$. On the Bergman space, of course, Theorem 6 has an easy function theoretic proof based on the observations that $\|\tilde{\varphi}\|_\infty = \|\varphi\|_\infty$ and that, when $\varphi$ is in $H^\infty$, the set of bounded measurable functions on the circle whose negative Fourier coefficients are zero, $\tilde{\varphi}$ is the analytic function in $D$ whose Taylor coefficients are the Fourier coefficients of $\varphi$.

**Theorem 6.** If $W$ is a weighted shift with $\|W\| \leq 1$ and $\varphi$ is a bounded measurable function on the unit circle, then $V_\varphi$ is a bounded operator and

$$\|V_\varphi\| \leq \|\varphi\|_\infty.$$

Moreover, if $\varphi$ is in $H^\infty$, then the spectrum of $V_\varphi$ satisfies

$$\sigma(V_\varphi) \subset \overline{\varphi(D)}.$$

**Proof.** Since $\|W\| \leq 1$, the weights satisfy $0 < w_n \leq 1$, so that

$$p_{n-1} = \beta(n) \geq w_n \beta(n) = \beta(n+1) = p_n.$$
Note that the diagonal entries of $M$ are all 1. This means that $M$ is positive, and by Theorem 2 and standard results on Toeplitz operators [2]

$$\|V_{\phi}\| = \|M \cdot T_{\phi}\| \leq \|T_{\phi}\| = \|\phi\|_{\infty}.$$ 

It is easily checked that for functions $\phi$ and $\psi$ in $H^\infty$, we have $V_{\phi}V_{\psi} = V_{\phi \psi}$ and $V_{\phi - \lambda I} = V_{\phi - \lambda}$. Thus, for $\phi$ in $H^\infty$ and $\lambda$ not in $\varphi(D)$, the identity $(T_{\phi - \lambda I})^{-1} = T_{(\varphi - \lambda)^{-1}}$, and the above inequality imply that $(V_{\phi - \lambda I})^{-1}$ is bounded. It follows that for $\varphi$ in $H^\infty$, we have $\sigma(V_{\varphi}) \subset \varphi(D)$. \hfill \square

**Corollary.** For $\varphi$ in $L^\infty$, $\sigma(V_{\varphi})$ is a subset of the closed convex hull of the essential range of $\varphi$. For $\varphi$ real valued, $\sigma(V_{\varphi})$ is a subset of the closed interval $[\text{ess inf}(\varphi), \text{ess sup}(\varphi)]$.

**Proof.** These results follow as in the classical case from the norm inequality (see [2, pp. 182, 183]). \hfill \square

Theorem 6 allows us to determine in what sense the operator series represents the operator. Recall that if $\varphi$ is a bounded measurable function on the circle with Fourier series $\varphi \sim \sum_{-\infty}^{\infty} a_k e^{ik\theta}$, the Cesàro means of $\varphi$ are the trigonometric polynomials

$$\sigma_n(e^{i\theta}) = \frac{1}{n} \left( a_0 + \sum_{1}^{n-1} (n-k)(a_k e^{ik\theta} + a_{-k} e^{-ik\theta}) \right).$$

Standard results about Cesàro means (see, for example, [3, p. 19]) say that $\|\sigma_n\|_{\infty} \leq \|\varphi\|_{\infty}$ for each $n$ and that the sequence $\sigma_n$ converges to $\varphi$ in the weak star topology on $L^\infty$. For a weighted shift $W$, denoting by $\sigma_n(W)$ the operator

$$\sigma_n(W) = \frac{1}{n} \left( a_0 I + \sum_{1}^{n-1} (n-k)\left[ a_k W^k + a_{-k} (W^*)^k \right] \right),$$

Theorem 6 implies $\|\sigma_n(W)\| \leq \|\varphi\|_{\infty}$ for each $n$. This, together with compactness of the unit ball in the set of bounded operators on $l^2(N)$ with respect to the weak operator topology, implies that the set of operators $\{\sigma_n(W): n \in N\}$ has an accumulation point. On the other hand, it is easily seen that

$$\lim_{n \to \infty} \langle \sigma_n(W) e_j, e_i \rangle = \langle V_{\varphi} e_j, e_i \rangle$$
for each i and j, so \( V_\varphi \) is the only possible accumulation point of the operator sequence. Thus, the series represents \( V_\varphi \) in an analogous way to that in which the Fourier series represents \( \varphi \). More precisely, we have the following corollary.

**Corollary.** If \( W \) is a weighted shift with \( \|W\| \leq 1 \) and \( \varphi \) is a bounded measurable function on the unit circle, then the sequence of operator Cesàro means \( \sigma_n(W) \) converges to \( V_\varphi \) in the weak operator topology.

Similar and stronger results hold for the special case in which \( \lim_{n \to \infty} w_n = 1 \) and the Fourier series for \( \varphi \) converges absolutely, since then \( T_\varphi - V_\varphi \) is compact.

These techniques can be applied to certain composition operators also.

**Definition.** For \( \eta \) analytic in the unit disk such that \( \eta(D) \subset D \) and \( \eta(0) = 0 \), let \( C_\eta^\beta \) denote the composition operator on \( H_\beta^2 \) defined by \( C_\eta^\beta(z^j) = \eta^j \).

It is easily seen that for each positive integer \( n \), the subspace \( \mathcal{K}_n^\beta = z^nH_\beta^2 \) is invariant for \( C_\eta^\beta \).

**Theorem 7.** Suppose \( H_a^2 \) and \( H_\beta^2 \) are such that

\[
\beta(j)/\alpha(j) \geq \beta(j+1)/\alpha(j+1)
\]

for \( j = 0, 1, 2, \ldots \). Then

\[
\|C_\eta^\beta|_{K_n^\beta}\| \leq \|C_\eta^\alpha|_{K_n^\alpha}\|
\]

and

\[
\sigma(C_\eta^\beta|_{K_n^\beta}) \subset \sigma(C_\eta^\alpha|_{K_n^\alpha})
\]

for all \( n \).

**Proof.** For each \( n \),

\[
C_\eta^\beta|_{K_n^\beta} = M \cdot C_\eta^\alpha|_{K_n^\alpha},
\]

where \( M \) is as in Theorem 5 for \( p_j = \beta(j + n - 1)/\alpha(j + n - 1) \). The first inequality follows from Theorem 2. The special form of \( M \) implies that the map \( A \mapsto M \cdot A \) is unital and multiplicative on the lower triangular matrices. Since the composition operators in question have lower triangular matrices,
the invertibility of \((C^\eta_\eta - \lambda I)|_{K_\eta^\varphi}\) implies \((C^\beta_\eta - \lambda I)|_{K_\eta^\varphi}\) is formally invertible and the norm equality shows that the inverse is bounded.

**Corollary.** Suppose \(\beta(0) \gg \beta(1) \gg \cdots\). Then

\[
\|C^\beta_\eta\| \leq 1,
\]

and if \(C^\eta_\eta\) is compact on the usual Hardy space \(H^2\), then \(C^\beta_\eta\) is compact also.

**Proof.** Take \(\alpha(j) = 1\) in the theorem. Since \(\eta(0) = 0\) implies \(\|C_\eta^\eta\| \leq 1\) for \(H^2\), the norm inequality follows. If \(C^\eta_\eta\) is compact on \(H^2\), then \(\|C^\eta_\eta|_{K_\eta^\varphi}\| \to 0\) as \(n \to \infty\), and this implies \(\|C^\beta_\eta|_{K_\eta^\varphi}\| \to 0\), which means \(C^\beta_\eta\) is compact.

Other results for composition operators on \(H^2\) generalize for these spaces; for example, results regarding the spectrum [5], spectral radius [1], and essential norm [9] for operators whose symbol satisfies \(\eta(0) = 0\) have immediate analogues. Moreover, if the composition operators whose symbols are inner linear fractional transformations are bounded on the space \(H^2_\beta\), then these results will further generalize to the case in which \(\eta\) has a fixed point in \(D\). Indeed, for these spaces and for \(H^2\), the operators for which \(\eta\) has a fixed point in \(D\) are similar to operators with \(\eta(0) = 0\).

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