The Banach manifold structure of the space of metrics on noncompact manifolds*

Jürgen Eichhorn
Ernst-Moritz-Arndt-Universität, Jahnstraße 15a, Greifswald, Germany

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Abstract: We study the $C^k$-structure of the space of Riemannian metrics of bounded geometry on open manifolds, the group of bounded diffeomorphisms, its action and the factor space. Each component of the space of metrics has a natural Banach manifold structure and the group of bounded diffeomorphisms is a completely metrizable topological group.

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1. Introduction

Consider the general procedure of gauge theory. There are given a Riemannian manifold $(M^n, g)$, a principal fibre bundle $P(M, G) \to M$ and the full gauge group. Special cases are the Yang-Mills theory with the gauge group $G_P = \text{Aut}_v(P)$ of vertical automorphisms (automorphisms over id$_M$), acting on the space $C_P$ of $G$-connections and the gauge theory of gravitation with $P = L(O(n), M)$, and the gauge group $\text{Diff} M \subset \text{Aut} L(O(n), M)$ acting on the space $M$ of metrics (Einstein theory) or on the space of metrics and on the space of connections (Hilbert-Palatini theory) or on the space of connections (Eddington theory). Moreover, there is given a functional invariant under $G$, i.e. a functional on the space of configurations $C_P/G_P$ resp. $M/\text{Diff} M$. Therefore the first step in the corresponding theory is the study of the configuration space $C_P/G_P$ resp. $M/\text{Diff} M$. For compact $M$ this has been done in [5, 2, 14, 15, 13] and many others. On open manifolds, the methods applied there completely break down. Consider $M, \mathcal{M}, \text{Diff} M, \mathcal{M}/\text{Diff} M$. At first one has to introduce Sobolev completions. But on an open manifold different metrics and different covariant derivatives generate in general different Sobolev spaces and completions.

Therefore it is not clear what are Sobolev completions. Furthermore, the image of a differential operator with injective symbol is not closed in general. Therefore the differential of the action $\text{Diff} M \cdot \{g\}, (f, g) \to f^*g$, has no closed image in general, i.e.

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is no immersion. Moreover the action is not proper, even not locally closed in general. Therefore $\text{Diff } M/T(M, g)$ and the orbit $\text{Diff } M \cdot g$ are not diffeomorphic in general. If we would like to introduce a weak or strong Riemannian metric on $M$ as usual, we run into troubles too. Fix $g$, then $\langle g, g \rangle = \int |g|^2_{g, e} \, d\text{vol}_e(g) = \int n \, d\text{vol}_e(g) = \infty$ if $\text{vol}(M^n, g) = \infty$. The same holds for all quasi isometric $g'$. On the other hand, the Riemannian geometry of $M$ is very important for many constructions.

We indicated a few of the difficulties which arise starting with an open manifold. All these troubles can be overcome if we restrict to metrics of bounded geometry, to diffeomorphisms bounded from below and above and if we introduce the "right" topology. In fact, the set of Riemannian metrics of bounded geometry and the set of bounded diffeomorphisms carry quite natural inner topologies, which coincide for compact manifolds with the usual one.

To carry out the whole programme, is a rather long history. In this paper, we restrict ourselves to the $C^k$-case. For further applications the $L_2$-case or $L_p$-case is much more important. But many of the main ideas can be exhibited in the simpler $C^k$-case. Moreover, applying Sobolev embedding theorems, one later partially returns to the $C^k$-case. The $L_2$-case will be carried out in a longer forthcoming paper.

This paper is organized as follows. We present in Section 2 the main technical results for the following sections. These are essentially Theorem 2.4 and Corollary 2.5. In Section 3 we introduce our topology on the space of metrics. The main result is Theorem 3.26. Section 4 is devoted to the group of bounded diffeomorphisms with Theorem 4.6 as the main result. Finally, we discuss shortly the isotropy group and the orbits. Here the $C^k$-methods are not efficient enough. One has to apply $L_2$-elliptic theory. This shall be done in a forthcoming paper.

2. Bounded geometry

We present in this section the key assertion which is of fundamental meaning for what follows. Let $(M^n, g)$ be open and complete. We say that $(M^n, g)$ has bounded geometry up to the order $k$ if it satisfies the following conditions (I) and (B$_k$).

(I) The injectivity radius $r_{\text{inj}}(M) = \inf_{x \in M} r_{\text{inj}}(x)$ is positive, $r_{\text{inj}}(M) > 0$.

(B$_k$) All covariant derivatives $\nabla^i R$ up to the order $k$ of the curvature tensor $R$ are bounded, i.e. there exist constants $C_i$ such that

$$|\nabla^i R| \leq C_i, \quad 0 \leq i \leq k,$$

where $|\cdot|$ denotes the pointwise norm.

Many classes of open manifolds are endowed with metrics of bounded geometry or admit such metrics in a natural manner, e.g. homogeneous spaces or coverings of closed manifolds.

In what follows, we need implications of bounded geometry for the Christoffel symbols and the metric in normal coordinates. The first result in this direction was obtained by Kaul [11].
Proposition 2.1. If \((M^n, g)\) satisfies \((B_1)\) then Christoffel symbols in normal coordinates of bounded radius are bounded by a constant depending on \(\sup_{x \in M} |R_1|\) and \(\sup_{x \in M} |\nabla R_1|\).

Corollary 2.2. \((B_1)\) implies the boundedness of the metric and its first derivatives in normal coordinates.

More generally, there holds:

Proposition 2.3. Assume \((B_0)\). Then the boundedness of the partial derivatives of the metric up to the order \(k \geq 1\), and of the Christoffel symbols up to the order \(k - 1\), in normal coordinates, are equivalent.

Therefore, in order to prove the boundedness of the derivatives of the Christoffel symbols, or what is the same, the higher derivatives of the exponential map, we should prove the boundedness of the partial derivatives of the metric in normal coordinates.

Theorem 2.4. If \((M^n, g)\) is open, complete and satisfies the condition \((B_k)\), then there exists a constant \(C_k\) independent of \(p \in M\) which bounds the \(C^k\)-norm of the \(g_{ij}\) in any normal coordinate system of radius \(\leq r_0\) at any \(p \in M\).

Corollary 2.5. Assume the hypotheses of Theorem 2.4. Then there exists a constant \(D_{k-1}\) independent of \(p \in M\) which bounds the \(C^{k-1}\)-norm of the \(\Gamma^m_{ij}\) in any normal coordinate system of radius \(\leq r_0\) at any \(p \in M\).

The proof of Theorem 2.4 which is rather long and technical shall appear in [6]. It uses extensively Jacobi field techniques and comparison theorems for inhomogeneous ordinary differential equations.

3. The space of Riemannian metrics on a noncompact manifold

The introduction of a topology on the set of Riemannian metrics on a compact manifold is a very simple matter. One can work with finite covers or take the induced topology from the Sobolev (vector) space \(H^s(S^2T^*)\) of symmetric covariant \(a\)-tensors. In any case, one gets the same topology. On open manifolds all these procedures do not work or give different topologies. Inspecting the set of all complete Riemannian metrics on an open manifold more carefully, one sees that this set carries a natural "intrinsic" topology which coincides on a compact manifold with the usual one. We describe in this paper this "intrinsic" topology at the \(C^k\)-level. The more difficult \(L_p\)-case will be studied in [6].

For a tensor \(t\) and a metric \(g\) we denote by \(|t|_{g,x}\) the pointwise norm at \(x \in M\) taken with respect to the metric \(g\). The global \(C^0\)-norm \(\|t\|\) is defined by \(\|t\| \equiv \sup_{x \in M} |t|_{g,x}\). Let \(\mathcal{M} = \mathcal{M}(M^n)\) be the set of smooth complete Riemannian metrics.
on $M$ and $g \in M$. We define a uniform structure $bU(g)$

$$bU(g) = \left\{ g' \in M \left| \sup_{x \in M} |g - g'|_{g,x} < \infty; \sup_{x \in M} |g - g'|_{g',x} < \infty \right. \right\}.$$ 

Lemma 3.1. $bU(g)$ coincides with the quasi isometry class of $g$, in particular, the conditions $g' \in bU(g)$ and $g \in bU(g')$ are equivalent, cf. [7]. □

Corollary 3.2. $g' \in bU(g)$ implies the existence of bounds $A_k(g,g'), B_k(g,g') > 0$ such that

$$A_k|t|_{g,x} \leq |t|_{g',x} \leq B_k|t|_{g,x}, \quad A_k^b|t|_g \leq b|t|_{g'} \leq B_k^b|t|_g$$

(3.1)

for every $(r,s)$-tensor field with $r + s \leq k$.

Set for $E > 0$, $g \in M$

$$bU_\varepsilon(g) = \left\{ g' \in bU(g) \left| b|g - g'|_g < \varepsilon \right. \right\}.$$ 

Lemma 3.3. The set of all $bU_\varepsilon(g)$, $\varepsilon > 0$, $g \in M$, defines a set of neighbourhood filter bases for a locally metrizable topology on $M$.

Proof. There remains only to show that for each $bU_\varepsilon(g)$ there exists a $bU_\varepsilon'(g)$ such that $bU_\varepsilon'(g)$ is a neighbourhood for each $g' \in bU_\varepsilon'(g)$, i.e. there exists a $\delta = \delta(g')$ such that $bU_\delta(g') \subset bU_\varepsilon'(g)$. Set $\varepsilon' = \varepsilon/2$ and $\delta < \varepsilon \cdot A_2/2$ and let $g'' \in bU_\delta(g')$. Then, since

$$|g' - g''|_g \leq 1/A_2 \cdot |g' - g''|_{g'} ,$$

$$b|g - g''|_g \leq b|g - g'|_g + b|g' - g''|_g < \frac{\varepsilon}{2} + \frac{1}{A_2} \frac{A_2}{2} \varepsilon = \varepsilon.$$  \hspace{1cm} (3.3)

The arising topology on $M$ shall be denoted as $C^{0,h}$-topology and we write $bM$ for $M$ endowed with this topology. In a similar manner we define the $C^{k,h}$-topology. For metrics $g, g'$ we set $B = g - g'$, $D = \nabla' - \nabla = \nabla a' - \nabla g$. Assume $g' \in bU(g)$.

Lemma 3.4. The boundedness of the following expressions is equivalent if $g' \in bU(g)$.

(a) $|\nabla^i g'|_g, 1 \leq i \leq k$; \hspace{1cm} (b) $|\nabla^i g'|_g, 1 \leq i \leq k$;
(c) $|\nabla^i g'|_g, 1 \leq i \leq k$; \hspace{1cm} (d) $|\nabla^i g'|_g, 1 \leq i \leq k$;
(e) $|\nabla^i B|_g, 1 \leq i \leq k$; \hspace{1cm} (f) $|\nabla^i B|_g, 1 \leq i \leq k$;
(g) $|\nabla^i B|_g, 1 \leq i \leq k$; \hspace{1cm} (h) $|\nabla^i B|_g, 1 \leq i \leq k$;
(i) $|\nabla^i D|_g, 0 \leq j \leq k - 1$; \hspace{1cm} (j) $|\nabla^i D|_g, 0 \leq j \leq k - 1$;
(k) $|\nabla^i D|_g, 0 \leq j \leq k - 1$; \hspace{1cm} (m) $|\nabla^i D|_g, 0 \leq j \leq k - 1$.

Proof. We use $|\nabla g'| = |\nabla B|, |\nabla' g| = |\nabla' B|$, always taken with respect to the same metric,

$$g'(D(X,Y),Z) = \frac{1}{2} \{ \nabla X B(Y,Z) + \nabla Y B(X,Z) - \nabla Z B(X,Y) \},$$

(3.4)
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\( \nabla_X B(Y, Z) = g(D(X, Y), Z) + g(Y, D(X, Z)). \)  
(3.5)

The proof for \( k = 1 \) follows immediately from formulas (3.4), (3.5) and is performed in [7, p. 35]. For \( k > 1 \) the proof follows by a simple induction, differentiating (3.4), (3.5). \( \square \)

We define now

\[ b^k U(g) = \{ g' \in b U(g) \mid b|\nabla^i D| < \infty, \ 0 \leq i \leq k - 1 \}. \]

Then Lemma 3.4 immediately implies

**Corollary 3.5.** \( g' \in b^k U(g) \) if and only if \( g \in b^k U(g') \).

**Remark 3.6.** \( b|\nabla^i D| < \infty \) means a condition for the \((i + 1)\)-th derivatives of the metric \( g' \) and it would be possible and natural to define \( b^k U(g) \) using the derivatives of the metric, but for several reasons, which become evident below, we prefer to work with the derivatives of \( D \).

Set for \( \varepsilon > 0, \ g \in \mathcal{M} \)

\[ b^k U_\varepsilon(g) = \left\{ g' \in b U(g) \mid b^k |g - g'| \equiv b|g - g'|_g + \sum_{j=0}^{k-1} b^j \nabla^j D|_g < \varepsilon \right\}. \]

**Lemma 3.7.** The \( b^k U(g), g \in \mathcal{M}, \varepsilon > 0 \), form a set of neighbourhood filter bases for a locally metrizable topology on \( \mathcal{M} \).

**Proof.** We have to show once again that for each \( b^k U_\varepsilon(g) \) there exists a \( b^k U_{\varepsilon'}(g) \) such that \( b^k U_\varepsilon(g) \) is a neighbourhood for each \( g' \in b^k U_{\varepsilon'}(g) \), i.e. there exists a \( \delta = \delta(g') \) such that \( b^k U_\delta(g') \subset b^k U_\varepsilon(g) \),

\[ b^k |g - g''|_g \equiv b|g - g''|_g + \sum_{j=0}^{k-1} b^j |\nabla^j (\nabla - \nabla'')|_g < \varepsilon \]  
(3.6)

for each \( g'' \in b^k U_\delta(g') \). The proof of formula (3.6) is not so easy as of (3.3). We start with \( k = 1 \). Set \( \varepsilon' = \frac{\varepsilon}{2} \) and \( \delta < \frac{1}{2} A_2 \cdot \frac{1}{2} \varepsilon \). Then

\[ b^1 |g - g''|_g = b^1 |g - g''|_b + b^1 |\nabla - \nabla''|_g \]

\[ \leq b^1 |g - g'|_b + b^1 |g' - g''|_g + b^1 |\nabla - \nabla'|_g + b^1 |\nabla' - \nabla''|_g \]

\[ = b^1 |g - g'|_b + b^1 |g' - g''|_g + b^1 |\nabla' - \nabla''|_g \]

\[ < \frac{\varepsilon}{2} + \frac{1}{A_2} b^1 |g' - g''|_g + \frac{1}{A_3} b^1 |\nabla' - \nabla''|_g \]

\[ < \frac{\varepsilon}{2} + \frac{1}{A_2} (b^1 |g' - g''|_g + b^1 |\nabla' - \nabla''|_{g'}) \]

\[ < \frac{\varepsilon}{2} + \frac{1}{A_2} \delta < \frac{\varepsilon}{2} + \frac{1}{A_2} \frac{1}{2} \varepsilon < \varepsilon. \]
Assume now $k > 1$ and consider

$$b^k |g - g''|_g \leq b |g - g'|_g + \sum_{j=0}^{k-1} b^j |\nabla^j (\nabla' - \nabla'')|_g$$

$$+ b^k |g' - g''|_g + \sum_{j=0}^{k-1} b^j |\nabla^j (\nabla' - \nabla'')|_g$$

$$= b^k |g - g'|_g + b^k |g' - g''|_g + \sum_{j=0}^{k-1} b^j |\nabla^j (\nabla' - \nabla'')|_g$$

$$\leq b^k |g - g'|_g + \frac{1}{A_2} b^k |g' - g''|_g + \sum_{j=0}^{k-1} b^j |\nabla^j (\nabla' - \nabla'')|_g.$$ 

We have done if there exists a bound $C = C(V, \nabla') > 0$ such that

$$\sum_{j=0}^{k-1} b^j |\nabla^j (\nabla' - \nabla'')|_g \leq C \sum_{j=0}^{k-1} b^j |\nabla^j (\nabla' - \nabla'')|_{g'}$$

(3.7)

for all $g'' \in b^k U(g)$.

Then, for $\epsilon' = \epsilon/2$, $\delta < \epsilon/2 \cdot 1/\max(1/A_2, C)$,

$$\frac{b^k |g - g''|_g}{2} + \frac{1}{A_2} b^k |g' - g''|_g + C \sum_{j=0}^{k-1} b^j |\nabla^j (\nabla' - \nabla'')|_{g'}$$

$$= \frac{\epsilon}{2} + \max(1/A_2, C) \cdot \delta$$

$$< \frac{\epsilon}{2} + \max(1/A_2, C) \cdot \frac{1}{\max(1/A_2, C)} \cdot \frac{\epsilon}{2} = \epsilon.$$

There remains to prove (3.7) for $k > 1$. For $k = 2$ we have for the pointwise norm $| \cdot | = | \cdot |_{g, x}$

$$| \nabla' - \nabla''| + |\nabla(\nabla' - \nabla'')|$$

$$\leq |\nabla' - \nabla''| + |(\nabla - \nabla')(\nabla' - \nabla'')| + |\nabla'(\nabla' - \nabla'')|$$

$$\leq \text{const} (\nabla, \nabla') |\nabla' - \nabla''| + |\nabla'(\nabla' - \nabla'')|.$$ 

Using (3.1) and taking the supremum norm, we obtain the assertion for $k = 2$.

Suppose now (3.7) for $\sum_{j=0}^{k-1}$ and consider $\sum_{j=0}^{k}$, assuming (3.7) for $\sum_{j=0}^{k-1}$. Then, setting for a moment $\nabla' - \nabla'' = \eta$,

$$\nabla^j \eta = \nabla^j (\nabla' - \nabla'') = (\nabla^j - \nabla' \nabla'^{-1}) \eta + \nabla' \nabla'^{-1} \eta$$

$$= (\nabla - \nabla') \nabla'^{-1} \eta + \nabla' \nabla'^{-1} \eta,$$(3.8)
more general,
\[ \nabla^\kappa \eta = \sum_{i=1}^{\kappa} \nabla^{i-1}(\nabla - \nabla') \nabla^{\kappa-i} \eta + \nabla^{i\kappa} \eta, \] (3.9)
i.e.
\[ b|\nabla^\kappa \eta|_g \leq \sum_{i=1}^{\kappa} b|\nabla^{i-1}(\nabla - \nabla') \nabla^{\kappa-i} \eta|_g + b|\nabla^{i\kappa} \eta|_g. \] (3.10)

For the term on the right hand side corresponding to \( i = 1 \) we have done, since by induction assumption,
\[ b|(\nabla - \nabla') \nabla^{\kappa-1} \eta|_g \leq b|\nabla - \nabla'|_g \cdot b|\nabla^{\kappa-1} \eta|_g \]
\[ = \frac{1}{A_3} \cdot \varepsilon' \cdot C(\nabla, \nabla', \kappa - 1) \sum_{j=0}^{\kappa-1} b|\nabla^{ij} \eta|_{g'}. \]

There remains to consider the terms \( \nabla^{i-1}(\nabla - \nabla') \nabla^{\kappa-i} \eta, i > 1 \). Applying the procedure (3.8) repeated to \( \nabla^{\kappa-i} \) in the terms \( \nabla^{i-1}(\nabla - \nabla') \nabla^{\kappa-i} \eta \), we have finally to estimate expressions
\[ \nabla^{i_1}(\nabla - \nabla')^{j_1} \nabla^{i_2}(\nabla - \nabla')^{j_2} \cdots (\nabla - \nabla')^{i_{\kappa-1}} \nabla^{i_{\kappa}} \eta \] (3.11)
i_1 + \cdots + i_\kappa = \kappa, i_1 \geq 1, 1 \leq i_\kappa < \kappa. By the Leibniz rule each term of (3.11) turns into a sum of products, where each factor can be estimated by
\[ C_{j, s}(\varepsilon') \cdot b|\nabla^{ij}(\nabla - \nabla')|_g \cdot b|\nabla^{is} \eta|_g, \] \( j \leq \kappa - 2, j + s \leq \kappa - 1 \). But
\[ C_{j, s}(\varepsilon') \cdot b|\nabla^{ij}(\nabla - \nabla')|_g \cdot b|\nabla^{is} \eta|_g \]
\[ \leq C_{j, s}(\varepsilon') \cdot D_{j, s}(\varepsilon') \cdot b|\nabla^{j}(\nabla - \nabla')|_g \cdot \frac{1}{A_{s+3}} b|\nabla^{s} \eta|_g \]
\[ \leq C_{j, s} \cdot D_{j, s} \cdot \frac{\varepsilon}{2 A_{s+3}} \cdot b|\nabla^{s} \eta|_{g'}, \]
and we have done. \( \Box \)

Denote \( B_k \mathcal{M} \) for \( \mathcal{M} \) endowed with this topology. Set
\[ \mathcal{M}(B_k) = \{ g \in \mathcal{M} \mid g \text{ satisfies } (B_k) \}. \]

**Proposition 3.8.** If \( g \in \mathcal{M}(B_k), g' \in b^{k+2}U(g) \), then \( g' \in \mathcal{M}(B_k) \), i.e. for \( g \in \mathcal{M}(B_k) \) holds \( b^{k+2}U(g) \subseteq \mathcal{M}(B_k) \).

**Proof.** We perform induction and start with \( k = 0 \). With \( R = R^g, R' = R^{g'} \) there holds
\[ R'(U, V)W = R(U, V)W + D(U, D(V, W)) \]
\[ - D(V, D(U, W)) - D(D(U, V), W) + D(D(V, U), W) \] (3.12)
\[ + \nabla U D(V, W) - \nabla V D(U, W), \]
i.e. $R, D, \nabla D$ bounded imply $R'$ bounded. Now the induction is easily completed, differentiating (3.12). \[ \square \]

**Corollary 3.9.** $\mathcal{M}(B_k)$ is open in $b^{k+2}\mathcal{M}$. \[ \square \]

**Proposition 3.10.** For $g \in \mathcal{M}(B_k)$, $\mathcal{M}(B_k) \subseteq b^k U(g)$.

**Proof.** If $g, g' \in \mathcal{M}(B_k)$, then according to Corollary 2.5 $b|g - g'|_g < \infty$ and the $C^{k-1}$-norm of the Christoffel symbols is bounded. As one easily sees, this implies $b|D|_g = b|\nabla - \nabla'|_g < \infty$, $b|\nabla D|_g < \infty$, $\ldots$, $b|\nabla^{k-1}D|_g < \infty$, and therefore $g' \in b^k U(g)$. \[ \square \]

We recall now a fundamental result of U. Abresch which implies the density of $\mathcal{M}(B_k)$ in $b^0\mathcal{M}(B_0)$.

Set for $\varepsilon > 0$ and $x \in M$

$$\Lambda(x) := \sup \left\{ \left| g(R(V, W)W, V) \right|^{1/2} \right\}_{V, W \in T_x M, |V| = |W| = 1},$$

$$r_{\Lambda, \varepsilon}(x) := \sup \{ \rho > 0 \mid \rho \cdot (\varepsilon + \Lambda(y) \leq 1 \text{ for all } y \in B_{10\rho}(y) \in M\}.$$

**Theorem 3.11.** There exists a family of smoothing operators $S_{\varepsilon_0, \varepsilon} : \mathcal{M} \to \mathcal{M}$, $\varepsilon_0$, $\varepsilon > 0$, with the following properties:

a) For $\bar{g} := S_{\varepsilon_0, \varepsilon}(g)$ there holds

$$\left(1 + \varepsilon_0\right)^{-2}g \leq \bar{g} \leq \left(1 + \varepsilon_0\right)^2g.$$ \[ (3.13) \]

b) $|\bar{\nabla}_V W - \nabla_V W|_x \leq \text{const}(n, \varepsilon_0) \cdot r_{\Lambda, \varepsilon}(x)^{-1}|V| \cdot |W|$ \[ (3.14) \]

c) $|\bar{\nabla}^i \bar{R}|_x = \text{const}(n, i, \varepsilon_0) \cdot r_{\Lambda, \varepsilon}(x)^{-(i+2)}$ \[ (3.15) \]

for all $x \in M$.

d) If $\phi : (M, g) \to (M, g)$ is an isometry, then $\phi$ is an isometry for the smoothed metric too, i.e. $S_{\varepsilon_0, \varepsilon}(g) = \phi^* S_{\varepsilon_0, \varepsilon}(g')$.

e) $S_{\varepsilon_0, \varepsilon}(g)$ depends only on $g$ restricted to $B_{12r_{\Lambda, \varepsilon}(x)}(x)$. This ball is contained in $B_{12r_{\Lambda, \varepsilon}(x)}(x)$.

f) $r_{\text{finj}}(x, S_{\varepsilon_0, \varepsilon}(g)) \geq \min \left\{ \left(1 + \varepsilon_0\right)^{-1} \cdot r_{\text{finj}}(x, g), \text{const} \cdot (n, \varepsilon_0) \cdot r_{\Lambda, \varepsilon}(x)\right\}$.\[ \]

According to (3.13), it is unimportant whether we take in (3.14), (3.15) the pointwise norm with respect to $g$ or $S_{\varepsilon_0, \varepsilon}(g) = \bar{g}$. For the proof we refer to [1]. \[ \square \]

**Corollary 3.12.** $\mathcal{M}(B_\infty) = \bigcap_k \mathcal{M}(B_k)$ is dense in $b^0\mathcal{M}(B_0)$.

**Proof.** If $g \in \mathcal{M}(B_0)$ then, according to the definition, $\inf_{x \in M} r_{\Lambda, \varepsilon}(x) > 0$. Therefore, $|\bar{\nabla}^i \bar{R}|_x \leq C(n, i, \varepsilon_0)$. Formula (3.13) implies the existence of a $C^0, b$-convergent sequence $\bar{g}_\nu \to g$, $\bar{g}_\nu \in \mathcal{M}(B_\infty)$, setting $\varepsilon_0 = 1/\nu$. \[ \square \]

This provides a certain justification for the restriction to $\mathcal{M}(B_k), k$ sufficiently large if necessary. In the forthcoming paper devoted to the $L_2$-case this fact is of important meaning.
Analogously we define
\[ M(I) = \{ g \in M \mid r_{\text{inj}}(M^n, g) > 0 \}, \]
and
\[ M(B_k, I) = M(B_k) \cap M(I). \]
Theorem 3.11 then immediately implies

**Proposition 3.13.** \( M(B_k, I) \) is dense in \( ^{b,0}M(B_0) \).

We need now some estimates on the injectivity radius and follow here [3, p. 47–48]. Let \((M^n, g)\) be open, complete, satisfying \((B_0)\), i.e. \( \delta \leq K \leq \Delta \) for the sectional curvature \( K \), \( p \in M \), \( x \in M \), \( \rho = \rho(p, x) \) (Riemannian distance), and fix \( r, r_0, s \) with \( r_0 + 2s < \pi/\sqrt{|\Delta|}, r_0 < \pi/4\sqrt{\Delta} \). Denote by \( V_d(p) \) the volume \( \text{vol}(B_d(p)) \) of the metric ball of radius \( d \) centered at \( p \) and by \( V^r_d \) the volume of the metric ball of radius \( d \) in a space of constant curvature \( \kappa \).

**Theorem 3.14.** The injectivity radius \( r_{\text{inj}}(x) \) satisfies the inequality

\[ r_{\text{inj}} \geq \frac{r_0}{2} \frac{1}{1 + (V^\delta_{r_0+s}/V_r(p))(V^\delta_{\rho+r}/V^\delta_s)}. \]  

Moreover, if \( r + s < \rho \), then
\[ r_{\text{inj}}(x) \geq \frac{r_0}{2} \frac{1}{1 + (V^\delta_{r_0+s}/V_r(p))(V^\delta_{\rho+r} - V^\delta_{\rho-s})/V^\delta_s}. \]  

For the proof we refer to [3].

This formula provides a lower bound for the injectivity radius. It is clear that the right hand side of (3.16) could tend to 0 but \( r_{\text{inj}}(M, g) > 0 \). But (3.16) indicates in many cases the actual behaviour of the injectivity radius. For instance, on cusps with \( K \equiv -1 \) we have \( r_{\text{inj}}(p) \xrightarrow{p \to \infty} 0 \), and the right hand side tends to zero (as it should be), since \( V_r(p) \xrightarrow{p \to \infty} 0 \), denominator \( \to \infty \). If \((M^n, g)\) satisfies \((B_0)\) and \((I)\), then there exists a constant \( b > 0 \) such that \( V_r(p) > b \) for all \( p \) and the other ingredients of the right hand side of (3.16) are positively bounded from below and above.

**Proposition 3.15.** \( M(B_k, I) \) is open in \( M(B_k) \subset ^{b,k+2}M \).

**Proof.** Consider a quotient \( 1/(1 + a) \), \( a > 0 \), and arbitrary \( \varepsilon > 0 \) such that \( 1 - \varepsilon \times (1 + a) > 0, \varepsilon < 1/(1 + a) \). For real \( \lambda \) with \( 1 + a + \lambda > 0 \) the conditions

\[ 0 < \frac{1}{1 + a} - \varepsilon < \frac{1}{1 + a + \lambda} < \frac{1}{1 + a} + \varepsilon \]  

and
\[ -\frac{\varepsilon (1 + a)^2}{1 + \varepsilon (1 + a)} < \lambda < \frac{\varepsilon (1 + a)^2}{1 - \varepsilon (1 + a)} \]  

(3.18) (3.19)
are equivalent.

Assume \( g \in \mathcal{M}(B_k, I) \), \( r_{\text{inj}}(M, g) > 0 \), \( \delta \leq K_g \leq \Delta \). Fix \( r_0, s \) as above and \( r < r_{\text{inj}}/2 \). According to the comparison theorems for metric balls there exists a \( b > 0 \) such that \( V_{r, \delta}(p) \geq b \). Then

\[
\frac{1}{1 + (V_{r_0 + s}^{\delta}(p)(V_{r_0 + s}^{\delta}/V_{r}^{\delta}))} \geq \frac{1}{1 + (V_{r_0 + s}^{\delta}/b)(V_{r_0 + s}^{\delta}/V_{r}^{\delta})} \equiv \frac{1}{1 + a}.
\]

Let \( \epsilon > 0 \) be arbitrary but \( < 1/(1 + a) \). The volume \( \text{vol}_g(B_r(p)) \) of a metric ball is a continuous function of the metric \( g \). The family \( \{V_{r, \delta}(p)\}_p \) is an equicontinuous family of continuous functions of the metric in the \( C^{b,k+2}_b \)-topology. Then there exists an \( \epsilon' > 0 \) such that with

\[
\lambda = (V_{r_0 + s}^{\delta}(p)(V_{r_0 + s}^{\delta}/V_{r}^{\delta})) - a,
\]

\( \lambda \) satisfies (3.19) for all \( g' \in b^{k+2}U_{\epsilon'}(g) \) and \( p \in \mathcal{M} \). Now the assertion follows from Theorem 3.14. \( \square \)

**Lemma 3.16.** \( b^k \mathcal{M}, b^{k+2} \mathcal{M}(B_k) \) and \( b^{k+2} \mathcal{M}(B_k, I) \) are locally arcwise connected.

**Proof.** We show the local contractibility which implies the locally arcwise connectedness. According to Lemma 3.4, we can work with neighbourhood bases

\[
b^{k}U_{\epsilon'}(g) = \{ g' \in bU(g) \mid b^k|g - g'|_{g} = \sum_{i=0}^{k} |\nabla^i(g - g')|_{g} < \epsilon \}.
\]

We show that \( g' \in b^kU_{\epsilon'}(g), 0 < t < 1 \), implies \( tg' + (1 - t)g \in b^kU_{\epsilon'}(g) \). But \( b^k|tg' + (1 - t)g - g|_{g} = b^k|t(g' - g)|_{g'} = t \cdot b^k|g' - g|_{g'} < t\epsilon < \epsilon \). Since \( \mathcal{M}(B_k), \mathcal{M}(B_k, I) \) are open in \( b^{k+2} \mathcal{M} \) we have done.

**Corollary 3.17.** In \( b^k \mathcal{M}, b^{k+2} \mathcal{M}(B_k), b^{k+2} \mathcal{M}(B_k, I) \) components and arc components coincide.

**Proposition 3.18.** The component of \( g \) in \( b^k \mathcal{M} \) coincides with \( b^kU(g) \).

**Proof.** According to Corollary 3.17, we have to consider arc components. Assume \( g' \) to be an element of the arc component of \( g \), \( g' \in \text{comp}(g) \), and let \( \{g_t\}_{0 \leq t \leq 1} \) be an arc between \( g \) and \( g' \), \( g_0 = g, g_1 = g' \). Then for every \( \epsilon > 0 \) the arc can be covered by a finite number of open neighbourhoods \( b^kU_{\epsilon}(g_0), b^kU_{\epsilon}(g_1), \ldots, b^kU_{\epsilon}(g_r) = b^kU_{\epsilon}(g') \), \( b^kU_{\epsilon}(g_{t-1}) \cap b^kU_{\epsilon}(g_t) = \emptyset \). If \( g_{t-1,t} \in b^kU_{\epsilon}(g_{t-1}) \cap b^kU_{\epsilon}(g_t) \), then

\[
b^kU(g_{t-1}) \ni b^kU_{\epsilon}(g_{t-1}) \ni g_{t-1} \in b^kU_{\epsilon}(g_t) \subseteq b^kU(g_t),
\]

i.e. according to Corollary 3.5, \( b^kU_{\epsilon}(g_{t-1}) = b^kU_{\epsilon}(g_t) \). Repeating this conclusion, we obtain \( b^kU(g) = b^kU(g') \), \( g \in b^kU(g) \), \( \text{comp}(g) \subseteq b^kU(g) \). We have to show that \( b^kU(g) \subseteq \text{comp}(g) \). Suppose now that \( g' \in b^kU(g) \). Set \( g_t = tg' + (1 - t)g \). Then we conclude immediately from the proof of 3.16 that \( \{g_t\}_{0 \leq t \leq 1} \) is an arc in \( b^kU(g) \) which connects \( g \) and \( g' \), i.e. \( g' \in \text{comp}(g) \), \( b^kU(g) \subseteq \text{comp}(g) \). \( \square \)
Corollary 3.19. \( b^k M \) has a representation as a topological sum
\[
b^k M = \sum_{i \in I} b^k U(g_i). \tag{3.20}
\]

Corollary 3.20. \( b^{k+2} M(B_k) \) resp. \( b^{k+2} M(B_k, I) \) have representations as a topological sum
\[
b^{k+2} M(B_k) = \sum_{j \in J} b^{k+2} U(g_j), \tag{3.21}
\]
\[
b^{k+2} M(B_k, I) = \sum_{j \in J} b^{k+2} U(g_j) \cap M(B_k, I). \tag{3.22}
\]

Proof. Clearly, \( b^{k+2} M(B_k) = M(B_k) \cap b^{k+2} M = M(B_k) \cap \sum_{i \in I} b^{k+2} U(g_i) = \sum_{i \in I} b^{k+2} U(g_i) \cap M(B_k) \). But according to 3.8 and 3.5, either \( b^{k+2} U(g_i) \cap M(B_k) = \emptyset \) or \( b^{k+2} U(g_i) \cap M(B_k) = b^{k+2} U(g_i) \). \( \square \)

Denote by \( b^k \overline{M} \) resp. \( b^{k+2} M(B_k) \) resp. \( b^{k+2} M(B_k, I) \) the corresponding completions. For the components \( b^k U(g) \) the completion and the closure in \( b^k M \) coincide. As \( b^k \overline{M} \) is still locally connected (since \( b^k M \) was locally arcwise connected, in particular locally connected), its components coincide with \( b^k \overline{U}(g) \) and
\[
b^k \overline{M} = \sum_{i \in I} b^k \overline{U}(g_i). \tag{3.23}
\]

Analogously,
\[
b^{k+2} M(B_k) = \sum_{j \in J} b^{k+2} \overline{U}(g_j), \tag{3.24}
\]
\[
b^{k+2} M(B_k, I) = \sum_{j \in J} b^{k+2} \overline{U}(g_j) \cap b^{k+2} M(B_k, I) \tag{3.25}
\]
\[= \sum_{j \in J} b^{k+2} \overline{U}(g_j) \cap M(B_k, I). \]

To describe the structure of the considered spaces and to prepare the next section we introduce Banach spaces of bounded tensors. Let \( (E, h) \to (M, g) \) be a Riemannian vector bundle, \( \nabla^h \) an associated Riemannian connection, \( \nabla^g \) the Levi-Civita connection, \( T^g \otimes E \) a tensor bundle with values in \( E \), \( \nabla \) the associated tensor product connection coming from \( \nabla^g, \nabla^h \). Denote by \( \Omega^0(T^g \otimes E) \) the 0-forms with values in \( T^g \otimes E = \text{the smooth section of} T^g \otimes E = \text{smooth tensor fields with values in} E \).

We set
\[
b^k \Omega^0(T^g \otimes E) = \left\{ \phi \in \Omega^0(T^g \otimes E) \mid b^k \| \phi \| = \sum_{i=0}^k b^i \| \nabla^i \phi \| = \sum_{i=0}^k \sup_{x \in M} |\nabla^i \phi|_x < \infty \right\}
\]
and \( b, k \Omega_0^0 (T_r^q \otimes E) \) = completion of \( b, k \Omega^0_k (T_r^q \otimes E) \) with respect to \( b, k | \phi | \). The space \( b, k \Omega^0_k (T_r^q \otimes E) \) is by construction a Banach space and coincides with

\[
b, k \Omega^0_k (T_r^q \otimes E) = \{ \phi \mid \phi \text{ is a } C^k \text{-section of } T_r^q \otimes E \text{ and } b, k | \phi | < \infty \},
\]

cf. [8, p. 12]. The same definition shall be applied to subbundles of \( T_r^q \otimes E \). Consider the space \( b, k \Omega^0_k (S^2 T^* \otimes \otimes) \subset b, k \Omega^0_k (T^* \otimes \otimes) \) of symmetric covariant 2-tensors of class \( C^k \)
which are bounded up to order \( k \). Then \( b, k \Omega^0_k (S^2 T^* \otimes \otimes) \) is a Banach space. For clarity we write \( b, k \Omega^0_k (S^2 T^* \otimes \otimes, g) \), thus indicating the used Riemannian metric.

**Proposition 3.21.** If \( g' \in b, k U (g) \) then

\[
b, k \Omega^0_k (S^2 T^* \otimes \otimes, g) = b, k \Omega^0_k (S^2 T^* \otimes \otimes, g').
\]

**Proof.** It is sufficient to show \( b, k \Omega^0_k (S^2 T^* \otimes \otimes, g) = b, k \Omega^0_k (S^2 T^* \otimes \otimes, g') \) with equivalent norms. According to Corollary 3.2, for \( k = 0 \) there is nothing to show. Consider \( k = 1 \). Assume \( h \in b, k \Omega^0_k (S^2 T^* \otimes \otimes, g) \), \( | \cdot | = | \cdot |_{g', x} \). Then

\[
| \nabla' h | \leq | (\nabla' - \nabla) h | + | \nabla h | \leq | \nabla' - \nabla | \| h \| + | \nabla h |.
\]

By assumption, \( b, k | \nabla h | g < \infty \), \( b, k | \nabla' - \nabla | g < \infty \), and we obtain by means of (3.2)

\[
b, k | \nabla h |_{g'} \leq C'(g, g')(b, k | h | g + b, k | \nabla h | g) = C'(g, g') \cdot b, k | h |_{g'},
\]

\[
b, k | h |_{g'} \leq C(g, g') \cdot b, k | h |_{g}.
\]

Changing the role of \( g \) and \( g' \), which is possible according to Corollary 3.5, we obtain for \( h \in b, k \Omega^0_k (S^2 T^* \otimes \otimes, g') \)

\[
b, k | h |_{g} \leq D(g, g') b, k | h |_{g'},
\]

and we have done. Assume \( b, i \Omega^0_k (S^2 T^* \otimes \otimes, g) = b, i \Omega^0_k (S^2 T^* \otimes \otimes, g') \) for \( i = \kappa - 1 \) with equivalent norms and \( h \in b, \Omega^0_k (S^2 T^* \otimes \otimes, g) \). Then, with \( | \cdot | = | \cdot |_{g', x} \)

\[
| \nabla' \kappa h | \leq | (\nabla' - \nabla) \nabla' \kappa - 1 h | + | \nabla' \kappa - 1 h |
\]

\[
\leq | \nabla' - \nabla | \cdot | \nabla' \kappa - 1 h | + | \nabla^{\kappa - 1} h |.
\]

By assumption resp. induction assumption \( b, k | \nabla' - \nabla | g < \infty \), \( b, k | \nabla' - \nabla | g' < \infty \), resp. \( b, k | \nabla' h | g' < \infty \). Therefore, we have to consider \( | \nabla' \kappa - 1 h | \) and we use (3.9), (3.10),

\[
\nabla' \kappa = \sum_{i=1}^{\kappa} \nabla^{i-1} (\nabla' - \nabla) \nabla' \kappa - i h + \nabla^{\kappa} h,
\]

and we recall word for word the arguments, i.e. finally we have to estimate pointwise

\[
\nabla^{j} (\nabla' - \nabla) \nabla' \kappa h, \quad j \leq \kappa - 2, \quad j + s = \kappa - 1.
\]
But by assumption \( g' \in b^k U(g) \), \( b^i |\nabla^i (\nabla' - \nabla)|_g < \infty \), \( b^i |\nabla^i (\nabla' - \nabla)|_{g'} < \infty \), \( b^i |\nabla^i h|_{g'} \leq C'' \cdot b^i |\nabla^i h|_g \), \( b^i |\nabla^i h|_{g'} \leq C \cdot b^i |\nabla^i h|_g \), \( \kappa \leq k \).

Exchanging the role of \( g \) and \( g' \), which is possible according to Corollary 3.5, we obtain for \( h \in b^k \Omega^0_k (S^2 T, g') \)

\[
b^k |h|_g \leq D \cdot b^k |h|_{g'},
\]

and we have done. \( \square \)

This proposition justifies the notation \( b^k \Omega^0 (S^2 T, U(g)) \) as a class of topological Banach spaces.

Denote by \( C^{k+2} M \) the set of all complete Riemannian metrics of differentiability class \( k + 2 \).

**Lemma 3.22.** If \( g \in M(B_k) \) then \( b^{k+2} U(g) \cap C^{k+2} M \) is open in \( b^{k+2} U(g) \subset b^{k+2} M(B_k) \).

**Proof.** Let \( g' \in b^{k+2} U(g) \cap C^{k+2} M \), i.e. \( g' \) is a complete Riemannian metric of class \( C^{k+2} \) satisfying (\( B_k \)). According to Proposition 3.8 this implies the existence of \( C_1, C_2 > 0 \) such that

\[
C_1 |X|^2_g = C_1 \cdot g(X, X) \leq g'(X, X) \leq C_2 \cdot g(X, X) = C_2 |X|^2_g
\]

for all \( X \in T_x M \) and for all \( x \in M \). Set \( \varepsilon = \varepsilon(g') = \sqrt{C_1} / 2 \) and let \( g'' \in b^{k+2} U_\varepsilon (g') \), \( g'' = g' + \tau, b^i |\tau|_g \leq b^{k+2} |\tau|_g < \varepsilon = \sqrt{C_1} / 2 \), where we have taken \( b^{k+2} U_\varepsilon (g') \) in the completed component \( b^{k+2} U(g) \). From \( b^i |\tau|_g < \sqrt{C_1} / 2 \) we obtain

\[
-C_1 / 2 |X|^2_g \leq \tau(X, X) \leq C_1 / 2 |X|^2_g,
\]

\[
C_1 / 2 \cdot |X|^2_g = -C_1 / 2 \cdot |X|^2_g + C_1 \cdot |X|^2_g
\]

\[
\leq \tau(X, X) + g'(X, X) = g''(X, X) = (C_1 / 2 + C_2) |X|^2_g
\]

for all \( X \in T_x M \) and for all \( x \in M \), i.e. \( g'' \) is positively definite of class \( C^{k+2} \). Moreover, replacing in (3.12) \( R' \) by \( R'' = R g'' \) and \( R \) by \( R' = R g' \), \( D = \nabla g'' - \nabla g' \) and using \( b^{k+2} g'' - g' \) we conclude as in the proof of 3.8 that \( b^i |\nabla^i R''| \leq C_i, 0 \leq i \leq k \), and \( b^{k+2} U_\varepsilon (g') \subset b^{k+2} U(g) \cap C^{k+2} M \). Here it is unimportant whether we use \( |\cdot|_g \) or \( |\cdot|_{g'} \) or \( |\cdot|_{g''} \).

In the same manner we conclude for \( b^{k+2} U(g) \cap C^{k+2} M, g \in M(B_k, I) \), using the proof of Proposition 3.15. \( \square \)

**Lemma 3.23.** If \( g \in M(B_k, I) \) then \( b^{k+2} U(g) \cap M(B_k, I) \cap C^{k+2} M \) is open in \( b^{k+2} U(g) \cap M(B_k, I) \), where the bar denotes the completion. \( \square \)
Proposition 3.24. Assume \( g \in M(B_k) \). The map \( b^{k+2}U(g) \cap C^{k+2}M \to b^{k+2}Q^0(S^2T^*, g), g' \in b^{k+2}U(g) \cap C^{k+2}M \to g' - g \in b^{k+2}Q^0(S^2T^*, g) \) is a homeomorphism of \( b^{k+2}U(g) \cap C^{k+2}M \) onto an open subset of \( b^{k+2}Q^0(S^2T^*, g) \).

**Proof.** The one-to-one property and the continuity of the map are clear from the definition of the topologies. We have to show the openness of the map. For this it is sufficient that for every \( g' \in b^{k+2}U(g) \cap C^{k+2}M \) there exists an \( \varepsilon = \varepsilon(g') > 0 \) such that \( b^{k+2}U\varepsilon(g') \cap C^{k+2}M \) maps onto an open subset of \( b^{k+2}Q^0(S^2T^*, g) \). Here \( b^{k+2}U\varepsilon(g') \) is taken in \( b^{k+2}U(g) \). As above, there exist constants \( C_1(g), C_2(g) > 0 \) such that \( C_1 |X|^2_g \leq g'(X, X) \leq C_2 |X|^2_g \). Setting \( \varepsilon(g') = \sqrt{C_1}/2 \), we see immediately that \( b^{k+2}U\varepsilon(g') \cap C^{k+2}M \) is mapped one-to-one onto \( U\varepsilon(0) \subset b^{k+2}Q^0(S^2T^*, g) \). \( \square \)

Proposition 3.25. Assume \( g \in M(B_k, I) \). The map \( b^{k+2}U(g) \cap M(B_k, I) \cap C^{k+2}M \to b^{k+2}Q^0(S^2T^*, g) \) is a homeomorphism onto an open subset of \( b^{k+2}Q^0(S^2T^*, g) \).

**Proof.** We proceed as in Proposition 3.24, choosing \( \varepsilon'(g') \) in general much smaller than \( \sqrt{C_1}/2 \), namely such that for all \( g'' \in U\varepsilon'(g') \) (taken in \( b^{k+2}U(g) \cap M(B_k, I) \)) (3.19) is satisfied, where we replace \( g \) in the proof of 3.15 by \( g' \). \( \square \)

From this proposition we obtain our first main result.

**Theorem 3.26.**

a) Each component of \( b^{k+2}M(B_k) \cap C^{k+2}M \) is a Banach manifold.

b) Each component of \( b^{k+2}M(B_k, I) \cap C^{k+2}M \) is a Banach manifold. \( \square \)

**Remark.** \( b^{k+2}U(g) \cap C^{k+2}M \) is not necessarily open in \( b^{k+2}U(g) \), for arbitrary \( g \in M \), and we have no evident manifold structure on the components. This and 3.12, 3.13 give a certain justification (among others) for the restriction to metrics satisfying \( (B_k) \) and \( (I) \). In the \( L_2 \)-category this becomes still more striking, and we refer to [6].

4. The group of bounded diffeomorphisms on a manifold of bounded geometry

We recall and develop in this section some results of [9] in the Banach category and study the group of bounded diffeomorphisms of an open complete manifold \( (M^n, g) \) satisfying \( (B_k) \) and \( (I) \).

Assume \( (M^n, g), (N^r, h) \) being open, complete and satisfying \( (B_k) \) and \( (I) \). This implies, in particular, the existence of numbers \( \delta_M, \delta_N > 0, \delta_M < r_{inj}(M), \delta_N < r_{inj}(N) \), and uniformly locally finite covers \( U_M = \{ U_{\delta_M}(x_i) \}; U_N = \{ U_{\delta_N}(y_j) \}; \) by normal coordinate neighbourhoods \( (U_{\delta_M}(x_i), x^1, \ldots, x^n), (U_{\delta_N}(y_j), y^1, \ldots, y^r) \). Let \( \nabla^g, \nabla^h \) be the Levi-Civita connections of \( M \) resp. \( N \) and \( f \in C^\infty(M, N) \). The differential \( df \equiv f_* \) can be considered as a section of \( T^*M \otimes f^*TN \). Then \( f \) induces the connection \( f^*\nabla^h \) in the induced bundle \( f^*TN \) which is locally given by

\[
\Gamma_{a j}^i = \partial_a f^m(x) \Gamma_{m j}^i(f(x)), \quad \partial_a = \frac{\partial}{\partial x^a}.
\]
A coordinate free description is given by

\[(f^*\nabla^h)_X(Y_{f(x)}, x) = (\nabla^h_{f_*X} Y_{f(x)}, x).\]

The \(\nabla^g\) and \(f^*\nabla^h\) induce connections \(\nabla\) in all tensor bundles \(T^a(M) \otimes f^*T^b(N)\). Therefore, \(\nabla^m df\) is well defined. Since \((B_0)\) implies the boundedness of the \(g_{ab}, g^{uv}, h_{ij}\) in normal coordinates, the conditions of \(df\) bounded, in local coordinates

\[|df|^2 = \text{tr}^g(f^*h) = g^{ab}h_{ij} \partial_a f^i \partial_b f^j < \infty,\]

and \(\partial_a f^i\) bounded, are equivalent.

**Proposition 4.1.** Assume \((M^n, g), (N^n, h)\) being open, complete, satisfying \((B_k), (I)\), \(f \in C^\infty(M, N), b|df| < \infty\). Then for \(k \geq 1\) the following conditions are equivalent:

a) All \(\partial^{\alpha}/\partial x^\alpha f^p\) are bounded, \(|\alpha| \leq k, \rho = 1, \ldots, r\), where the derivatives are taken with respect to some uniformly locally finite atlas for \(M\) resp. \(N\).

b) All \(\nabla^\kappa df\) are bounded, \(\kappa \leq k - 1\).

c) All \(\partial^{\alpha}/\partial x^\alpha \Gamma^j_{ai}\) are bounded, \(|\alpha| \leq k - 1, a = 1, \ldots, n, i, j = 1, \ldots, r\).

For the simple proof we refer to [9, p. 143]. □

Denote for \((M^n, g), (N^n, h)\) of bounded geometry up to order \(k\) by \(C^{\infty,k}(M, N)\) the set of all \(f \in C^\infty(M, N)\) which satisfy

\[b^{k-1} |df| = \sum_{\alpha=0}^{k-1} \sup_{x \in M} |\nabla^\alpha df|_x < \infty.\]

Suppose now that \(0 < \delta < \delta_N < r_{\text{inj}}(N)\) and \(Y \in C^\infty(f^*TN) = \Omega^0(f^*TN)\) with \(b|Y|_{f^*h} = \sup_{x \in M} |Y_{f(x)}|_{h,x} < \delta\). Then the map \(x \Rightarrow \exp_{f(x)} Y_{f(x)}, \text{i.e. } g_Y = \exp_f Y \circ f\), defines an element of \(C^\infty(M, N)\). More generally, if \(b^{k}|Y| \equiv \sum_{\alpha=0}^{k} \sup_{x \in M} |\nabla^\alpha Y| < \delta\), then the map \(f(x) \rightarrow \exp_{f(x)} Y_{f(x)} \in C^{\infty,k}(M, N)\). This follows from the chain rule and the fact that \((B_k)\) implies \(d^i(\exp)\) bounded, \(0 \leq i \leq k\) \((d^i(\exp)\) is bounded for \(0 \leq i \leq k\) if and only if \(d^i\Gamma\) is bounded, \(0 \leq j \leq k - 1\), apply 2.5). If additionally \(b^{k-1} |df| < \infty\), then \(g_Y = \exp_f Y \circ f \in C^{\infty,k}(M, N)\).

We define for \(0 \leq \epsilon \leq \delta_N < r_{\text{inj}}(N)\) and \(f \in C^{\infty,k}(M, N)\)

\[b^{k}U_\epsilon(f) = \left\{ g \in C^{\infty,k}(M, N) \left| \begin{array}{l} \text{There is an } Y \in b^{k}\Omega^0(f^*TN) \text{ such that } \left. \begin{array}{l} g = g_Y = \exp_f Y \circ f \text{ and } b^{k} |Y| < \epsilon \end{array} \right. \end{array} \right\} \right\}
\]

and set

\[b^{k}d(f, g) := b^{k} |Y|.\]
Lemma 4.2. The system of all $b^kV_\varepsilon(f)$, where $0 < \varepsilon \leq \delta_N < \operatorname{r}_{\operatorname{inj}}(N)$, and $f \in C^{\infty,k}(M, N)$, forms a base of neighbourhood filters for a locally metrizable topology on $C^{\infty,k}(M, N)$.

For the proof we refer to [9, p. 145]. □

Let $b^k\Omega(M, N)$ be the completion of $C^{\infty,k}(M, N)$.

Theorem 4.3. Suppose $(M^n, g), (N^r, h)$ being open, complete and of bounded geometry up to the order $k \geq 1$. Then each component of $b^k\Omega(M, N)$ is a Banach manifold.

Proof. The space $b^k\Omega(M, N)$ is locally contractible, in particular locally arcwise connected and therefore components coincide with arc components, i.e. with bounded homotopy classes in $b\Omega(M, N)$. Let $f \in b^k\Omega(M, N)$ and set $T_f^b\Omega(M, N) := b\Omega_0^0(f^*TN)$. Then $Y \rightarrow g_Y, g_Y(x) = \exp f(x) Y_f(x) = (\exp f \circ f)(x)$ is for $0 < \delta \leq \delta_N$ a homeomorphism between $B_{\delta}(0) \subset b\Omega^0(f^*TN)$ and $b\Omega_0(f^*TN)$, i.e. a chart. It can be established in the same manner as for $M, N$ closed that the transition maps are $C^\infty$ (in the Banach category). If $f' \in \operatorname{comp}(f)$, there exists a bounded homotopy $\{f_t\}_{0 \leq t \leq 1}, f_t \in b^k\Omega(M, N), f_0 = f, f_1 = f'$, and $f^*TN - f_0^*TN \cong f_1^*TN - f_*^*TN$, i.e. $b\Omega_0^0(f^*TN)$ and $b\Omega_0^0(f_1^*TN)$ are equivalent Banach spaces. Here the equivalence of the norms essentially follows from Corollary 2.5 and Proposition 4.1. Therefore each component of $b^k\Omega(M, N)$ is locally modelled over the same Banach space. □

All the concepts above are applicable to sections of fibre bundles $E \rightarrow M$ if we assume that $M, E$ and the vertical bundle have bounded geometry up to the order $k$ and the fibres are totally geodesic. The case of the product bundle $M \times F \rightarrow M$ corresponds just to the case above: A section can be identified with a map $(\operatorname{id}, f) : E \rightarrow M \times F$. For an arbitrary fibre bundle which satisfies the above geometric assumptions we consider the set $C^{\infty,k}(E)$ of smooth $C^k$-bounded sections, define metrizable neighbourhoods by composition with the exponential map in the vertical bundle of the fibres and perform completion as above. Thus we get a Banach manifold structure on each component of $b^k\Omega(E)$. The following special case is of particular meaning for the gauge theory on open manifolds.

Proposition 4.4. Let $(M^n, g)$ be open, complete, $P(M, G) \rightarrow M$ a principal fibre bundle, $\tilde{P} = P \times_{\operatorname{Ad}_G} G$. If $M, P, T_vP$ have bounded geometry up to the order $k$ and the fibres are totally geodesic, then $b^k\Omega(\tilde{P})$ is a Banach manifold.

As we have seen until now, the assumption of bounded geometry allowed to establish a Banach manifold property for the space of $C^k$-bounded maps between open manifolds $(M^n, g), (N^r, h)$. If we restrict to $C^k$-bounded diffeomorphisms, there arise several difficulties. This subset is no longer an open subset. But, if we restrict to diffeomorphisms which are bounded from below too, then we get an open subset and a manifold structure. Assume $M^n$ additionally oriented.
Set for \( k \geq 1 \)

\[
bb,k \mathcal{D}(M) = \left\{ f \in b,k \Omega(M,M) \left| f \text{ is injective, surjective, preserves orientation and } \inf_{x \in M} |df|_x > 0 \right. \right\}.
\]

The "\( bb \)" indicates that the differential is bounded from above and positively from below.

**Theorem 4.5.** The \( bb,k \mathcal{D}(M) \) is open in \( b,k \Omega(M,M) \). In particular, each component of \( bb,k \mathcal{D}(M,M) \) is a Banach manifold.

For the proof we refer to that of Lemma 5.3 of [9, p. 148–149]. \( \square \)

We want to show, that \( bb,k \mathcal{D}(M) \) is a completely metrizable topological group, i.e. we want to show three facts, that \( bb,k \mathcal{D}(M) \) is a group, a topological group and completely metrizable.

**Theorem 4.6.** \( bb,k \mathcal{D}(M) \) is a completely metrizable topological group.

**Proof.** At first we show that \( bb,k \mathcal{D}(M) \) is a group. Clearly, \( id_M \in bb,k \mathcal{D}(M) \). Let \( f \in bb,k \mathcal{D}(M) \). We obtain that \( f^{-1} \) is injective and surjective since the same holds for \( f \).

We have to show that \( \inf_{x \in M} |df^{-1}|_x > 0 \) and that \( |\nabla^i df^{-1}| \) is bounded, \( 0 \leq i \leq k-1 \). But, by assumption, \( \sup_{x \in M} |df|_x < \infty \), and \( df^{-1} = (f^{-1})_* = f^{-1} = (df)^{-1} \), therefore \( \inf_{x} |df^{-1}|_x > 0 \). In the same manner, \( \inf_{x} |df|_x > 0 \) implies \( \sup_{x} |df|_x < \infty \), i.e. \( |df^{-1}| \) is bounded. As Proposition 4.1 immediately implies, \( \nabla^k df^{-1} \) is bounded for \( 0 \leq \kappa \leq k-1 \) if and only if all \( \partial^\kappa / \partial x^\alpha (df^{-1})_{\mu,\nu} \) are bounded for \( |\alpha| \leq k-1, \mu,\nu = 1,\ldots,n \) and a uniformly locally finite atlas of normal charts. The \( (df^{-1}) \) are certain linear combinations of the \( (df)_{\rho\sigma} \), divided by \( \det(df) \). Applying now the quotient rule and using \( \partial^\kappa / \partial x^\alpha (df)_{\rho\sigma} \) bounded, \( 0 \leq |\alpha| \leq k-1, \inf_{x} |\det(df)|_x > 0 \), we conclude that \( |\nabla^k df^{-1}| \) is bounded, \( 0 \leq \kappa \leq k-1 \). For \( g, f \in bb,k \mathcal{D}(M) \), we use \( d(f \circ g) = (f \circ g)_* = f_* \circ g_* = (df \circ g) \circ dg \) and conclude by the Leibniz rule.

Assume now that \( Y_\nu \in b,k \Omega(TM), b,k |Y_\nu| \leq \delta < \tau_{inj}(M), f \in bb,k \mathcal{D}(M), f_\nu = \exp_f Y_\nu \circ f \in bb,k \mathcal{D}(M) \) and \( f_\nu \to f \) in the sense of \( b,k d(f_\nu, f) \xrightarrow{\nu \to \infty} 0 \). We remark that

\[
\tag{4.1}
\begin{align*}
\begin{array}{c}
b,k d(id, \exp Y) = b,k d(f, \exp_f Y) = b,k |Y| \end{array}
\end{align*}
\]

for arbitrary \( f \in bb,k \mathcal{D}(M) \). Then

\[
\begin{align*}
b,k (f^{-1}, f_\nu^{-1}) = & b,k d(f^{-1}, (\exp_f Y_\nu \circ f)^{-1}) \\
= & b,k d(f^{-1}, f^{-1} \circ (\exp_f Y_\nu)^{-1}) \\
= & b,k d(id, \exp(-Y_\nu)) = b,k |Y| \xrightarrow{\nu \to \infty} 0,
\end{align*}
\]

i.e. the forming of the inverse is continuous. Consider now \( f_\nu = \exp_f Y_\nu \circ f, g_\mu = \exp_g Z_\mu \circ g, b,k |Y_\nu|, b,k |Z_\mu| \) sufficiently small, \( Y_\nu \to 0, Z_\mu \to 0, f_\nu \to f, g_\mu \to g, f_\nu, f, g_\mu, g \in bb,k \mathcal{D}(M) \). We want to show that \( f_\nu \cdot g_\mu \to f \cdot g \) with respect to \( b,k d \).
But $f, g, f_\nu, g_\mu$ are bounded diffeomorphisms, $f$ maps the geodesic ball of radius $\delta_\mu$ centered at $g(x)$ into a metric ball of the radius $^b|df|\cdot\delta_\mu$ centered at $f(g(x))$, $f_\nu$ maps the same ball into the metric ball of radius $^b|df_\nu|\cdot\delta_\mu$ centered at $f_\nu(g(x))$, i.e. for $\mu, \nu$ sufficiently large ($^b|df_\nu|\cdot\delta_\mu < \delta < r_{\text{mij}}(M)$), $^b d(f_\nu g_\mu, f \cdot g)$ is well defined. Furthermore, $^b r d(f_\nu g_\mu, f \cdot g)$ is locally defined and locally

\[ ^b r d(f_\nu g_\mu, f \cdot g) \leq ^b r d(f_\nu g_\mu, f \nu g) + ^b r d(f_\nu g, f \cdot g) . \]  (4.2)

But globally $g_\mu \to g$, $^b r d(g_\mu y, y) \to 0$, by (4.1) $^b r d(e, y, y) \to 0$, $^b r d(e, y^{-1} f_\nu y) \to 0$, $^b r d(f_\nu g_\mu, f_\nu g) \to 0$ and globally $^b r d(f_\nu^{-1}, f^{-1}) \to 0$, $^b r d(g^{-1} f_\nu^{-1}, g^{-1} f^{-1}) \to 0$ (by (4.1)), $^b r d(f_\nu g, f \cdot y) \to 0$. Therefore the right hand side of (4.2) globally tends to zero and the same holds for the left hand side. $^{bb,k} D(M)$ is a topological group. Concerning the complete metrizability of $^{bb,k} D(M)$, we cite the following proposition from [4, p. 49].

**Proposition 4.7.** The topology of a topological group is metrizable if and only if there exists a countable filter base $U_\nu$ of neighbourhoods of the unit element with $\bigcap_\nu U_\nu = \{e\}$. Then the topology can be defined by a left or right invariant metric.

**Proof.** At first one defines inductively a sequence $(V_\nu)_\nu$ of symmetric neighbourhoods of $e$ such that $V_1 \subset U_1, V_{\nu+1} \subset U_\nu \cap V_\nu$. The $V_\nu$ also form a base for the neighbourhood filter. Then one defines $g : G \times G \to \mathbb{R}$ as follows: $g(x, x) = 0$. Assume $x \neq y$. Then either $x^{-1} y \in V_1$ and $g(x, y) := 1$ or there exists a largest $k \in \mathbb{Z}$ such that $x^{-1} y \in V_k$.

In this case $g(x, x) := 2^{-k}$. From the definition, $g(x, y) = g(y, x), g(x, y) \geq 0$ and $g(zx, zy) = g(x, y)$ for all $x, y, z \in G$. Finally one sets

\[ d(x, y) := \inf_{i=0}^{p-1} g(x_i, x_{i+1}) , \]  (4.3)

where the infimum is taken with respect to the set of all finite sequences $(x_i)_{0 \leq i \leq p}$ ($p$ arbitrary) such that $x_0 = x, x_p = y$. Then $d$ is a left invariant metric which satisfies

\[ \frac{1}{2} g(x, y) \leq d(x, y) \leq g(x, y) . \]

In our case, a countable base for the neighbourhood filter of the unit element $e = \text{id}_M$ is given by

\[ U_\nu = \text{bigl}\{ f \in ^{bb,k} D(M) \mid f = g_\nu = \exp Y, ^b r |Y| < \min\{1/\nu, r_{\text{mij}}(M)\}\}. \]

Then $^{bb,k} D(M)$ is complete by construction (or one applies 12.9.5. of [4] which leads to the same conclusion).

**Remark.** In our case, a metric on each component of $^{bb,k} D(M)$ which is equivalent to that of (4.3) can be explicitly constructed by covering an arc by $\varepsilon$-neighbourhoods and taking the infimum over all covers and arcs.
The space of metrics on noncompact manifolds

Clearly, different \( g, g' \) of the same component of \( b^{k+3}M(B_k, I) \cap C^{k+3}M \) give the same group \( b^{k+3}D(M) \) with equivalent metrics. This follows immediately from 3.8, 3.10 and 4.1. Moreover, \( b^{k+3}D(M) \) acts on each component of \( b^{k+3}M(B_k, I) \cap C^{k+3}M \), if one defines \( b^{k+3}D(M) \) and its metric with respect to this component.

If \( g \in M(B_k, I) \) is a smooth metric, then the isotropy group \( b^{k+3}D_g(M) \subset b^{k+3}D(M) \) is a subgroup of smooth diffeomorphisms. This follows from the fact that each element of the isotropy group is an element of \( I(M^n, g) \) and each \( C^1 \)-isometry of a smooth metric is smooth according to a theorem of Palais (cf. [16]). Therefore, \( b^{k+3}D_g(M) \subset I(M^n, g) \).

**Lemma 4.7.** For \( g \) smooth, \( b^{k+3}D_g(M) = I(M, g) \).

**Proof.** We have to show \( I(M, g) \subset b^{k+3}D_g(M) \), i.e. the differential \( df \) of each \( f \in I(M, g) \) has bounded covariant derivatives up to the order \( k + 2 \). There holds by definition

\[
(\nabla_X df)Y = \nabla_X df(Y) - df(\nabla_X Y),
\]

but the right hand side of (4.4) equals to zero since \( f \) is an isometry, i.e. \( \nabla = f^*\nabla \). Hence \( \nabla df = 0 \), \( \nabla^\kappa df = 0 \) for all \( \kappa \). \( \square \)

If \( f \) is an isometry of an \( C^{k+2} \)-metric, then according to a theorem of Calabi-Hartman, \( f \) is of class \( C^{k+3} \). Altogether, we obtained:

**Proposition 4.8.** For \( g \in b^{k+2}M(B_k, I) \cap C^{k+2}M \), \( b^{k+3}D_g(M) = I(M, g) \). \( \square \)

The isotropy group \( I(M^n, g) \) is a finite dimensional Lie group (cf. [12]). It inherits its manifold structure from the following imbedding cf. [12, II, §1].

Let \( O(M) \) be the orthonormal bundle of frames of \( (M^n, g) \), \( x_0 \in M \), and \( u_0 \in \pi^{-1}(x_0) \subset O(M) \) a fixed frame. Then \( I(M^n, g) \to O(M), f \to f_*u_0 \), is an imbedding. The induced topology on \( b^{k+3}D_g(M) = I(M, g) \) is the same as the original topology on \( b^{k+3}D_g(M) \) (i.e. the trace topology of \( b^{k+3}D(M) \)). This follows from 5.1 of [5].

**Corollary 4.9.** The group \( b^{k+3}D_g(M) \) is a closed, finite dimensional subgroup of \( b^{k+3}D(M) \) and a Lie group. \( \square \)

The next task would be to describe the orbit \( b^{k+3}D(M) \cdot g \subset \text{comp}(g) \) and to establish a diffeomorphism \( b^{k+3}D(M)/I(M, g) \to b^{k+3}D(M) \cdot g \).

To do this, one would like to apply the following well known

**Theorem.** If \( G \) is a Lie group, \( X \) a smooth manifold, \( G \times X \to X \) a smooth action, \( x \in X, G_x \) the isotropy group and if \( G \to X, g \to g \cdot x \), is a subimmersion, then \( G_x \) is a submanifold. If additionally \( G \cdot x \) is locally closed and the topology of \( G \) has a countable base, then \( G/G_x \to G \cdot x \) is a diffeomorphism of manifolds.

In our case, Banach theory for \( M \) and \( \text{Diff } M \), this theorem is not immediately applicable. The same holds for the \( L_2 \)-case. But we are able to prove the diffeomor-
phism under an additional assumption concerning the closedness of the image of the differential of the map $G \to X, g \to g \cdot x$, in the $L_2$-case. The main subject of this paper was to introduce into the problems concerning $\mathcal{M}/\text{DiffM}$ for open manifolds and their solution in the $C^k$ Banach case.

References