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On extensions of Sobolev functions defined on regular subsets of metric measure spaces

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Abstract

We characterize the restrictions of first-order Sobolev functions to regular subsets of a homogeneous metric space and prove the existence of the corresponding linear extension operator.

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1. Main definitions and results

Let (X, d, μ) be a metric space (X, d) equipped with a Borel measure μ, which is non-negative and outer regular, and is finite on every bounded subset. In this paper we describe the restrictions of first-order Sobolev functions to measurable subsets of X which have a certain regularity property.

There are several known ways of defining Sobolev spaces on abstract metric spaces, where of course we cannot use the notion of derivatives. Of particular interest to us, among these definitions, is the one introduced by Hajłasz [13]. But let us first consider a classical characterization of classical Sobolev spaces due to Calderón. Since it does not use derivatives, it can lead to yet another way of defining Sobolev spaces on metric spaces. In [2] (see also [3]) Calderón characterizes the Sobolev spaces $W^{k,p}(\mathbf{R}^n)$ in terms of L^p-properties of sharp maximal functions. To generalize this characterization to the setting of a metric measure space (X, d, μ), let f be a locally integrable

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real-valued function on X and let α be a positive number. Then the *fractional sharp maximal function* of f, is defined by

$$f^{\sharp}_{\alpha}(x) := \sup_{r>0} \frac{r^{-\alpha}}{\mu(B(x,r))} \int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu.$$

Here $B(x,r) := \{y \in X : d(y,x) < r\}$ denotes the open ball centered at x with radius r, and, for every Borel set $A \subset X$ with $\mu(A) < \infty$, f_A denotes the average value of f over A

$$f_A := \frac{1}{\mu(A)} \int_A f \, d\mu.$$

If $A = \emptyset$, we put $f_A := 0$.

In [2] Calderón proved that, for $1 < p \leqslant \infty$, the function u is in $W^{1,p}(\mathbf{R}^n)$, if and only if u and $u^{\sharp}_1$ are both in $L^p(\mathbf{R}^n)$. This result motivates us to introduce the space $CW^{1,p}(X,d,\mu)$, which we will call the *Calderón–Sobolev space*. We define it to consist of all functions u defined on X such that $u, u^{\sharp}_1 \in L^p(X)$. We equip this space with the Banach norm

$$\|u\|_{CW^{1,p}(X,d,\mu)} := \|u\|_{L^p(X)} + \|u^{\sharp}_1\|_{L^p(X)}.$$

Let us now recall the details of the definition of Hajłasz mentioned above. Hajłasz [13] introduced the Sobolev-type space on a metric space, $M^{1,p}(X,d,\mu)$ for $1 < p \leqslant \infty$. It consists of all functions $u \in L^p(X)$ for which there exists a function $g \in L^p(X)$ (depending on u) such that the inequality

$$|u(x) - u(y)| \leqslant d(x,y)(g(x) + g(y)) \tag{1.1}$$

holds μ-a.e. (This means that there is a set $E \subset X$ with $\mu(E) = 0$ such that (1.1) holds for every $x, y \in X \backslash E$.) As in [14] we will refer to all functions g which satisfy the inequality (1.1) as *generalized gradients* of u. $M^{1,p}(X,d,\mu)$ is normed by

$$\|u\|_{M^{1,p}(X,d,\mu)} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over the family of all generalized gradients of u.

In the case where $X = \Omega \subset \mathbf{R}^n$ is an open bounded domain with a Lipschitz boundary, d is the Euclidean distance and μ is n-dimensional Lebesgue measure on Ω, Hajłasz [13] showed that the space $M^{1,p}(\Omega,d,\mu)$ coincides with the Sobolev space $W^{1,p}(\Omega)$ and, moreover, that every function $u \in W^{1,p}(\mathbf{R}^n)$ satisfies (1.1) with $g = c\mathcal{M}\|\nabla u\|$. Here $\mathcal{M}$ is the Hardy–Littlewood maximal operator and $c = c(n)$. (For further development and application of this approach to Sobolev spaces on metric space see, e.g., [9–11,13–15,17,19,22] and references therein).

It turns out that for a *doubling* measure μ, the Hajłasz–Sobolev space coincides with the Calderón–Sobolev space, i.e.,

$$CW^{1,p}(X,d,\mu) = M^{1,p}(X,d,\mu), \quad 1 < p \leqslant \infty,$$

and, moreover, for every $u \in M^{1,p}(X,d,\mu)$, the function $g = cu^{\sharp}_1$ (with some constant $c = c(X)$) is a generalized gradient of u. This is an immediate consequence of a result of Hajłasz and Kinnunen. (See [14], Theorem 3.4).

We recall that a measure μ satisfies the *doubling condition* if there exists a constant $C_d \geqslant 1$ such that, for every $x \in X$ and $r > 0$,

$$\mu(B(x,2r)) \leqslant C_d \mu(B(x,r)). \tag{1.2}$$

As usual, see [5], we call a metric measure space (X, d, μ) with a doubling measure μ a *metric space of homogeneous type* and refer to C_d as a *doubling constant*.

In this paper we will only consider such metric measure spaces, which also satisfy an additional condition, namely that there exists a constant $C_{rd} > 1$ such that, for every $x \in X$ and $r > 0$,

$$C_{rd}\,\mu(B(x,r)) \leqslant \mu(B(x,2r)). \tag{1.3}$$

We call this condition the reverse doubling condition and refer to C_{rd} as a reverse doubling constant.

We will characterize the restrictions of Calderón–Sobolev and Hajłasz–Sobolev functions to *regular* subsets of a homogeneous metric space (X, d, μ).

Definition 1.1. A measurable set $S \subset X$ is said to be regular if there are constants $\theta_S \geqslant 1$ and $\delta_S > 0$ such that for every $x \in S$ and $0 < r \leqslant \delta_S$

$$\mu(B(x,r)) \leqslant \theta_S\,\mu(B(x,r) \cap S).$$

As non-trivial examples of regular subsets of $\mathbf{R}^n$ we can mention Cantor-like sets and Sierpiński-type gaskets (or carpets) of positive Lebesgue measure. (Regular subsets of $\mathbf{R}^n$ are often also referred to as Ahlfors n-regular or n-sets [21].) For properties of metric spaces supporting doubling measures and sets satisfying regularity conditions we refer to [1,20,21,30] and references therein.

Given a Borel set $A \subset X$, a function $f \in L^{1,\mathrm{loc}}(A)$ and $\alpha > 0$ we let $f^{\sharp}_{\alpha,A}$ denote the fractional sharp maximal function of f on A,

$$f^{\sharp}_{\alpha,A}(x) := \sup_{r>0} \frac{r^{-\alpha}}{\mu(B(x,r))} \int_{B(x,r)\cap A} |f - f_{B(x,r)\cap A}|\,d\mu, \quad x \in A. \tag{1.4}$$

Thus, $f^{\sharp}_{\alpha} = f^{\sharp}_{\alpha,X}$.

As usual, for a Banach space $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ of measurable functions defined on X and a Borel set $S \subset X$, we let $\mathcal{A}|_S$ denote the restriction of $\mathcal{A}$ to S, i.e., the Banach space

$$\mathcal{A}|_S := \{f : S \to \mathbf{R} : \exists F \in \mathcal{A} \text{ such that } F|_S = f\}$$

equipped with the standard quotient space norm

$$\|f\|_{\mathcal{A}|_S} := \inf\{\|F\|_{\mathcal{A}} : F \in \mathcal{A},\ F|_S = f\}.$$

We can now state the first main result of this paper.

Theorem 1.2. *Let (X, d, μ) be a metric space of homogeneous type satisfying the reverse doubling condition* (1.3) *and let S be a regular subset of X. Then a function $u \in L^p(S)$, $1 < p \leqslant \infty$, can be extended to a function $\tilde{u} \in CW^{1,p}(X, d, \mu)$ if and only if $u^{\sharp}_{1,S} \in L^p(S)$. In fact,*

$$\|u\|_{CW^{1,p}(X,d,\mu)|_S} \approx \|u\|_{L^p(S)} + \|u^{\sharp}_{1,S}\|_{L^p(S)}$$

with constants of equivalence depending only on C_d, C_{rd}, θ_S, δ_S and p. Moreover, there exists a linear continuous extension operator

$$\mathrm{Ext}_S : CW^{1,p}(X,d,\mu)|_S \to CW^{1,p}(X,d,\mu).$$

Its operator norm is bounded by a constant depending only on C_d, C_{rd}, θ_S, δ_S and p.

Let us apply this result to $X = \mathbf{R}^n$ with Lebesgue measure (clearly, in this case (1.3) is satisfied with $C_{rd} = 2^n$). Then, for every regular subset $S \subset \mathbf{R}^n$, we have:

(i) $W^{1,p}(\mathbf{R}^n)|_S = \{u : S \to \mathbf{R} : u, u^{\sharp}_{1,S} \in L^p(S)\},\ 1 < p \leqslant \infty$.
(ii) There is a linear continuous extension operator from $W^{1,p}(\mathbf{R}^n)|_S$ into $W^{1,p}(\mathbf{R}^n)$.

Observe that (ii) follows from a general result of Rychkov [27].

There is an extensive literature devoted to description of the restrictions of Sobolev functions to different classes of subsets of $\mathbf{R}^n$. We refer the reader to the books by Maz'ya [25] and by Maz'ya and Poborchi [26], the article of Farkas and Jakob [7] and also references in [25], [26] and [7], for numerous results and methods related to this topic. We also observe that the criterion (i) can be useful for description of Sobolev extension domains, i.e., domains $\Omega \subset \mathbf{R}^n$ such that $W^{1,p}(\mathbf{R}^n)|_\Omega = W^{1,p}(\Omega)$. For instance, it follows from a result of Koskela [23], that every Sobolev extension domain is a regular subset of $\mathbf{R}^n$ whenever $n - 1 < p < \infty$.

The second main result of the paper is the following.

Theorem 1.3. *Let* (X, d, μ) *be a homogeneous metric space satisfying condition* (1.3). *Then, for every regular subset S of X*,

$$M^{1,p}(X, d, \mu)|_S = M^{1,p}(S, d, \mu).$$

Moreover, there exists a linear continuous extension operator

$$\mathrm{Ext}_S : M^{1,p}(S, d, \mu) \to M^{1,p}(X, d, \mu)$$

such that $\|\mathrm{Ext}_S\| \leqslant C(C_d, C_{rd}, \theta_S, \delta_S, p)$.

For families of bounded domains in $\mathbf{R}^n$ satisfying a certain "plumpness" condition (the so-called $A(c)$-condition) Theorem 1.3 was proved by Hajłasz and Martio [17]. Harjulehto [18] has generalized this result to the case of homogeneous metric spaces (X, d, μ) and domains Ω satisfying the so-called $A^*(\varepsilon, \delta)$-condition. Observe that both $A(c)$- and $A^*(\varepsilon, \delta)$-sets are regular, but Cantor-type sets of positive Lebesgue measure in $\mathbf{R}^n$ provide examples of a regular subset which satisfies neither the $A(c)$- nor the $A^*(\varepsilon, \delta)$-condition.

In the particular case where $X = \mathbf{R}^n$, d is the Euclidean distance and μ is n-dimensional Lebesgue measure, Theorem 1.3 has been proved independently by Hajłasz et al. [16].

The proofs of Theorems 1.2 and 1.3 are based on a modification of the Whitney extension method suggested in the author's work [28] for the case of regular subsets of $\mathbf{R}^n$. (See also [29]).

We conclude this introduction by briefly describing the contents of the other sections of this paper. The crucial step of our approach is presented in Section 2. Without loss of generality we may assume that S is closed (see Lemma 2.1) so that $X \backslash S$ is open. Since μ is doubling, $X \backslash S$ admits a Whitney covering which we denote by W_S (Theorem 2.4).

To each ball $B = B(x_B, r_B) \in W_S$ we assign a measurable subset $H_B \subset S$, which we call the "*reflected quasi-ball* associated with the Whitney ball B". These sets H_B have the properties that $H_B \subset B(x_B, \gamma_1 r_B) \cap S$ and $\mu(B) \leqslant \gamma_2 \mu(H_B)$ whenever $r_B \leqslant \delta_S$. Furthermore, the family

$$\mathcal{H}_S := \{H_B : B \in W_S\}$$

has *finite multiplicity*, i.e., every point $x \in S$ belongs to at most γ_3 sets of the family $\mathcal{H}_S$. Here $\gamma_1, \gamma_2, \gamma_3$ are positive constants depending only on C_d, C_{rd} and θ_S. The existence of this family $\mathcal{H}_S$ of reflected quasi-balls is proved in Theorem 2.6.

The second step of the extension method and the proof of Theorem 1.3 are presented in Section 3. We fix functions $u \in M^{1,p}(S,d,\mu)$ and $g \in L^p(S)$ which satisfy the inequality (1.1) on *S*. Then we define an extension $\tilde{u}$ of *u* by the formula

$$\tilde{u}(x) = (\mathrm{Ext}_S\, u)(x) := \sum_{B \in W_S} u_{H_B} \varphi_B(x), \quad x \in X \setminus S. \tag{1.5}$$

Here $\{\varphi_B : B \in W_S\}$ is a partition of unity associated with the Whitney covering.

Finally, we define an extension $\tilde{g}$ of *g* by setting

$$\tilde{g}(x) := \sum_{B \in W_S} (g_{H_B} + |u_{H_B}|)\chi_{B^*}(x), \quad x \in X \setminus S,$$

where $B^* := B(x_B, \frac{9}{8} r_B)$. We show that $\tilde{u} \in L^p(X)$, $\tilde{g} \in L^p(X)$ and $\tilde{g}$ is a generalized gradient of $\tilde{u}$, i.e., the pair $(\tilde{u}, \tilde{g})$ satisfies the inequality (1.1) on *X*. Since $\tilde{u}|_S = u$, this proves that $u \in M^{1,p}(X,d,\mu)|_S$ so that Ext_S provides a *linear extension operator* from $M^{1,p}(S,d,\mu)$ into $M^{1,p}(X,d,\mu)$.

Section 4 is devoted to estimates of the sharp maximal function of the extension $\tilde{u} := \mathrm{Ext}_S\, u$. Given a function *f* defined on *S* we let $f^{\curlywedge}$ denote the extension of f to all of *X* which is obtained by simply setting $f^{\curlywedge}(x) = 0$ for all $x \in X \setminus S$. In Theorem 4.7 we show that, for every $\alpha > 0$ and $x \in X$,

$$(\tilde{u})^{\sharp}_{\alpha}(x) \leqslant C(\mathcal{M}(u^{\sharp}_{\alpha,S})^{\curlywedge}(x) + \mathcal{M}u^{\curlywedge}(x)).$$

Using this estimate and the Hardy–Littlewood maximal theorem we then prove a slightly more general version of Theorem 1.2 related to the function space $\mathcal{C}^{\alpha}_p(X,d,\mu)$. This space consists of all functions *u* defined on *X* such that $u, u^{\sharp}_{\alpha} \in L^p(X)$. $\mathcal{C}^{\alpha}_p(X,d,\mu)$ is normed by

$$\|u\|_{\mathcal{C}^{\alpha}_p(X,d,\mu)} := \|u\|_{L^p(X)} + \|u^{\sharp}_{\alpha}\|_{L^p(X)}.$$

For the case $X = \mathbf{R}^n$ with Lebesgue measure this space was introduced and investigated by DeVore and Sharpley [6] and Christ [4]. Clearly,

$$CW^{1,p}(X,d,\mu) = \mathcal{C}^1_p(X,d,\mu).$$

Our generalization of Theorem 1.2 is as follows:

Theorem 1.4. *Let* (X,d,μ) *be a metric space of homogeneous type satisfying condition* (1.3) *and let S be a regular subset of X. A function* $u \in L^p(S)$, $1 < p \leqslant \infty$, *belongs to the trace space* $\mathcal{C}^{\alpha}_p(X,d,\mu)|_S$ *if and only if* $u^{\sharp}_{\alpha,S} \in L^p(S)$. *In fact,*

$$\|u\|_{\mathcal{C}^{\alpha}_p(X,d,\mu)|_S} \approx \|u\|_{L^p(S)} + \|u^{\sharp}_{\alpha,S}\|_{L^p(S)} \tag{1.6}$$

and there exists a linear continuous extension operator

$$\mathrm{Ext}_S : \mathcal{C}^{\alpha}_p(X,d,\mu)|_S \to \mathcal{C}^{\alpha}_p(X,d,\mu)$$

whose operator norm is bounded by a constant depending only on C_d, C_{rd}, θ_S, δ_S *and p.*

Observe that in the case where $X = \mathbf{R}^n$ and *S* is a Lipschitz or an (ε, δ)-domain this result follows from extension theorems proved by DeVore and Sharpley [6, pp. 99–101], (Lipschitz domains), and Christ [4] ((ε, δ)-domains).

2. The Whitney covering and a family of reflected quasi-balls

We will use the following notation. Throughout the paper $C, C_1, C_2, \dots$ will be generic positive constants which depend only on $C_d, C_{rd}, \theta_S, \delta_S$ and p. These constants can change, even in a single string of estimates. We write $A \approx B$ if there is a constant C such that $A/C \leqslant B \leqslant CA$. For a ball $B = B(x,r)$ we let x_B and r_B denote the center and radius of B. Given a constant $\lambda > 0$ we let λB denote the ball $B(x, \lambda r)$. For $A, B \subset X$ and $x \in X$ we put

$$\operatorname{dist}(A, B) := \inf\{d(a,b) : a \in A, b \in B\}$$

and $d(x, A) := \operatorname{dist}(\{x\}, A)$. Finally, by $\operatorname{cl}(A)$ we denote the closure of A in X.

Lemma 2.1. *For every regular subset $S \subset X$*

$$\mu(\operatorname{cl}(S)\backslash S) = 0.$$

Proof. Denote $Y := \operatorname{cl}(S)\backslash S$ and fix $y \in Y$. Then for every r, $0 < r \leqslant \delta$, there is a point $\tilde{y} \in S$ such that $\operatorname{dist}(y, \tilde{y}) \leqslant r/4$. Clearly, $B(\tilde{y}, r/4) \subset B(y, r)$. Since S is regular and $\tilde{y} \in S$, we obtain

$$\mu(B(y,r) \cap S) \geqslant \mu(B(\tilde{y}, r/4) \cap S) \geqslant \theta_S \mu(B(\tilde{y}, r/4)).$$

On the other hand, $B(y,r) \subset B(\tilde{y}, 5r/4)$ so that by the doubling condition

$$\mu(B(y,r)) \leqslant \mu(B(\tilde{y}, 5r/4)) \leqslant C_d^3 \mu(B(\tilde{y}, r/4)).$$

Hence $\mu(B(y,r) \cap S) \geqslant \theta_S C_d^{-3} \mu(B(y,r))$. We let D_A denote the family of density points of the set $A := X\backslash S$. Then

$$\frac{\mu(B(y,r) \cap A)}{\mu(B(y,r))} < 1 - \theta_S C_d^{-3}, \quad y \in Y,$$

which implies $Y \cap D_A = \emptyset$. Thus, $Y \subset A\backslash D_A$ so that by Lebesgue's theorem, see, e.g. [8, Section 2.9], $\mu(Y) \leqslant \mu(A\backslash D_A) = 0$. $\square$

In the remaining part of the paper we will assume that S is a *closed* regular subset of X.

We will need the following technical lemma.

Lemma 2.2. *Let $B(b,r) \subset X\backslash S$ be a ball in X and let C_1, C_2 be two positive constants such that $C_1 r \leqslant \operatorname{dist}(B, S) \leqslant C_2 r$. Then for every $\lambda > 0$ and every $x \in \lambda B$ we have*

$$(C_1 - \lambda)r \leqslant d(x, S) \leqslant (C_2 + 1 + \lambda)r.$$

Proof. Since $\operatorname{dist}(\cdot, S)$ is a Lipschitz function, for every $a \in B$ we have

$$d(x,S) \leqslant d(a,S) + d(a,x) \leqslant d(a,S) + d(a,b) + d(b,x) \leqslant d(a,S) + (1+\lambda)r.$$

Since $a \in B$ is arbitrary, we obtain

$$d(x,S) \leqslant \operatorname{dist}(B,S) + (1+\lambda)r \leqslant (C_2 + 1 + \lambda)r.$$

On the other hand,

$$d(x,S)\geqslant d(b,S)-d(b,x)\geqslant \operatorname{dist}(B,S)-\lambda r\geqslant (C_1-\lambda)r$$

proving the lemma. □

The next lemma easily follows from inequalities (1.2) and (1.3).

Lemma 2.3. *For every* $x\in X, r>0, 1\leqslant t<\infty$,

$$\mu(B(x,r))\leqslant C_{rd}t^{-\alpha}\mu(B(x,tr)) \tag{2.1}$$

and

$$\mu(B(x,tr))\leqslant C_d t^{\beta}\mu(B(x,r)), \tag{2.2}$$

where $\alpha:=\log_2 C_{rd}$ *and* $\beta:=\log_2 C_d$.

Theorem 2.4. *There is a countable family of balls* W_S *such that*

(i) $X\backslash S=\cup\{B : B\in W_S\}$.
(ii) *For every ball* $B=B(x,r)\in W_S$

$$3r\leqslant \operatorname{dist}(B(x,r),S)\leqslant 25r. \tag{2.3}$$

(iii) *Every point of* $X\backslash S$ *is covered by at most* $N=N(C_d)$ *balls from* W_S.

Proof. Since μ is a doubling measure, there exists a constant $M_\varepsilon=M(\varepsilon,C_d)$ such that in every ball $B(x,r)$ there are at most M_ε points $\{x_j\}$ satisfying the inequality $d(x_i,x_j)\geqslant\varepsilon r,\ i\neq j$ (one can put $M_\varepsilon:=(4\varepsilon^{-1})^{\log_2 C_d}$). In [12], Theorem 2.3, it was shown that for every metric space with this property the following is true: for every open subset $G\subset X$ with a non-empty boundary there is a countable family of balls $\mathcal{W}_G$ such that $G=\cup\{B : B\in\mathcal{W}_G\}$, every point of G is covered by at most $9M_{\frac{1}{2}}$ sets from $\mathcal{W}_G$ and $r\leqslant\operatorname{dist}(B(x,r),\partial G)\leqslant 4r$ for every $B=B(x,r)\in\mathcal{W}_G$.

Let us apply this result to the open set $G=X\backslash S$. We conclude that there exists a countable family of balls $\widetilde{W}_S$ which covers $X\backslash S$ with multiplicity at most $9M_{\frac{1}{2}}$. Moreover, every ball $\widetilde{B}=B(x,r)\in\widetilde{W}_S$ satisfies the following inequality:

$$r\leqslant\operatorname{dist}(\widetilde{B},S)\leqslant 4r. \tag{2.4}$$

Let us slightly modify $\widetilde{W}_S$ and construct a family of balls W_S satisfying the inequality (2.3) rather than (2.4). To this end we put $\varepsilon:=\frac{1}{4}$ and given $\widetilde{B}\in\widetilde{W}_S$ fix a maximal ε-net in $\widetilde{B}$, i.e., a family of points $\{x_i,\ i\in I_{\widetilde{B}}\}\subset\widetilde{B}$ satisfying the following conditions:

(a) $d(x_i,x_j)\geqslant\varepsilon$ for all $i,j\in I_{\widetilde{B}}, i\neq j$;
(b) for every $z\in\widetilde{B}$ there x_i such that $d(x_i,z)<\varepsilon$.

As we have noted above this family of points consists of at most M_ε elements.

We let $\mathcal{A}_{\widetilde{B}}$ denote a family of balls $\{B(x_i,\varepsilon r_{\widetilde{B}}),\ i\in I_{\widetilde{B}}\}$. Then, clearly, $\mathcal{A}_{\widetilde{B}}$ also consists of at most M_ε elements and by (b)

$$\widetilde{B}\subset\cup\{B(x_i,\varepsilon r_{\widetilde{B}}) : i\in I_{\widetilde{B}}\}.$$

We put $W_S := \cup\{\mathcal{A}_{\widetilde{B}} : \widetilde{B} \in \widetilde{W}_S\}$. Then property (i) is obvious. Since multiplicity of $\widetilde{W}_S$ is bounded by $9M_{\frac{1}{2}}$, the family W_S has multiplicity at most $9M_{\frac{1}{2}} \cdot M_\varepsilon$. This proves property (iii) of the theorem. To prove (ii) fix a ball $B = B(x_i, r) \in \mathcal{A}_{\widetilde{B}}$ with $r = \varepsilon r_{\widetilde{B}}$ and a point $a \in B$. Recall that $x_i \in \widetilde{B}$. Then by (2.4) and Lemma 2.2 (with $\lambda = 1$)

$$r_{\widetilde{B}} \leqslant d(x_i, S) \leqslant 6r_{\widetilde{B}}$$

so that

$$d(a, S) \leqslant d(x_i, S) + d(x_i, a) \leqslant 6r_{\widetilde{B}} + r = (6/\varepsilon + 1)r = 25r.$$

On the hand, by (2.4)

$$d(a, S) \geqslant d(x_i, S) - d(x_i, a) \geqslant \operatorname{dist}(\widetilde{B}, S) - r \geqslant r_{\widetilde{B}} - r = 3r.$$

Thus, for every $a \in B$ we have $3r \leqslant d(a, S) \leqslant 25r$ which implies property (ii). The theorem is proved. □

Theorem 2.4 and Lemma 2.2 imply the following additional properties of Whitney's balls.

Lemma 2.5. (a) *For every $B = B(x_B, r_B) \in W_S$ there is a point $y_B \in S$ such that*

$$B(y_B, r_B) \subset B(x_B, 30r_B) \quad and \quad B = B(x_B, r_B) \subset B(y_B, 30r_B). \tag{2.5}$$

Moreover, $\mu(B(x_B, r_B)) \approx \mu(B(y_B, r_B))$.

(b) *For every $B \in W_S$ and every $x \in B^*$*

$$r_B \leqslant d(x, S) \leqslant 28r_B. \tag{2.6}$$

(*Recall that $B^* := \frac{9}{8}B$*).

(c) *If $B, K \in W_S$ and $B^* \cap K^* \neq \emptyset$, then*

$$\tfrac{1}{28} r_B \leqslant r_K \leqslant 28r_B. \tag{2.7}$$

(d) *For every ball $K \in W_S$ there are at most N balls from the family $W_S^* := \{B^* : B \in W_S\}$ which intersect K^*.*

Here N is a positive constant depending only on C_d.

Proof. By Lemma 2.2 $d(x_B, S) \leqslant 27r_B$ so that there is a point $y_B \in B(x_B, 28r_B) \cap S$. Now the statement of part (a) easily follows from the inequality $d(x_B, y_B) \leqslant 28r_B$ and the doubling condition.

Property (b) immediately follows from the inequality (2.3) and Lemma 2.2. In turn, property (c) is a simple corollary of (b).

To prove (d) we put $\mathcal{A}_K := \{B \in W_S : B^* \cap K^* \neq \emptyset\}$ and $M := \operatorname{card} \mathcal{A}_K$. Then by (c) $\frac{1}{28} r_B \leqslant r_K \leqslant 28r_B$ for every $B \in \mathcal{A}_K$. Since $B^* \cap K^* \neq \emptyset$, we have

$$B \subset (9/8 + (9/8) \cdot 28 + 1)K \subset 34K,$$

so that

$$\mathcal{B}_K := \cup\{B : B \in \mathcal{A}_K\} \subset 34K. \tag{2.8}$$

In a similar way we prove that $K \subset 34B$ for each $B \in \mathcal{A}_K$ so that by Lemma 2.3 $\mu(K) \leqslant C_1\mu(B)$, $B \in \mathcal{A}_K$. Hence

$$C_1^{-1} M\mu(K) \leqslant \sum\{\mu(B) : B \in \mathcal{A}_K\}. \tag{2.9}$$

On the other hand, by property (iii) of Theorem 2.4

$$\sum\{\mu(B) : B \in \mathcal{A}_K\} = \int_{\mathcal{B}_K} \sum\{\chi_B : B \in \mathcal{A}_K\}\, d\mu \leqslant N(C_d)\mu(\mathcal{B}_K)$$

so that by (2.8) and Lemma 2.3

$$\sum\{\mu(B) : B \in \mathcal{A}_K\} \leqslant N(C_d)\mu(34K) \leqslant C_2\mu(K).$$

This and (2.9) imply $M \leqslant C_1C_2$ proving property (d). □

Let us formulate the main result of the section.

Theorem 2.6. *There is a family of Borel sets* $\mathcal{H}_S = \{H_B : B \in W_S\}$ *such that*:

(i) $H_B \subset (\gamma_1 B) \cap S,\ B \in W_S$.
(ii) $\mu(B) \leqslant \gamma_2\mu(H_B)$ *whenever* $B \in W_S$ *and* $r_B \leqslant \delta_S$.
(iii) $\sum_{B\in W_S} \chi_{H_B} \leqslant \gamma_3$.

Here $\gamma_1, \gamma_2, \gamma_3$ *are positive constants depending only on* C_{rd}, C_d *and* θ_S.

Proof. Let $K = B(x_K, r_K) \in W_S$ and let y_K be a point on S satisfying condition (a) of Lemma 2.5. Thus, $B(y_K, r_K) \subset CK$ and $K \subset B(y_K, Cr_K)$ with $C = 30$.

Given ε, $0 < \varepsilon \leqslant 1$, we denote $K_\varepsilon := B(y_K, \varepsilon r_K)$. Let $B = B(x_B, r_B)$ be a ball from W_S with $r_B \leqslant \delta_S$. Set

$$\mathcal{A}_B := \{K = B(x_K, r_K) \in W_S : K_\varepsilon \cap B_\varepsilon \neq \emptyset,\ r_K \leqslant \varepsilon r_B\}. \tag{2.10}$$

Recall that $B_\varepsilon := B(y_B, \varepsilon r_B)$. We define a "quasi-ball" H_B by letting

$$H_B := (B_\varepsilon \cap S)\backslash(\cup\{K_\varepsilon : K \in \mathcal{A}_B\}). \tag{2.11}$$

If $r_B > \delta_S$ we put $H_B := \emptyset$.

Prove that for some $\varepsilon := \varepsilon(C_{rd}, C_d, \theta_S)$ small enough the family of subsets $\mathcal{H}_S := \{H_B : B \in W_S\}$ satisfies conditions (i)–(iii). By (2.11) and (2.5)

$$H_B \subset B_\varepsilon := B(y_B, \varepsilon r_B) \subset B(y_B, r_B) \subset B(x_B, Cr_B) = CB.$$

In addition, by (2.11) $H_B \subset S$ so that $H_B \subset (CB) \cap S$ proving property (i).

Let us prove (ii). Suppose that $B = B(x_B, r_B) \in W_S$ and $r_B \leqslant \delta_S$. If $K \in \mathcal{A}_B$, then by (2.10) $K_\varepsilon \cap B_\varepsilon \neq \emptyset$ and $r_K \leqslant \varepsilon r_B$. Hence

$$r_{K_\varepsilon} (= \varepsilon r_K) \leqslant \varepsilon r_{B_\varepsilon} (:= \varepsilon^2 r_B) \leqslant r_{B_\varepsilon}$$

so that $y_K \in 2B_\varepsilon$. But $r_K \leqslant \varepsilon r_B = r_{B_\varepsilon}$ and $K \subset B(y_K, Cr_K)$ which implies $K \subset (C+2)B_\varepsilon$. Thus

$$\mathcal{U}_B := \cup\{K : K \in \mathcal{A}_B\} \subset (C+2)B_\varepsilon. \tag{2.12}$$

By property (iii) of Theorem 2.4

$$\sum_{K\in\mathcal{A}_B}\chi_K(x)\leqslant\sum_{K\in W_S}\chi_K(x)\leqslant N=N(C_d),\quad x\in X,$$

so that by (2.12) and (2.2)

$$\sum_{K\in\mathcal{A}_B}\mu(K)=\int_{\mathcal{U}_B}\sum_{K\in\mathcal{A}_B}\chi_K\,d\mu\leqslant\int_{(C+2)B_\varepsilon}N\,d\mu=N\mu((C+2)B_\varepsilon)\leqslant C_1\mu(B_\varepsilon).$$

On the other hand, for every $K\in\mathcal{A}_B$ by (2.1) and by (a), Lemma 2.5

$$\mu(K_\varepsilon)=\mu(B(y_K,\varepsilon r_K))\leqslant C_{rd}\varepsilon^\alpha\mu(B(y_K,r_K))\leqslant C_2\varepsilon^\alpha\mu(K).$$

Hence

$$\mu(\cup\{K_\varepsilon:K\in\mathcal{A}_B\})\leqslant\sum_{K\in\mathcal{A}_B}\mu(K_\varepsilon)\leqslant C_2\varepsilon^\alpha\sum_{K\in\mathcal{A}_B}\mu(K)\leqslant C_3\varepsilon^\alpha\mu(B_\varepsilon).$$

Since S is regular and $r_{B_\varepsilon}=\varepsilon r_B\leqslant\delta_S$, $\mu(B_\varepsilon\cap S)\geqslant\theta_S^{-1}\mu(B_\varepsilon)$ so that

$$\begin{aligned}\mu(H_B)&=\mu((B_\varepsilon\cap S)\backslash(\cup\{K_\varepsilon:K\in\mathcal{A}_B\}))\\&\geqslant\mu(B_\varepsilon\cap S)-\mu(\cup\{K_\varepsilon:K\in\mathcal{A}_B\})\geqslant(\theta_S-C_3\varepsilon^\alpha)\mu(B_\varepsilon).\end{aligned}$$

By (2.2) and by property (a) of Lemma 2.5

$$\begin{aligned}\mu(B_\varepsilon)=\mu(B(y_B,\varepsilon r_B))&\geqslant C_d^{-1}\varepsilon^\beta\mu(B(y_B,r_B))\\&\geqslant C^{-1}C_d^{-1}\varepsilon^\beta\mu(B(x_B,r_B))=C_4\varepsilon^\beta\mu(B)\end{aligned}$$

so that

$$\mu(H_B)\geqslant C_4(\theta_S^{-1}-C_3\varepsilon^\alpha)\varepsilon^\beta\mu(B).$$

We define ε by setting $\varepsilon:=(2C_3\theta_S)^{-\frac{1}{\alpha}}$. Then the inequality $\mu(B)\leqslant\gamma_2\mu(H_B)$ holds with $\gamma_2:=2C_4^{-1}\theta_S^{\frac{\beta}{\alpha}+1}(2C_3)^{\frac{\beta}{\alpha}}$ proving property (ii) of the theorem.

Let us prove (iii). Let $B=B(x_B,r_B)$, $B'=B(x_{B'},r_{B'})\in W_S$ be Whitney's balls such that $r_B,r_{B'}\leqslant\delta_S$ and $H_B\cap H_{B'}\neq\emptyset$. Since $H_B\subset B_\varepsilon$, $H_{B'}\subset B'_\varepsilon$, we have $B_\varepsilon\cap B'_\varepsilon\neq\emptyset$.

On the other hand, $B\notin\mathcal{A}_{B'}$ and $B'\notin\mathcal{A}_B$, otherwise by (2.10) and (2.11) $H_B\cap H_{B'}=\emptyset$. Since $B_\varepsilon\cap B'_\varepsilon\neq\emptyset$, by definition (2.10) $r_B>\varepsilon r_{B'}$ and $r_{B'}>\varepsilon r_B$ so that $r_B\approx r_{B'}$. By (2.5)

$$B_\varepsilon=B(y_B,\varepsilon r_B)\subset B(y_B,r_B)\subset CB$$

and similarly $B'_\varepsilon\subset CB'$. But $B_\varepsilon\cap B'_\varepsilon\neq\emptyset$ so that $CB\cap CB'\neq\emptyset$ as well. Moreover, since $r_B\approx r_{B'}$, we have $B'\subset C_5B$ and $B\subset C_5B'$. This and the doubling condition imply $\mu(B')\approx\mu(B)$.

We denote

$$\mathcal{T}_B:=\{B'\in W_S:H_B\cap H_{B'}\neq\emptyset,\ r_{B'}\leqslant\delta_S\}$$

and $\mathcal{V}_B:=\cup\{B':B'\in\mathcal{T}_B\}$. Thus, we have proved that $\mathcal{V}_B\subset C_5B$ and $\mu(B')\approx\mu(B)$ for every $B'\in\mathcal{T}_B$.

Let $M_B := \operatorname{card} \mathcal{T}_B$ be the cardinality of $\mathcal{T}_B$. Clearly, to prove (iii) it suffices to show that $M_B \leqslant \gamma_3$. We have

$$M_B \mu(B) \leqslant C \sum_{B' \in \mathcal{T}_B} \mu(B') = C \int_{\mathcal{V}_B} \sum_{B' \in \mathcal{T}_B} \chi_{B'} \, d\mu \leqslant C \int_{C_5 B} \sum_{B' \in \mathcal{T}_B} \chi_{B'} \, d\mu.$$

By the property (iii) of Theorem 2.4

$$\sum \{\chi_{B'} : B' \in \mathcal{T}_B\} \leqslant \sum \{\chi_{B'} : B' \in W_S\} \leqslant N = N(C_d)$$

so that

$$M_B \mu(B) \leqslant C \int_{C_5 B} N \, d\mu = C N \mu(C_5 B) \leqslant C \mu(B)$$

proving the required inequality $M_B \leqslant \gamma_3$. $\square$

3. The extension operator: proof of Theorem 1.3

For every $u \in M^{1,p}(X, d, \mu)$ and every generalized gradient g of u the restriction $g|_S$ is a generalized gradient of $u|_S$ so that $M^{1,p}(X, d, \mu)|_S \subset M^{1,p}(S, d, \mu)$.

Let us prove that formula (1.5) provides a linear continuous extension operator from $M^{1,p}(S, d, \mu)$ into $M^{1,p}(X, d, \mu)$. Obviously, this will imply the converse imbedding as well.

Recall that for every $u \in M^{1,p}(S, d, \mu)$ its generalized gradient g belongs to $L^p(S)$ and satisfies the inequality

$$|u(x) - u(y)| \leqslant d(x, y)(g(x) + g(y)), \quad x, y \in S \setminus E, \tag{3.1}$$

where E is a subset of S of measure 0. We may suppose that g is almost optimal, i.e., $\|g\|_{L^p(S)} \leqslant 2\|u\|_{M^{1,p}(S,d,\mu)}$.

The extension operator Ext_S, see (1.5), is determined by the family of Borel subsets $\mathcal{H}_S = \{H_B : B \in W_S\}$ introduced in the previous section. We recall that $\mu(H_B) > 0$ for every ball $B \in W_S$ with $r_B \leqslant \delta_S$ and $H_B := \emptyset$ whenever $r_B > \delta_S$. Therefore, according to our notation u_{H_B} is the average of u over H_B whenever $r_B \leqslant \delta_S$ and $u_{H_B} := 0$ otherwise.

We let $\Phi_S = \{\varphi_B : B \in W_S\}$ denote a partition of unity associated to the Whitney covering W_S, see, e.g. [24]. We recall that Φ_S is a family of functions defined on X which have the following properties: For every ball $B \in W_S$ (a) $0 \leqslant \varphi_B \leqslant 1$; (b) $\operatorname{supp} \varphi_B \subset B^* (:= \frac{9}{8} B)$; (c) $\sum \{\varphi_B(x) : B \in W_S\} = 1$ on $X \setminus S$; (d) for some constant $C = C(C_d)$

$$|\varphi_B(x) - \varphi_B(y)| \leqslant C \frac{d(x, y)}{r_B}, \quad x, y \in X.$$

Recall that the extension operator $\tilde{u} = \operatorname{Ext}_S u$ is defined by the formula

$$\tilde{u}(x) := \sum_{B \in W_S} u_{H_B} \varphi_B(x), \quad x \in X \setminus S, \tag{3.2}$$

and $\tilde{u}(x) := u(x)$, $x \in S$. We also define an extension $\tilde{g}$ of g by letting

$$\tilde{g}(x) := \sum_{B \in W_S} (g_{H_B} + |u_{H_B}|) \chi_{B^*}(x), \quad x \in X \setminus S, \tag{3.3}$$

and $\tilde{g}(x) := g(x)$ for $x \in S$.

To prove that Ext_S satisfies conditions of Theorem 1.3 it suffices to show that

$$\|\tilde{u}\|_{L^p(X)} \leqslant C\|u\|_{L^p(S)}, \quad \|\tilde{g}\|_{L^p(X)} \leqslant C(\|g\|_{L^p(S)} + \|u\|_{L^p(S)}) \tag{3.4}$$

and the inequality

$$|\tilde{u}(x) - \tilde{u}(y)| \leqslant Cd(x,y)(\tilde{g}(x) + \tilde{g}(y)) \tag{3.5}$$

holds μ-a.e. on X. Then

$$\|\tilde{u}\|_{M^{1,p}(X,d,\mu)} \leqslant \|\tilde{u}\|_{L^p(X)} + \|\tilde{g}\|_{L^p(X)} \leqslant C(\|u\|_{L^p(S)} + \|g\|_{L^p(S)})$$

proving that $\|\tilde{u}\|_{M^{1,p}(X,d,\mu)} \leqslant C\|u\|_{M^{1,p}(S,d,\mu)}$ and $\|\mathrm{Ext}_S\| \leqslant C$.

Proofs of inequalities (3.4) and (3.5) are based on a series of auxiliary lemmas.

Lemma 3.1. *Let $H, H' \subset S$ and let $0 < \mu(H), \mu(H') < \infty$. Then*

$$|u_H - u_{H'}| \leqslant \mathrm{diam}(H \cup H')(g_H + g_{H'}) \tag{3.6}$$

and for every $y \in S$

$$|u_H - u(y)| \leqslant \mathrm{diam}(H \cup \{y\})(g_H + g(y)). \tag{3.7}$$

Proof. We have

$$I := |u_H - u_{H'}| \leqslant \frac{1}{\mu(H)} \frac{1}{\mu(H')} \int_H \int_{H'} |u(x) - u(y)|\, d\mu(x)\, d\mu(y)$$

so that by (3.1)

$$I \leqslant \frac{1}{\mu(H)} \frac{1}{\mu(H')} \int_H \int_{H'} d(x,y)(g(x) + g(y))\, d\mu(x)\, d\mu(y).$$

Since $d(x,y) \leqslant \mathrm{diam}(H \cup H')$ for every $x \in H$, $y \in H'$, we have

$$I \leqslant \frac{\mathrm{diam}(H \cup H')}{\mu(H)\mu(H')} \int_H \int_{H'} (g(x) + g(y))\, d\mu(x)\, d\mu(y) = \mathrm{diam}(H \cup H')(g_H + g_{H'})$$

proving (3.6). In a similar way we prove the inequality (3.7). □

Lemma 3.2. *Let $\widetilde{B} \in W_S$ and let $x \in \widetilde{B}$. Then for every $y \in X \setminus S$ and every ball $B \in W_S$ such that $B^* \cap \{x, y\} \neq \emptyset$ we have*

$$|u_{H_B} - u_{H_{\widetilde{B}}}| \leqslant C(d(x,S) + d(x,y) + d(y,S))(\tilde{g}(x) + \tilde{g}(y)). \tag{3.8}$$

If $y \in S$, then for every $B \in W_S$ such that $B^* \ni x$

$$|u_{H_B} - u(y)| \leqslant Cd(x,y)(\tilde{g}(x) + \tilde{g}(y)). \tag{3.9}$$

Proof. First, we prove (3.8). Suppose that $y \in X \setminus S$ and consider the case $r_B \leqslant \delta_S, r_{\widetilde{B}} \leqslant \delta_S$. Since $\mu(H_B), \mu(H_{\widetilde{B}}) > 0$, by (3.6)

$$|u_{H_B} - u_{H_{\widetilde{B}}}| \leqslant \mathrm{diam}(H_B \cup H_{\widetilde{B}})(g_{H_B} + g_{H_{\widetilde{B}}}). \tag{3.10}$$

By (2.6) $r_B \approx d(x,S)$ whenever $x \in B^*$ and by property (i) of Theorem 2.6, $H_B \subset \gamma_1 B$ so that $H_B \subset B(x, Cd(x,S))$. Since $x \in \widetilde{B} \subset (\widetilde{B})^*$, we also have $H_{\widetilde{B}} \subset B(x, C_2 d(x,S))$.

In a similar way we prove that $H_B \subset B(y, C_2 d(y, S))$ whenever $y \in B^*$. Hence

$$\operatorname{diam}(H_B \cup H_{\widetilde{B}}) \leqslant \operatorname{diam}(B(x, C_2 d(x, S)) \cup B(y, C_2 d(x, S)))$$

so that

$$\operatorname{diam}(H_B \cup H_{\widetilde{B}}) \leqslant C_2(d(x, S) + d(x, y) + d(y, S)). \tag{3.11}$$

Since $r_B, r_{\widetilde{B}} \leqslant \delta_S$ and $x \in B^*$ or $y \in B^*$, by definition of $\tilde{g}$, see (3.3), we have $g_{H_{\widetilde{B}}} \leqslant \tilde{g}(x)$, $g_{H_B} \leqslant \tilde{g}(x)$ (whenever $x \in B^*$) or $g_{H_B} \leqslant \tilde{g}(y)$ (if $y \in B^*$). Hence

$$g_{H_B} + g_{H_{\widetilde{B}}} \leqslant 2(\tilde{g}(x) + \tilde{g}(y)).$$

Combining this inequality with (3.10) and (3.11) we obtain (3.8) for the case $r_B, r_{\widetilde{B}} \leqslant \delta_S$.

Let us prove (3.8) for the case $r_B > \delta_S, r_{\widetilde{B}} \leqslant \delta_S$. By (2.6) $\delta_S \leqslant r_B \leqslant C d(y, S)$ whenever $y \in B^*$ or $\delta_S \leqslant r_B \leqslant C d(x, S)$, if $x \in B^*$. Hence

$$\frac{C}{\delta_S}(d(x, S) + d(x, y) + d(y, S)) \geqslant 1.$$

Since $x \in \widetilde{B}$, by (3.3) $|u_{H_{\widetilde{B}}}| \leqslant \tilde{g}(x)$, and since $r_B > \delta_S$, $u_{H_B} := 0$. Hence

$$|u_{H_B} - u_{H_{\widetilde{B}}}| = |u_{H_{\widetilde{B}}}| \leqslant \tilde{g}(x) \leqslant \frac{C}{\delta_S}(d(x, S) + d(x, y) + d(y, S))(\tilde{g}(x) + \tilde{g}(y))$$

proving (3.8). In the same way we prove (3.8) for the case $r_{\widetilde{B}} > \delta_S,\ r_B \leqslant \delta_S$. The remaining case $r_{\widetilde{B}} > \delta_S,\ r_B > \delta_S$ is trivial because here $u_{H_B} = u_{H_{\widetilde{B}}} = 0$.

We prove (3.9) by a slight modification of the proof given above. Using estimate (3.7) rather than (3.6) we have

$$|u_{H_B} - u(y)| \leqslant C(d(x, S) + d(x, y))(\tilde{g}(x) + \tilde{g}(y)).$$

But $d(x, S) \leqslant d(x, y)$, and (3.9) follows. □

Lemma 3.3. *Let $\widetilde{B} \in W_S$ and let $x \in \widetilde{B}^* (:= \frac{9}{8}\widetilde{B})$. Then for every $y \in X \backslash S$ we have*

$$|\tilde{u}(x) - \tilde{u}(y)| \leqslant C \max_{B \in A} \min\{1, d(x, y)/r_B\} |u_{H_B} - u_{H_{\widetilde{B}}}|, \tag{3.12}$$

where $A := \{B \in W_S : B^* \cap \{x, y\} \neq \emptyset\}$.

If $y \in S$, then

$$|\tilde{u}(x) - u(y)| \leqslant C \max\{|u_{H_B} - u(y)| : B \in W_S, B^* \ni x\}. \tag{3.13}$$

Proof. By definition (3.2) and properties of the partition of unity we have

$$\begin{aligned} I := |\tilde{u}(x) - \tilde{u}(y)| &= \left| \sum_{B \in W_S} u_{H_B} \varphi_B(x) - \sum_{B \in W_S} u_{H_B} \varphi_B(y) \right| \\ &= \left| \sum_{B \in W_S} (u_{H_B} - u_{H_{\widetilde{B}}})(\varphi_B(x) - \varphi_B(y)) \right| \\ &\leqslant \sum_{B \in A} |u_{H_B} - u_{H_{\widetilde{B}}}| |\varphi_B(x) - \varphi_B(y)| \end{aligned}$$

so that by property (d) of Lemma 2.5 for every $y \in X\setminus S$

$$I \leqslant 2N \max_{B\in A} |u_{H_B} - u_{H_{\widetilde{B}}}||\varphi_B(x) - \varphi_B(y)|. \qquad (3.14)$$

Since $0 \leqslant \varphi_B \leqslant 1$, this implies

$$I \leqslant C \max\{|u_{H_B} - u_{H_{\widetilde{B}}}| : B \in A\}.$$

On the other hand, by property (d) of partition of unity we have

$$I \leqslant C_1 d(x, y) \max_{B\in A} r_B^{-1} |u_{H_B} - u_{H_{\widetilde{B}}}|.$$

Clearly, these inequalities imply (3.12). Similarly to (3.14), for $y \in S$ we have

$$|\tilde{u}(x) - u(y)| \leqslant N \max\{|u_{H_B} - u(y)| : B \in W_S, B^* \ni x\}$$

proving (3.13). □

We are in a position to prove that for some C the function $C\tilde{g}$ is a generalized gradient of $\tilde{u}$.

Lemma 3.4. *The inequality*

$$|\tilde{u}(x) - \tilde{u}(y)| \leqslant C d(x, y)(\tilde{g}(x) + \tilde{g}(y))$$

holds μ-a.e. on X.

Proof. We will suppose that $x, y \in S\setminus E$, where E is a subset of S from the inequality (3.1) (recall that $\mu(E) = 0$). Clearly, for $x, y \in S$ the result follows from (3.1) so we may assume that $x \in X\setminus S$. We let $\widetilde{B} \in W_S$ denote a Whitney ball such that $\widetilde{B} \ni x$.

Denote $I := |\tilde{u}(x) - \tilde{u}(y)|$ and consider two cases.

The first case: $y \in \widetilde{B}^*$. Since $x \in \widetilde{B}$, we have $d(x, y) \leqslant 2r_{\widetilde{B}^*} \leqslant 3r_{\widetilde{B}}$. Moreover, by (2.6) $r_{\widetilde{B}} \approx d(x, S) \approx d(y, S)$ and by the inequality (3.12)

$$I \leqslant C d(x, y) \max\{r_B^{-1} |u_{H_B} - u_{H_{\widetilde{B}}}| : B \in W_S, B^* \cap \{x, y\} \neq \emptyset\}.$$

Since $x, y \in \widetilde{B}^*$, for every ball $B \in W_S$ such that $B^* \cap \{x, y\} \neq \emptyset$ we have $B^* \cap \widetilde{B}^* \neq \emptyset$. Therefore by (2.7) $r_{\widetilde{B}} \approx r_B$. In addition, by Lemma 3.2

$$|u_{H_B} - u_{H_{\widetilde{B}}}| \leqslant C(d(x, S) + d(x, y) + d(y, S))(\tilde{g}(x) + \tilde{g}(y))$$

so that

$$|u_{H_B} - u_{H_{\widetilde{B}}}| \leqslant C r_{\widetilde{B}} (\tilde{g}(x) + \tilde{g}(y)).$$

Hence

$$\begin{aligned} I &\leqslant C d(x, y) r_{\widetilde{B}}^{-1} \max\{|u_{H_B} - u_{H_{\widetilde{B}}}| : B \in W_S, B^* \cap \{x, y\} \neq \emptyset\} \\ &\leqslant C d(x, y)(\tilde{g}(x) + \tilde{g}(y)). \end{aligned}$$

The second case: $y \notin \widetilde{B}^*$. Since $x \in \widetilde{B}$, this implies $d(x, y) \geqslant \frac{1}{8} r_{\widetilde{B}}$. Recall that $r_{\widetilde{B}} \approx d(x, S)$ so that $d(x, S) \leqslant C d(x, y)$. Since the distance function $d(\cdot, S)$ satisfies the Lipschitz condition, we have

$$d(y, S) \leqslant d(x, S) + d(x, y) \leqslant C d(x, y).$$

Let $y \notin S$. Then by (3.12)

$$I \leqslant C \max\{|u_{H_B} - u_{H_{\widetilde{B}}}| : B \in W_S, B^* \cap \{x, y\} \neq \emptyset\}$$

so that by (3.8)

$$I \leqslant C(d(x, S) + d(x, y) + d(y, S))(\tilde{g}(x) + \tilde{g}(y)) \leqslant C d(x, y)(\tilde{g}(x) + \tilde{g}(y)).$$

In the remaining case, i.e., for $y \in S$, the lemma follows from estimates (3.9) and (3.13). □

Let $f \in L^p(S)$, $1 \leqslant p \leqslant \infty$. We define an extension F of f by letting $F(x) := f(x),\ x \in S$, and

$$F(x) := \sum_{B \in W_S} |f_{H_B}| \chi_{B^*}, \quad x \in X \setminus S. \tag{3.15}$$

Lemma 3.5. $\|F\|_{L^p(X)} \leqslant C \|f\|_{L^p(S)}$.

Proof. We will prove the lemma for the case $1 \leqslant p < \infty$; corresponding changes for $p = \infty$ are obvious. By property (d) of Lemma 2.5 for every $x \in X \setminus S$ at most $N = N(C_d)$ terms of the sum in (3.15) are not equal zero. Therefore

$$|F(x)|^p \leqslant C \sum_{B \in W_S} |f_{H_B}|^p \chi_{B^*}(x), \quad x \in X \setminus S.$$

This inequality and the doubling condition imply

$$\int_{X \setminus S} |F|^p \, d\mu \leqslant C \sum_{B \in W_S} |f_{H_B}|^p \mu(B^*) \leqslant C \sum_{B \in W_S} |f_{H_B}|^p \mu(B).$$

Recall that $\mu(H_B) \approx \mu(B)$ whenever $r_B \leqslant \delta$, see (i), (ii), Theorem 2.6, so that

$$|f_{H_B}|^p = \left| \frac{1}{\mu(H_B)} \int_{H_B} f \, d\mu \right|^p \leqslant \frac{1}{\mu(H_B)} \int_{H_B} |f|^p \, d\mu \leqslant C \frac{1}{\mu(B)} \int_{H_B} |f|^p \, d\mu.$$

Recall also that $H_B = \emptyset$ if $r_B > \delta$. Hence

$$\int_{X \setminus S} |F|^p \, d\mu \leqslant C \sum_{B \in W_S} \int_{H_B} |f|^p \, d\mu = C \int_S |f|^p \left(\sum_{B \in W_S} \chi_{H_B} \right) d\mu$$

so that by property (iii) of Theorem 2.6

$$\int_{X \setminus S} |F|^p \, d\mu \leqslant C \int_S |f|^p \, d\mu.$$

It remains to note that $F|_S = f$ and the lemma follows. □

Let us prove that

$$\|\tilde{u}\|_{L^p(X)} \leqslant C\|u\|_{L^p(S)}. \tag{3.16}$$

Since $0 \leqslant \varphi_B \leqslant 1$ for every $B \in W_S$, and $\operatorname{supp} \varphi_B \subset B^*$, by (3.2) for every $x \in X \setminus S$ we have

$$|\tilde{u}(x)| = \left| \sum_{B \in W_S} u_{H_B} \varphi_B(x) \right| \leqslant \sum_{B \in W_S} |u_{H_B}| \varphi_B(x) \leqslant \sum_{B \in W_S} |u_{H_B}| \chi_{B^*}(x).$$

Hence $|\tilde{u}| \leqslant |F|$ where $F(x) := u(x)$ for $x \in S$ and

$$F(x) := \sum_{B \in W_S} |u_{H_B}| \chi_{B^*}(x), \quad x \in X \setminus S.$$

Thus $\|\tilde{u}\|_{L^p(X)} \leqslant \|F\|_{L^p(X)}$. But by Lemma 3.5 $\|F\|_{L^p(X)} \leqslant C\|u\|_{L^p(S)}$, and (3.16) follows.

It remains to estimate L^p-norm of $\tilde{g}$. To this end we define a function G by letting $G(x) := g(x),\ x \in S$ and

$$G(x) := \sum_{B \in W_S} |g_{H_B}| \chi_{B^*}(x), \quad x \in X \setminus S.$$

Then by (3.3) $|\tilde{g}| \leqslant |G| + |F|$. By Lemma 3.5 $\|G\|_{L^p(X)} \leqslant C\|g\|_{L^p(S)}$ so that

$$\|\tilde{g}\|_{L^p(X)} \leqslant \|G\|_{L^p(X)} + \|F\|_{L^p(X)} \leqslant C(\|g\|_{L^p(S)} + \|u\|_{L^p(S)}).$$

Theorem 1.3 is completely proved.

4. The sharp maximal function: proof of Theorems 1.2 and 1.4

Let us fix a ball $K = B(z, r)$ such that $K \cap S \neq \emptyset$. We denote two families of balls associated to K by letting $\mathcal{B}_K := \{B \in W_S : B^* \cap K \neq \emptyset\}$ and

$$\widetilde{\mathcal{B}}_K := \{B \in W_S : B^* \cap K \neq \emptyset,\ r_B \leqslant \delta_S\}.$$

Lemma 4.1. (i) *For every* $c \in \mathbf{R}$

$$\int_{K \setminus S} |\tilde{u} - c|\, d\mu \leqslant C \sum_{B \in \mathcal{B}_K} \mu(B)|u_{H_B} - c|.$$

(ii) *For every ball* $B \in \mathcal{B}_K$ *we have* $r_B \leqslant \eta_1 r$.

(iii) *For every* $c \in \mathbf{R}$

$$\sum_{B \in \widetilde{\mathcal{B}}_K} \mu(B)|u_{H_B} - c| \leqslant C \int_{(\eta_2 K) \cap S} |u - c|\, d\mu.$$

Here η_1, η_2 *are constants depending only on the doubling constant* C_d.

Proof. Let us prove property (i). Recall that $\sum\{\varphi_B(x) : B \in W_S\} = 1$ for every $x \in X \setminus S$. Then by definition (3.2)

$$I := \int_{K \setminus S} |\tilde{u} - c|\, d\mu = \int_{K \setminus S} \left| \sum_{B \in W_S} u_{H_B} \varphi_B - c \right| d\mu$$

$$\leqslant \int_{K \setminus S} \sum_{B \in W_S} |u_{H_B} - c| \varphi_B d\mu = \sum_{B \in W_S} \int_{K \setminus S} |u_{H_B} - c| \varphi_B d\mu.$$

Hence, by properties (a), (b) of the partition of unity and by the doubling condition

$$I \leqslant \sum_{B \in \mathcal{B}_K} \int_{B^*} |u_{H_B} - c| \varphi_B d\mu \leqslant \sum_{B \in \mathcal{B}_K} \mu(B^*)|u_{H_B} - c| \leqslant C \sum_{B \in \mathcal{B}_K} \mu(B)|u_{H_B} - c|.$$

Prove (ii) Let $B \in \mathcal{B}_K$ and let $y \in B^* \cap K$. Then by (2.6) $B^* \subset X \backslash S$ so that $y \notin S$. Therefore, there is a ball $B' \in W_S$ which contains y. Since $K \cap S \neq \emptyset$ and $B' \cap K \neq \emptyset$, we have $\operatorname{dist}(B', S) \leqslant 2r$. But by Theorem 2.4 $r_{B'} \leqslant \operatorname{dist}(B', S)$ so that $r_{B'} \leqslant 2r$. In addition, $(B')^* \cap B^* \neq \emptyset$ so that by (2.7) $r_{B'} \approx r_B$. This implies the required inequality $r_B \leqslant \eta_1 r$ with some constant $\eta_1 = \eta_1(C_d)$.

Prove (iii) We denote $A := \cup\{H_B : B \in \widetilde{\mathcal{B}}_K\}$ and

$$m_K(x) := \sum \{\chi_{H_B}(x) : B \in \widetilde{\mathcal{B}}_K\}.$$

Since $|u_{H_B} - c| \leqslant |u - c|_{H_B}$ and $\mu(H_B) \approx \mu(B)$, see (ii), Theorem 2.6,

$$\sum_{B \in \widetilde{\mathcal{B}}_K} \mu(B)|u_{H_B} - c| \leqslant \sum_{B \in \widetilde{\mathcal{B}}_K} \frac{\mu(B)}{\mu(H_B)} \int_{H_B} |u - c|\, d\mu \leqslant \gamma_2 \sum_{B \in \widetilde{\mathcal{B}}_K} \int_{H_B} |u - c|\, d\mu$$
$$= \gamma_2 \int_A |u - c| m_K d\mu.$$

By property (i) of Theorem 2.6 for every $B \in \widetilde{\mathcal{B}}_K$ we have $H_B \subset (\gamma_1 B) \cap S$. Since $B^* \cap K \neq \emptyset$ and $r_B \leqslant \eta_1 r$, we obtain

$$(\gamma_1 B) \subset (1 + (\gamma_1 + 9/8)\eta_1)K = \eta_2 K$$

so that $H_B \subset (\eta_2 K) \cap S$. Thus, $A \subset (\eta_2 K) \cap S$.

It remains to note that by property (iii) of Theorem 2.6 $m_K \leqslant \gamma_3$ and the required property (iii) follows. □

Lemma 4.2. *For every ball $K = B(z, r)$ such that $z \in S$ and $r \leqslant \delta_S/\eta_1$ we have*

$$\frac{r^{-\alpha}}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_K|\, d\mu \leqslant C u^{\sharp}_{\alpha,S}(z).$$

Proof. We denote $D := (\eta_2 K) \cap S$ where η_2 is the constant from the inequality (iii) of Lemma 4.1. Let us prove that

$$\int_K |\tilde{u} - \tilde{u}_K|\, d\mu \leqslant C \int_D |u - u_D|\, d\mu. \tag{4.1}$$

Since $r \leqslant \delta_S/\eta_1$, by (ii) of Lemma 4.1 we have $r_B \leqslant \delta_S$ for every ball $B \in \mathcal{B}_K$. Thus, $\mathcal{B}_K = \widetilde{\mathcal{B}}_K$ so that $\{H_B : B \in \mathcal{B}_K\}$ is a subfamily of the family $\mathcal{H}_S$ satisfying properties (i)–(iii) of Theorem 2.6.

Applying property (i) of Lemma 4.1 with $c := u_D$ we obtain

$$\int_{K \backslash S} |\tilde{u} - c|\, d\mu \leqslant C_d \sum_{B \in \widetilde{\mathcal{B}}_K} \mu(B)|u_{H_B} - c|$$

so that by (iii) of Lemma 4.1

$$\int_{K\setminus S} |\tilde{u} - c|\, d\mu \leqslant C \int_D |u - c|\, d\mu.$$

This implies

$$\int_K |\tilde{u} - c|\, d\mu = \int_{K\cap S} |u - c|\, d\mu + \int_{K\setminus S} |\tilde{u} - c|\, d\mu \leqslant C \int_D |u - c|\, d\mu$$

so that

$$\int_K |\tilde{u} - \tilde{u}_K| \leqslant 2 \int_K |\tilde{u} - c|\, d\mu \leqslant C \int_D |u - c|\, d\mu$$

proving (4.1). Since $\mu(K) \approx \mu(\eta_2 K)$, we finally obtain

$$\frac{r^{-\alpha}}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_K|\, d\mu \leqslant C(\eta_2 r)^{-\alpha} \left(\frac{1}{\mu(\eta_2 K)} \int_D |u - u_D|\, d\mu \right) \leqslant C u^{\sharp}_{\alpha,S}(z). \qquad \square$$

Recall that given a function u defined on S we let $u^{\curlywedge}$ denote its extension by 0 to all of X. As usual given $f \in L^{1,\mathrm{loc}}(X)$ we let $\mathcal{M}f$ denote the Hardy–Littlewood maximal operator:

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\, d\mu.$$

Lemma 4.3. *Let $K = B(z, r)$ be a ball such that $z \in S$ and $r > \delta_S/\eta_1$. Then*

$$\frac{r^{-\alpha}}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_K|\, d\mu \leqslant C \mathcal{M} u^{\curlywedge}(z).$$

Proof. Applying property (i) of Lemma 4.1 with $c := 0$ we obtain

$$\int_{K\setminus S} |\tilde{u}|\, d\mu \leqslant C \sum_{B \in \mathcal{B}_K} \mu(B) |u_{H_B}|.$$

Since $u_{H_B} := 0$ whenever $r_B > \delta_S$, we have

$$\int_{K\setminus S} |\tilde{u}|\, d\mu \leqslant C \sum_{B \in \widetilde{\mathcal{B}}_K} \mu(B) |u_{H_B}|.$$

Applying (iii) of Lemma 4.1 with $c := 0$ we obtain

$$\int_{K\setminus S} |\tilde{u}|\, d\mu \leqslant C \int_{(\eta_2 K)\cap S} |u|\, d\mu.$$

Since $r > \delta_S/\eta_1$, this implies

$$I := \frac{r^{-\alpha}}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_K|\, d\mu \leqslant 2 \frac{r^{-\alpha}}{\mu(K)} \int_K |\tilde{u}|\, d\mu \leqslant \frac{2\eta_1^{\alpha}}{\delta_S^{\alpha}} \frac{1}{\mu(K)} \int_K |\tilde{u}|\, d\mu$$

$$\leqslant \frac{2\eta_1^{\alpha}}{\delta_S^{\alpha}} \frac{1}{\mu(K)} \left(\int_{K\cap S} |u|\, d\mu + C \int_{(\eta_2 K)\cap S} |u|\, d\mu \right)$$

so that

$$I \leqslant \frac{4\eta_1^\alpha C}{\delta_S^\alpha} \frac{1}{\mu(K)} \int_{(\eta_2 K)\cap S} |u|\, d\mu.$$

Since $\mu(K) \approx \mu(\eta_2 K)$, we have

$$I \leqslant \frac{C}{\mu(\eta_2 K)} \int_{(\eta_2 K)\cap S} |u|\, d\mu = \frac{C}{\mu(\eta_2 K)} \int_{\eta_2 K} |u^\curlywedge|\, d\mu \leqslant C\mathcal{M}u^\curlywedge(z). \qquad \square$$

Lemmas 4.2 and 4.3 imply the following.

Proposition 4.4. *For every $z \in S$*

$$(\tilde{u})_\alpha^\sharp(z) \leqslant C(u_{\alpha,S}^\sharp(z) + \mathcal{M}u^\curlywedge(z)).$$

Let us estimate the value of $(\tilde{u})_\alpha^\sharp(z)$ for $z \in X\backslash S$. We will put $\inf_{H_Q} u_{\alpha,S}^\sharp := 0$ whenever $H_Q = \emptyset$ (recall that $H_Q = \emptyset$ iff $r_Q > \delta_S$).

Lemma 4.5. *Let $Q = B(x_Q, r_Q) \in W_S$ and let $z \in Q$. Then for every ball $K := B(z, r)$ with $r \leqslant \frac{1}{8} r_Q$ we have*

$$\frac{r^{-\alpha}}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_K|\, d\mu \leqslant C \left(\inf_{H_Q} u_{\alpha,S}^\sharp + \mathcal{M}u^\curlywedge(z) \right). \tag{4.2}$$

Proof. We have to prove that for arbitrary $s \in H_Q$

$$I := \frac{r^{-\alpha}}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_K|\, d\mu \leqslant C\,(u_{\alpha,S}^\sharp(s) + \mathcal{M}u^\curlywedge(z)). \tag{4.3}$$

Since $r \leqslant \frac{1}{8} r_Q$, the ball $K = B(z,r) \subset \frac{9}{8} Q =: Q^*$. By Lemma 3.3 for every $x, y \in K (\subset Q^*)$

$$|\tilde{u}(x) - \tilde{u}(y)| \leqslant C d(x,y) \max\{r_B^{-1} |u_{H_B} - u_{H_Q}| : B \in W_S,\ B^* \cap \{x,y\} \neq \emptyset\}.$$

Since $x, y \in Q^*$, for every ball $B \in W_S$ such that $B^* \cap \{x,y\} \neq \emptyset$ we have $B^* \cap Q^* \neq \emptyset$. Therefore by (2.7)

$$\frac{1}{C_1} r_Q \leqslant r_B \leqslant C_1 r_Q, \tag{4.4}$$

where one can put $C_1 = 28$. We denote $A := \{B \in W_S : B^* \cap Q^* \neq \emptyset\}$. Then

$$|\tilde{u}(x) - \tilde{u}(y)| \leqslant C \frac{d(x,y)}{r_Q} \max_{B \in A} |u_{H_B} - u_{H_Q}|.$$

Hence

$$\begin{aligned} \frac{1}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_K|\, d\mu &\leqslant \frac{1}{\mu(K)^2} \int_K \int_K |\tilde{u}(x) - \tilde{u}(y)|\, d\mu(x) d\mu(y) \\ &\leqslant C \frac{d(x,y)}{r_Q} \max_{B \in A} |u_{H_B} - u_{H_Q}|. \end{aligned}$$

Since $d(x, y) \leqslant \operatorname{diam} K \leqslant 2r$ and $r \leqslant \frac{1}{8} r_Q$, this implies

$$I \leqslant C r_Q^{-\alpha} \max_{B \in A} |u_{H_B} - u_{H_Q}|. \tag{4.5}$$

Let us consider two cases.

The first case: $r_Q \leqslant \delta_S / C_1$, where C_1 is the constant from the inequality (4.4). Then for each $B \in A$ we have $r_B \leqslant \delta_S$ so that H_B, H_Q satisfy properties (i), (ii) of Theorem 2.6. Thus, $H_B \subset (\gamma_1 B) \cap S$, $H_Q \subset (\gamma_1 Q) \cap S$, $\mu(H_B) \approx \mu(B)$, and $\mu(H_Q) \approx \mu(Q)$.

Since $B^* \cap Q^* \neq \emptyset$ and $r_B \approx r_Q$, for some positive $C_2 = C_2(\gamma_1)$ we have

$$B \cup Q \cup H_B \cup H_Q \subset D := B(s, C_2 r_Q).$$

(Recall that s is an arbitrary point of H_Q.) These inequalities and the doubling condition imply $\mu(H_B) \approx \mu(H_Q) \approx \mu(D)$. Hence

$$|u_{H_B} - u_{D \cap S}| \leqslant \frac{1}{\mu(H_B)} \int_{H_B} |u - u_{D \cap S}| \, d\mu \leqslant C \frac{1}{\mu(D)} \int_{D \cap S} |u - u_{D \cap S}| \, d\mu.$$

A similar estimate is true for H_Q so that

$$|u_{H_B} - u_{H_Q}| \leqslant |u_{H_B} - u_{D \cap S}| + |u_{H_Q} - u_{D \cap S}| \leqslant C \frac{1}{\mu(D)} \int_{D \cap S} |u - u_{D \cap S}| \, d\mu.$$

Applying this inequality to (4.5) we obtain

$$I \leqslant C \frac{r_Q^{-\alpha}}{\mu(D)} \int_{D \cap S} |u - u_{D \cap S}| \, d\mu \leqslant C \frac{r_D^{-\alpha}}{\mu(D)} \int_{D \cap S} |u - u_{D \cap S}| \, d\mu,$$

where $r_D := C_2 r_Q$ is the radius of the ball $D := B(s, C_2 r_Q)$. Hence by definition (1.4) we have $I \leqslant C u^{\sharp}_{\alpha,S}(s)$ proving (4.3).

The second case: $r_Q > \delta_S / C_1$. By (4.5) $I \leqslant C \max\{|u_{H_B}| : B \in A\}$. Recall that $u_{H_B} := 0$ if $r_B > \delta_S$ so that

$$I \leqslant C \max\{|u_{H_B}| : B \in A, r_B \leqslant \delta_S\}.$$

By Theorem 2.6 for every $B \in A$ such that $r_B \leqslant \delta_S$ we have $H_B \subset (\gamma_1 B) \cap S$, $\mu(H_B) \approx \mu(B)$. Since $r_B \approx r_Q$ and $z \in Q$, for some positive $C_3 = C_3(\gamma_1)$ we have $H_B \subset B(z, C_3 r_Q)$. Put $\widetilde{D} := B(z, C_3 r_Q)$. Since $\mu(\widetilde{D}) \approx \mu(Q)$ and $\mu(B) \approx \mu(Q)$, we have $\mu(H_B) \approx \mu(\widetilde{D})$. Hence

$$|u_{H_B}| \leqslant \frac{1}{\mu(H_B)} \int_{H_B} |u| \, d\mu \leqslant C \frac{1}{\mu(\widetilde{D})} \int_{\widetilde{D} \cap S} |u| \, d\mu \leqslant C \mathcal{M} u^{\curlywedge}(z)$$

proving that $I \leqslant C \mathcal{M} u^{\curlywedge}(z)$. $\square$

Lemma 4.6. *Inequality* (4.2) *is true for every* $r > \frac{1}{8} r_Q$.

Proof. We denote $\eta_3 := 8(\gamma_1 + 10)$, $\tilde{r} := \eta_3 r$ and $\widetilde{K} := \eta_3 K = B(z, \tilde{r})$. Recall that γ_1 is the constant from Theorem 2.6. Prove that $\widetilde{K} \cap S \neq \emptyset$. In fact, let $a_Q \in Q$ and $b_Q \in S$ be points satisfying the inequality $d(a_Q, b_Q) \leqslant 2d(Q, S)$. Then by (ii), Theorem 2.4, $d(a_Q, b_Q) \leqslant 2d(Q, S) \leqslant 8 r_Q$.

But $z \in Q$ so that

$$d(z, b_Q) \leqslant d(z, a_Q) + d(z, b_Q) \leqslant 2r_Q + 8r_Q = 10r_Q \leqslant 80r \leqslant \eta_3 r = \tilde{r}.$$

Thus, $b_Q \in \widetilde{K} \cap S$ proving that $\widetilde{K} \cap S \neq \emptyset$.

Let us consider two cases.

The first case: $\tilde{r} := \eta_3 r > \delta_S/\eta_1$. Since $\tilde{r}$ is the radius of the ball $\widetilde{K} = \eta_3 K$, $\tilde{r} > \delta_S/\eta_1$ and $\widetilde{K} \cap S \neq \emptyset$, by Lemma 4.3

$$\frac{\tilde{r}^{-\alpha}}{\mu(\widetilde{K})} \int_{\widetilde{K}} |\tilde{u} - \tilde{u}_{\widetilde{K}}| \, d\mu \leqslant C \mathcal{M} u^{\curlywedge}(z).$$

By the doubling condition $\mu(K) \approx \mu(\widetilde{K})$ so that

$$\begin{aligned} I := \frac{r^{-\alpha}}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_K| \, d\mu &\leqslant \frac{2r^{-\alpha}}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_{\widetilde{K}}| \, d\mu \\ &\leqslant C \frac{\tilde{r}^{-\alpha}}{\mu(\widetilde{K})} \int_{\widetilde{K}} |\tilde{u} - \tilde{u}_{\widetilde{K}}| \, d\mu \leqslant C \mathcal{M} u^{\curlywedge}(z) \end{aligned}$$

proving (4.2).

The second case: $\tilde{r} := \eta_3 r \leqslant \delta_S/\eta_1$. Since $8r > r_Q$, we have

$$r_Q < 8\delta_S/(\eta_1 \eta_3) < \delta_S.$$

Therefore by Theorem 2.6 $H_Q \neq \emptyset$, $\mu(H_Q) \approx \mu(Q)$ and $H_Q \subset (\gamma_1 Q) \cap S$.

Take $s \in H_Q$ and put $V := B(s, \eta_3 r)$. Since $H_Q \subset \gamma_1 Q$, $d(s, x_Q) \leqslant \gamma_1 r_Q$ so that for every $a \in K = B(z, r)$

$$\begin{aligned} d(s, a) &\leqslant d(s, x_Q) + d(x_Q, z) + d(z, a) \\ &\leqslant \gamma_1 r_Q + r_Q + r \leqslant 8\gamma_1 r + 8r + r = (8\gamma_1 + 9)r \leqslant \eta_3 r \end{aligned}$$

proving that $K \subset V$. On the other hand, $V \subset 2\eta_3 K$ so that by the doubling condition $\mu(V) \approx \mu(K)$. Hence

$$I \leqslant 2r_K^{-\alpha} \frac{1}{\mu(K)} \int_K |\tilde{u} - \tilde{u}_V| \, d\mu \leqslant C r_V^{-\alpha} \frac{1}{\mu(V)} \int_V |\tilde{u} - \tilde{u}_V| \, d\mu.$$

But $r_V := \eta_3 r \leqslant \delta_S/\eta_1$ so that by Lemma 4.2 $I \leqslant C u^{\sharp}_{\alpha,S}(s)$. This finishes the proof of (4.2) and the lemma. □

Theorem 4.7. *For every $z \in X$*

$$(\tilde{u})^{\sharp}_{\alpha}(z) \leqslant C(\mathcal{M}(u^{\sharp}_{\alpha,S})^{\curlywedge}(z) + \mathcal{M} u^{\curlywedge}(z)).$$

Proof. For $z \in S$ this follows from Proposition 4.4.

Let $Q \in W_S$ and let $z \in Q$. Then by Lemmas 4.5 and 4.6

$$(\tilde{u})^{\sharp}_{\alpha}(z) \leqslant C \left(\inf_{H_Q} u^{\sharp}_{\alpha,S} + \mathcal{M} u^{\curlywedge}(z) \right). \tag{4.6}$$

Recall that in this formula we put the infimum to be equal 0 whenever $H_Q = \emptyset$, i.e., $r_Q > \delta_S$. Therefore in the remaining part of the proof we may assume that $r_Q \leqslant \delta_S$. Then by Theorem 2.6 $\mu(H_Q) \approx \mu(Q)$ and $H_Q \subset (\gamma_1 Q) \cap S$.

Let us denote $B := B(z, (\gamma_1+1)r_Q)$ and $h := (u^{\sharp}_{\alpha,S})^{\curlywedge}$. Since $z \in Q$, we have $H_Q \subset \gamma_1 Q \subset B$. In addition, by the doubling condition $\mu(H_Q) \approx \mu(B)$. Hence

$$\inf_{H_Q} u^{\sharp}_{\alpha,S} = \inf_{H_Q} h \leqslant \frac{1}{\mu(H_Q)} \int_{H_Q} h\, d\mu \leqslant \frac{1}{\mu(H_Q)} \int_B h\, d\mu \leqslant \frac{C}{\mu(B)} \int_B h\, d\mu \leqslant C\mathcal{M}h(z).$$

This inequality and (4.6) imply the proposition. □

Remark 4.8. Similar estimates and definition of $\tilde{u}$, see (3.2), easily imply that $|\tilde{u}(x)| \leqslant C\mathcal{M}u^{\curlywedge}(x)$ for every $x \in X$.

Proof of Theorem 1.4. It can be easily shown that for any extension U of a function $u \in L^p(S)$ to all of X we have $u^{\sharp}_{\alpha,S}(x) \leqslant 2U^{\sharp}_{\alpha}(x),\ x \in S$. This immediately implies the inequality

$$\|u\|_{L^p(S)} + \|u^{\sharp}_{\alpha,S}\|_{L^p(S)} \leqslant 2\|u\|_{\mathcal{C}^{\alpha}_p(X,d,\mu)|_S}.$$

Now let $u, u^{\sharp}_{\alpha,S} \in L^p(S)$, $1 < p \leqslant \infty$. Prove that $\tilde{u} = \mathrm{Ext}_S\, u \in \mathcal{C}^{\alpha}_p(X, d, \mu)$. By Theorem 4.7

$$\|(\tilde{u})^{\sharp}_{\alpha}\|_{L^p(X)} \leqslant C(\|\mathcal{M}(u^{\sharp}_{\alpha,S})^{\curlywedge}\|_{L^p(X)} + \|\mathcal{M}u^{\curlywedge}\|_{L^p(X)}).$$

Recall that the operator $\mathcal{M}$ is bounded in $L^p(X)$ whenever $1 < p \leqslant \infty$ and (X, d, μ) is a metric space of a homogeneous type, see, e.g. [19, p. 10]. Hence

$$\|(\tilde{u})^{\sharp}_{\alpha}\|_{L^p(X)} \leqslant C(\|(u^{\sharp}_{\alpha,S})^{\curlywedge}\|_{L^p(X)} + \|u^{\curlywedge}\|_{L^p(X)}) = C(\|u^{\sharp}_{\alpha,S}\|_{L^p(S)} + \|u\|_{L^p(S)}).$$

Since $\|\tilde{u}\|_{L^p(X)} \leqslant C\|u\|_{L^p(S)}$, see (3.16), we finally obtain

$$\|\tilde{u}\|_{\mathcal{C}^{\alpha}_p(X,d,\mu)} := \|\tilde{u}\|_{L^p(X)} + \|(\tilde{u})^{\sharp}_{\alpha}\|_{L^p(X)} \leqslant C(\|u\|_{L^p(S)} + \|u^{\sharp}_{\alpha,S}\|_{L^p(S)})$$

proving that $\tilde{u} \in \mathcal{C}^{\alpha}_p(X, d, \mu)$ and equivalence (1.6) holds.

The proof of Theorem 1.4 is complete. □

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