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Determination of an unknown coefficient in a nonlinear heat equation

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Abstract

The problem of determining an unknown term $k(u)$ in the equation $k(u)u_t = (k(u)u_x)_x$ is considered in this paper. Applying Tikhonov's regularization approach, we develop a procedure to find an approximate stable solution to the unknown coefficient from the overspecified data. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

Cannon and DuChateau considered in [1] the following overspecified problem:

$$k(u)u_t = \big(k(u)u_x\big)_x, \quad 0 < x,\ 0 < t < T, \tag{1.1}$$

$$u(x,0) = 0, \quad 0 < x, \tag{1.2}$$

$$u(0,t) = f(t), \quad f(0) = 0, \quad 0 < t < T, \tag{1.3}$$

$$-k\big(u(0,t)\big)u_x(0,t) = g(t), \quad 0 < t < T. \tag{1.4}$$

Here f, g are known functions and $k(\lambda)$, a positive function, is to be determined. It was assumed that f, g are strictly increasing and at least continuously differ-

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entiable. Under these conditions, Cannon and DuChateau proved that there exists a unique $k(\cdot) > 0$ that depends on the data f, g continuously. A similar problem on a finite interval was also considered in the article.

A practically more interesting situation is that the input functions f, g are not smooth as most data are gathered through measurement or other laboratory methods. When f, g are not differentiable, the method used in [1] does not apply. Let us see:

Integral transform

$$v(x,t) = \int_0^{u(x,t)} k(\lambda)\,d\lambda \tag{1.5}$$

converts (1.1)–(1.4) to the following linearized problem:

$$v_t = v_{xx}, \quad 0 < x,\ 0 < t < T, \tag{1.6}$$

$$v(x,0) = 0, \quad 0 < x, \tag{1.7}$$

$$v(0,t) = \int_0^{f(t)} k(\lambda)\,d\lambda, \quad 0 < t < T, \tag{1.8}$$

$$-v_x(0,t) = g(t), \quad 0 < t < T. \tag{1.9}$$

If the pair (v,k) solves the problem (1.6)–(1.9),

$$v(x,t) = \int_0^t H(x,t-\tau)g(\tau)\,d\tau, \tag{1.10}$$

$$H(x,t) = \frac{1}{\sqrt{\pi t}} e^{-x^2/(4t)}. \tag{1.11}$$

Letting x approach zero and using (1.8) yields

$$\int_0^{f(t)} k(\lambda)\,d\lambda = \frac{1}{\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}\,d\tau. \tag{1.12}$$

Therefore, the problem turns out to be an integral equation on k:

$$A(k)(\lambda) = G(\lambda), \quad G(0) = 0, \tag{1.13}$$

where

$$A(k)(\lambda) = \int_0^{\lambda} k(\lambda)\,d\lambda, \qquad G(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^s \frac{g(\tau)}{\sqrt{s-\tau}}\,d\tau \bigg|_{s=f^{-1}(\lambda)}. \tag{1.14}$$

Under the hypothesis of smoothness of f and g, it was proven in [1] that Eq. (1.13) has a uniquely determined solution that depends on the data continuously.

In practice, the boundary data f, g are obtained, in general, by physical measurements and are thus not smooth. The problem (1.13) is ill-posed for non-smooth data. We need to find new ways of determining the unknown coefficient in Eq. (1.1).

In Section 2, we will apply the regularization approach adapted in [2–4] to define and construct approximate stable solution $k(\cdot)$ of (1.1). Then in Section 3, we will demonstrate the applicability of our approximation approach by comparing the exact solution and the approximate solution numerically.

2. Regularization approach

We assume f to be strictly increasing. f, g are L^2 functions on $[0, T]$.

Based on the basic concept of Tikhonov's regularizing operators, we will define and construct an approximate solution of the integral equation (1.13). We will show that the solution is stable under small changes of the data.

First of all, we introduce a smooth functional

$$\begin{aligned} M^\alpha(k, G) &= \left\| A(k) - G \right\|^2_{L^2[0,T]} + \alpha \left\| k(\lambda) \right\|^2_{W_2^1[0,T]} \\ &= \int_0^T \left(\int_0^\lambda k(\lambda)\, d\lambda - G(\lambda) \right)^2 d\lambda + \alpha \int_0^T \left(k^2(\lambda) + k'^2(\lambda) \right) d\lambda \end{aligned} \tag{2.1}$$

and establish the following result:

Theorem 2.1. *For every $G \in L^2[0, T]$ and every $\alpha > 0$, there exits a unique minimizer $k_\alpha(\lambda) \in C[0, T]$ such that*

$$M^\alpha(k_\alpha, G) = \inf M^\alpha(k, G). \tag{2.2}$$

Proof. By considering the first variation of the functional M^α, we easily obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{d\varepsilon} M^\alpha(k + \varepsilon\phi, G) \bigg|_{\varepsilon=0} \\ &\quad = \int_0^T \left(\left(\int_0^\lambda k(\lambda)\, d\lambda - G(\lambda) \right) \int_0^\lambda \phi(\lambda)\, d\lambda \right) d\lambda \\ &\qquad + \alpha \int_0^T \left(k(\lambda)\phi(\lambda) + k'(\lambda)\phi'(\lambda) \right) d\lambda \end{aligned}$$

$$= \int_0^T \phi(\lambda)\left(\int_\lambda^T \int_0^\lambda k(\lambda)\,d\lambda\,d\lambda - \int_\lambda^T G(\lambda)\,d\lambda - \alpha\big(k''(\lambda) - k(\lambda)\big)\right)d\lambda$$
$$+ \alpha\big\|k'(\lambda)\phi(\lambda)\big\|_0^T.$$

It follows then that the minimizer $k_\alpha(\lambda)$ is determined by the following Euler integrodifferential equation:

$$\int_\lambda^T \int_0^\lambda k(\lambda)\,d\lambda\,d\lambda - \int_\lambda^T G(\lambda)\,d\lambda = \alpha\big(k''(\lambda) - k(\lambda)\big) \tag{2.3}$$

with the boundary conditions

$$k'(0) = k'(T) = 0. \tag{2.4}$$

The solution of (2.3), (2.4) is unique. In fact, if k is a solution of the homogeneous equation

$$\int_\lambda^T \int_0^\lambda k(\lambda)\,d\lambda\,d\lambda = \alpha\big(k''(\lambda) - k(\lambda)\big), \tag{2.5}$$

we multiply both sides by k and integrate on λ from 0 to T to obtain the equality

$$-\alpha\big\|k(\lambda)\big\|^2_{W_2^1} = \big\|A(k)\big\|^2_{L^2},$$

which implies that, for $\alpha > 0$, $k \equiv 0$. □

Now, we denote the function obtained above as an operator $k_\alpha = R(G, \alpha)$, depending on the parameter α. For positive small number δ, measuring the error in data, if we can select an appropriate function $\alpha = \alpha(\delta)$ as regularization parameter, $R(G_\delta, \alpha(\delta))$ is then a regularizing operator and hence the minimizer $k_{\alpha(\delta)} = R(G_\delta, \alpha(\delta))$, can be taken as an approximate solution of the problem, as discussed in detail below.

Theorem 2.2. *Let $k_T(\lambda) \in C^1[0, T]$ be the exact solution of Eq.* (1.13) *with the exact right-hand member $G_t(\lambda)$*:

$$A(k_T)(\lambda) = \int_0^\lambda k_T(\lambda)\,d\lambda = G_T(\lambda). \tag{2.6}$$

Then, for every $\varepsilon > 0$, there exists a positive number $\delta(\varepsilon)$ such that for any function $G_\delta(\lambda) \in L^2[0, T]$ satisfying

$$\big\|G_\delta(\lambda) - G_T(\lambda)\big\|_{L^2[0,T]} \leqslant \delta \leqslant \delta(\varepsilon), \tag{2.7}$$

the following inequality holds:

$$\left\| k_{\alpha(\delta)}(\lambda) - k_T(\lambda) \right\|_{C[0,T]} < \varepsilon, \tag{2.8}$$

where

$$k_{\alpha(\delta)} = R\big(G_\delta, \alpha(\delta)\big), \quad \alpha(\delta) = \delta^r, \quad 0 < r < 2. \tag{2.9}$$

Proof. By the definition of $k_\alpha(\delta)$, we have

$$M^{\alpha(\delta)}(k_{\alpha(\delta)}, G_\delta) \leqslant M^{\alpha(\delta)}(k_T, G_\delta). \tag{2.10}$$

It follows then that

$$\begin{aligned} &\left\| A(k)_{\alpha(\delta)} - G_\delta \right\|_{L^2}^2 \\ &\quad \leqslant \int_0^T \left(\int_0^\lambda k_T(\lambda)\, d\lambda - G_\delta(\lambda) \right)^2 d\lambda + \delta^r \int_0^T \big(k_T^2(\lambda) + k_T'^2(\lambda)\big)\, d\lambda \\ &\quad = \int_0^T \big(G_T(\lambda) - G_\delta(\lambda)\big)^2 d\lambda + \delta^r \int_0^T \big(k_T^2(\lambda) + k_T'^2(\lambda)\big)\, d\lambda \\ &\quad \leqslant \delta^2 + \delta^r \int_0^T \big(k_T^2(\lambda) + k_T'^2(\lambda)\big)\, d\lambda \leqslant \delta^r d, \end{aligned} \tag{2.11}$$

where $d = 1 + \int_0^T (k_T^2(\lambda) + k_T'^2(\lambda))\, d\lambda$ and $\|k_{\alpha(\delta)}\|_{W_2^1}^2 \leqslant d$.

So both k_T and $k_{\alpha(\delta)}$ belong to the compact subset of $C[0,T]$:

$$E = \big\{k(\lambda)\colon \|k\|_{W_2^1}^2 \leqslant d\big\}.$$

The continuity of the inverse A^{-1} on AE implies that

$$\|k_T - k_{\alpha(\delta)}\|_C \leqslant \|A^{-1}\| \cdot \|Ak_T - Ak_{\alpha(\delta)}\|_{L^2}. \tag{2.12}$$

Moreover,

$$\begin{aligned} \|Ak_T - Ak_{\alpha(\delta)}\|_{L^2}^2 &\leqslant 2\big(\|Ak_T - G_\delta\|_{L^2}^2 + \|G_\delta - Ak_{\alpha(\delta)}\|_{L^2}^2\big) \\ &= 2\big(\|G_T - G_\delta\|_{L^2}^2 + \|G_\delta - Ak_{\alpha(\delta)}\|_{L^2}^2\big) \\ &\leqslant 2(\delta^2 + \delta^r d) \leqslant 2\delta^r(1+d). \end{aligned} \tag{2.13}$$

Then (2.12), (2.13) show that the $\delta(\varepsilon)$ needed in the theorem should be chosen in the form

$$\delta(\varepsilon) \leqslant \left[\frac{\varepsilon^2}{2\|A_{-1}\|^2(1+d)} \right]^{1/r} \tag{2.14}$$

to end the proof of the theorem. □

Finally, we show that $G(\lambda)$ depends on the initial data f, g continuously.

Theorem 2.3. *For every positive* δ*, there exists a number* $\Delta(\delta) > 0$ *such that the inequalities*

$$\| f_\delta^{-1} - f_T^{-1} \|_C \leqslant \Delta \leqslant \Delta(\delta), \qquad \|g_\delta - g_T\|_C \leqslant \Delta \leqslant \Delta(\delta) \tag{2.15}$$

imply

$$\|G_\delta - G_T\|_{L^2} \leqslant \delta, \tag{2.16}$$

where

$$G_T(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^s \frac{g_T(\tau)}{\sqrt{s-\tau}}\, d\tau \bigg|_{s = f_T^{-1}(\lambda)},$$

$$G_\delta(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^s \frac{g_\delta(\tau)}{\sqrt{s-\tau}}\, d\tau \bigg|_{s = f_\delta^{-1}(\lambda)}. \tag{2.17}$$

Proof. Since

$$\begin{aligned}
&\pi \|G_\delta - G_T\|_{L^2} \\
&\quad \leqslant \int_0^T \left(\int_0^{f_\delta^{-1}(\lambda)} \frac{g_\delta(\tau)}{\sqrt{f_\delta^{-1}(\lambda) - \tau}}\, d\tau - \int_0^{f_T^{-1}(\lambda)} \frac{g_T(\tau)}{\sqrt{f_T^{-1}(\lambda) - \tau}}\, d\tau \right)^2 d\lambda \\
&\quad \leqslant 2 \int_0^T \left(\int_0^{f_\delta^{-1}(\lambda)} \frac{g_\delta(\tau)}{\sqrt{f_\delta^{-1}(\lambda) - \tau}}\, d\tau - \int_0^{f_T^{-1}(\lambda)} \frac{g_\delta(\tau)}{\sqrt{f_T^{-1}(\lambda) - \tau}}\, d\tau \right)^2 d\lambda \\
&\qquad + 2 \int_0^T \left(\int_0^{f_T^{-1}(\lambda)} \frac{g_\delta(\tau) - g_T(\tau)}{\sqrt{f_T^{-1}(\lambda) - \tau}}\, d\tau \right)^2 d\lambda \\
&\quad \leqslant 80T \| f_\delta^{-1} - f_T^{-1} \|_C \|g_\delta\|_C^2 + 8\|g_\delta - g_T\|_C^2 \int_0^T f_t^{-1}(\lambda)\, d\lambda,
\end{aligned}$$

where the estimate of the first term on the right-hand side follows from the following computation:

$$\left(\int_0^{f_\delta^{-1}(\lambda)} \frac{g_\delta(\tau)}{\sqrt{f_\delta^{-1}(\lambda) - \tau}}\, d\tau - \int_0^{f_T^{-1}(\lambda)} \frac{g_\delta(\tau)}{\sqrt{f_T^{-1}(\lambda) - \tau}}\, d\tau \right)^2$$

$$\leqslant 4\|g_\delta\|_C^2\Big(2\sqrt{\big|f_\delta^{-1}(\lambda)-f_T^{-1}(\lambda)\big|}-\Big|\sqrt{f_\delta^{-1}(\lambda)}-\sqrt{f_T^{-1}(\lambda)}\Big|\Big)^2$$
$$\leqslant 8\|g_\delta\|_C^2\Big(\Big|\sqrt{f_\delta^{-1}(\lambda)}-\sqrt{f_T^{-1}(\lambda)}\Big|^2+4\big|f_\delta^{-1}(\lambda)-f_T^{-1}(\lambda)\big|^2\Big)$$
$$\leqslant 40\|g_\delta\|_C^2+\big|f_\delta^{-1}(\lambda)-f_T^{-1}(\lambda)\big|^2.$$

We then choose $\Delta(\delta)=\pi\delta^2/(80TM_1+8M_2)$, where $M_1=(1+\|g_T\|_C)^2$, $M_2=\int_0^T f_T^{-1}(\lambda)\,d\lambda$, to end the proof. □

Theorems 2.2 and 2.3 imply then the following stability result.

Theorem 2.4. *The solution of* (1.13) *depends on nonsmooth data* f, g *continuously.*

3. An example

In this section, we will apply the regularization approach presented in Section 2 to Euler equation (2.3) to demonstrate the applicability and the accuracy of the approach.

First of all, we replace Eq. (2.3) by its finite difference approximation on a given uniform grid with step $h=T/(n+1)$, $T=1$:

$$\alpha\left[\frac{k_{j+1}^h-2k_j^h+k_{j-1}^h}{h^2}-k_j^h\right]=\sum_{i=j}^n h\sum_{l=1}^i hk_l^h-\sum_{i=j}^n hG_i^h,\quad j=1,2,\ldots,n, \tag{3.1}$$

where

$$G_i^h=\frac{2}{\sqrt{\pi}}\sum_{j=1}^i\big(\sqrt{\tau_i-\tau_{j-1}}-\sqrt{\tau_i-\tau_j}\big)g_j^h,\qquad \tau_i=f^{-1}(ih),\quad i=1,2,\ldots,n, \tag{3.2}$$

$k^h=(k_1^h,\ldots,k_n^h)$ with $k_0^h=k_1^h$, $k_n^h=k_{n+1}^h$, $f_h^{-1}=(f_1^h,\ldots,f_n^{-1})$ and $g^h=(g_1^h,\ldots,g_n^h)$ are difference solutions.

The above equations can be rewritten in matrix form:

$$\big(h^4B+\alpha D\big)k^h=\frac{2}{\sqrt{\pi}}h^3RQg^h, \tag{3.3}$$

where

$$B = \begin{pmatrix} n & n-1 & \dots & 2 & 1 \\ n-1 & n-1 & \dots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} h^2+1 & -1 & & & & & \\ -1 & h^2+2 & -1 & & & & \\ & -1 & h^2+2 & \dots & & & \\ & & & \ddots & & & \\ & & & \dots & h^2+2 & -1 & \\ & & & & -1 & h^2+2 & -1 \\ & & & & & -1 & h^2+2 \end{pmatrix},$$

$$R = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$$Q = (q_{ij}), \quad q_{ij} = \begin{cases} \sqrt{\tau_i - \tau_{j-1}}, & i \geqslant j, \\ 0, & i < j, \end{cases} \quad i, j = 1, 2, \dots, n.$$

Now, let $f_T(t) = t$, $g_T(t) = (2/35)\sqrt{t/\pi}(32t^3 - 56t^2 + 35)$. The exact solution of (1.13) is

$$k_T(\lambda) = 2\lambda^3 - 3\lambda^2 + 1.$$

In practice, exact smooth data are not given. Let us try to recover, by applying our regularizing approach, the solution of (1.13) from the perturbations of the data:

$$f_\delta(t) = t^{1-\delta}, \qquad g_\delta(t) = g_T\,dt(t) + \delta \sin(50t).$$

Table 1
($n = 79$, $\delta = 10^{-8}$, $r = 1.78$)

λ	0.05	0.10	0.15	0.20	0.25	0.30	0.35
$k_T(\lambda)$	0.99275	0.97200	0.93925	0.89600	0.84375	0.78400	0.71825
$k_\delta(\lambda)$	1.04908	0.99454	0.94404	0.89616	0.84421	0.78530	0.71985
λ	0.40	0.45	0.50	0.55	0.60	0.65	0.70
$k_T(\lambda)$	0.64800	0.57475	0.50000	0.42525	0.35200	0.28175	0.21600
$k_\delta(\lambda)$	0.64949	0.57602	0.50108	0.42617	0.35278	0.28239	0.21644
λ	0.75	0.80	0.85	0.90	0.95		
$k_T(\lambda)$	0.15625	0.10400	0.06075	0.02800	0.00725		
$k_\delta(\lambda)$	0.15637	0.10374	0.06059	0.02968	0.01363		

The comparison of the approximate solution k_δ and the solution k_T with exact data is shown in Table 1. One can see from this comparison that, when there is a small change in data, the solution of (1.13) does not change much, which shows that, using our regularizing approach, we can obtain stable approximate solution of (1.13) from nonsmooth approximate data.

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