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The confluent hypergeometric functions $M(a, b; z)$ and $U(a, b; z)$ for large b and z

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ABSTRACT

We obtain new and complete asymptotic expansions of the confluent hypergeometric functions $M(a, b; z)$ and $U(a, b; z)$ for large b and z. The expansions are different in the three different regions: $z + a + 1 - b > 0$, $z + a + 1 - b < 0$ and $z + a + 1 - b = 0$. The expansions are not of Poincaré type and we give explicit expressions for the terms of the expansions. In some cases, the expansions are valid for complex values of the variables too. We give numerical examples which show the accuracy of the expansions.

1. Introduction

Asymptotic approximations of the confluent hypergeometric functions $M(a, b; z)$ and $U(a, b; z)$ for large b and z have been studied by several authors using different methods. For example, uniform methods for integrals are used in [1], where the author obtains uniform expansions of $M(a, b; z)$ and $U(a, b; z)$ in terms of parabolic cylinder functions. See also [2] for a similar expansion of the Whittaker function $W_{\kappa,\mu}(z) = e^{-z/2}z^{\mu+1/2}M(1/2 + \mu - \kappa, 1 + 2\mu, z)$ when its parameters and variable are large. Asymptotic methods for differential equations are used in [3,4], where the author derives uniform asymptotic expansions of $M_{\kappa,i\mu}(z) = e^{-z/2}z^{\mu+1/2}U(1/2 + \mu - \kappa, 1 + 2\mu, z)$ and $W_{\kappa,i\mu}(z)$ for real κ, μ and z in terms of Bessel and Airy functions. See also [5] and [6] for similar expansions of the Whittaker functions when their parameters are large. Using the Laplace's method for integrals, non uniform expansions of $M(a, b; z)$ and $U(a, b; z)$ for large b and z are obtained in [7] for real a, b and z in terms of elementary functions.

Recently, it has been proposed in [8–10] a variant of the Laplace's method for integrals that avoids the change of variables of the standard method and simplifies the computations. As a difference with the standard method, this technique gives an explicit formula for the coefficients of the expansion. In this paper we apply this method to two integral representations of $M(a, b; z)$ and $U(a, b; z)$ and obtain non-uniform expansions of these functions for large b and z in terms of elementary functions. A review of the method is given in the following section. The asymptotics of $M(a, b; z)$ and $U(a, b; z)$ in given in Sections 3 and 4 respectively.

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2. Preliminaries

Consider the integral

$$F(z) := \int_a^b e^{-zf(t)} g(t)\mathrm{d}t, \quad z > 0, \tag{1}$$

where (a, b) is a real interval (finite or infinite) and z is a large parameter. Let the functions $f(t)$ and $g(t)$ above be continuous on (a, b), with (a, b) finite or infinite and suppose that the above integral exists for large enough z. Let t_0 be the unique minimum of $f(t)$ on $[a, b]$ and let $f(t)$ and $g(t)$ be analytic at $t = t_0$. If $t_0 = a$, we let $g(t)$ to possess a power branch point at $t = a$ in such a way that $(t-a)^{-s} g(t)$ is analytic at $t = a$, with $s \in (-1, 0]$ if $t_0 = a$ and $s = 0$ if $t_0 \in (a, b]$. Then [9], [10, Theorem 1],

$$F(z) \sim e^{-zf(t_0)} \sum_{n=0}^{\infty} \Psi_n(z), \quad \text{as } z \to \infty, \tag{2}$$

with

$$\Psi_n(z) := \sum_{k=\lfloor \frac{n-m}{p-m} \rfloor}^{\lfloor \frac{n}{p-m} \rfloor} a_{n+mk}(z) \frac{\alpha + (-1)^{n+mk}\beta}{m} \Gamma\left(k + \frac{n+s+1}{m}\right) \left|\frac{m!}{f^{(m)}(t_0)z}\right|^{k+\frac{n+s+1}{m}}. \tag{3}$$

In this sum, $\lfloor -n \rfloor$ with n natural must be understood as zero and

$$\alpha = \begin{cases} 1 & \text{if } t_0 \in [a, b) \\ 0 & \text{if } t_0 = b, \end{cases} \qquad \beta = \begin{cases} 1 & \text{if } t_0 \in (a, b] \\ 0 & \text{if } t_0 = a. \end{cases}$$

The integer m is the degree of the first non-vanishing derivative of $f(t)$ at $t = t_0$ and p is the degree of the next non-vanishing derivative. For $n = 0, 1, 2, \ldots$,

$$a_n(z) := e^{zf(t_0)} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{z^k}{k!} \left(\frac{f^{(m)}(t_0)}{m!}\right)^k \times \sum_{j=0}^{n-mk} A_j(z) B_{n-mk-j}, \tag{4}$$

where $A_j(z)$ and B_j are the Taylor coefficients at $t = t_0$ of $e^{-zf(t)}$ and $(t-a)^{-s} g(t)$ respectively:

$$e^{-zf(t)} = \sum_{n=0}^{\infty} A_n(z)(t-t_0)^n, \qquad g(t) = \sum_{n=0}^{\infty} B_n (t-t_0)^{n+s}, \quad |t-t_0| < r.$$

The terms of the expansion are of the order $\Psi_n(z) = \mathcal{O}\left(z^{-(n+s+1)/m}\right)$ as $z \to \infty$.

Observe that the terms $a_n(z)$ are polynomials of degree $\lfloor n/p \rfloor$. Then, from (3) we see that the asymptotic sequence $\Psi_n(z)$ is a sum of $\lfloor n/(p-m) \rfloor - \lfloor (n-m)/(p-m) \rfloor + 1$ negative powers of z.

3. Asymptotic expansions of $M(a, b; z)$

The starting point is the integral representation of $M(a, b; z)$ [11, Eq. (13.2.1)]:

$$M(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1} (1-t)^{b-a-1} e^{zt}\, \mathrm{d}t, \quad \Re b > \Re a > 0. \tag{5}$$

We define $\alpha = (b-a-1)/z$ and we consider that both, b and z are large and of the same order, which means that a and α are fixed. We consider three different cases: $z + a + 1 - b > 0$, $z + a + 1 - b < 0$ and $z + a + 1 - b = 0$.

3.1. Case I: $z + a + 1 - b > 0$

After the change of variable $t \to 1 - t$, the above integral can be written in the form (1),

$$M(a, b; z) = \frac{\Gamma(b) e^z}{\Gamma(a)\Gamma(b-a)} \int_0^1 g(t)\, e^{-zf(t)}\, \mathrm{d}t,$$

Table 1
All the columns represent the relative error in the approximation of $M(3/2, b; z)$ taking the first n terms of the expansion (6) for the given values of b and z.

b, z	n			
	0	1	2	3
40, 100	0.00717631	−0.00222531	−0.00401425	−0.00171174
80, 200	0.00370395	−0.000522916	−0.000495289	−0.000193111
100, 500	0.00046581	−0.000296132	−0.000227519	−0.0000804347
300, 700	0.00119686	−0.00003598	-9.44441×10^{-6}	-1.28325×10^{-6}
600, 1200	0.000898967	-9.05179×10^{-6}	-1.24016×10^{-6}	-8.68476×10^{-8}

with $g(t) = (1-t)^{a-1}$ and $f(t) = t - \alpha \log t$. We have that $0 < \alpha < 1$, the unique critical point of $f(t)$ is $t_0 = \alpha$ and $f(t)$ attains its unique minimum on $[0, 1]$ at $t_0 = \alpha$. Both functions $e^{-zf(t)}$ and $g(t)$ have a Taylor expansion at $t = \alpha$ (with $s = 0$):

$$\begin{aligned} e^{-zf(t)} &= e^{-zf(t_0)} e^{-z(t-t_0)} \left(1 + \frac{t-t_0}{t_0}\right)^{\alpha z} \\ &= e^{-zf(t_0)} \sum_{n=0}^{\infty} \left[(-z)^n \sum_{k=0}^{n} \frac{(-\alpha z)_k}{k!(n-k)!(\alpha z)^k}\right] (t-t_0)^n, \quad \text{with } |t-\alpha| < \alpha, \\ g(t) &= \sum_{n=0}^{\infty} \frac{(1-a)_n (1-t_0)^{a-1-n}}{n!} (t-t_0)^n \quad \text{with } |t-\alpha| < 1-\alpha. \end{aligned}$$

We have $f(t_0) = \alpha - \alpha \ln \alpha$, $f''(t_0) = 1/\alpha$ and $f'''(t_0) \neq 0$. With the notation of Section 2 we have $m = 2$, $p = 3$ and

$$a_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{n-2k} \sum_{l=0}^{j} \frac{(-1)^j (-\alpha z)_l (1-a)_{n-2k-j} z^{j+k-l}}{2^k (j-l)! l! k! (n-2k-j)! \alpha^{k+l} (1-\alpha)^{n-2k-j}}.$$

Therefore, from Eqs. (2) and (3) we have that, for large b and z with fixed α, $0 < \alpha < 1$:

$$\begin{aligned} M(a, b; z) \sim{}& \frac{\Gamma(b) e^z (1-\alpha)^{a-1}}{\Gamma(a)\Gamma(b-a)} \left(\frac{\alpha}{e}\right)^{\alpha z} \sqrt{\frac{2\alpha\pi}{z}} \\ &\times \left[1 + \sum_{n=1}^{\infty} \sum_{k=0}^{2} \frac{a_{6n+2k-4}(z)}{\sqrt{\pi}} \left(\frac{2\alpha}{z}\right)^{3n+k-2} \Gamma\left(3n+k-\frac{3}{2}\right)\right]. \end{aligned} \tag{6}$$

The first few terms of the expansion are

$$M(a, b; z) \sim \frac{\Gamma(b) e^z (1-\alpha)^{a-1}}{\Gamma(a)\Gamma(b-a)} \left(\frac{\alpha}{e}\right)^{\alpha z} \sqrt{\frac{2\alpha\pi}{z}} \left[1 + \left(\frac{(2-a\alpha)(1-a)}{2(1-\alpha)^2} + \frac{1}{12\alpha}\right)\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right].$$

Table 1 shows a numerical experiment about the accuracy of the above approximation.

3.2. *Case II:* $b > z + a + 1$

The integral (5) is of the form (1) with $g(t) = t^{a-1}$, $f(t) = -t - \alpha \log(1-t)$ and $\alpha = (b-a-1)/z$, with $\alpha > 1$. The unique critical point of $f(t)$ is $t = 1 - \alpha < 0$ and $f(t)$ attains its unique minimum on $[0, 1]$ at $t_0 = 0$. We have $f(0) = 0$, $f'(0) = \alpha - 1$ and $f''(0) \neq 0$. With the notation of Section 2 we have $m = 1$ and $p = 2$. Both functions $e^{-zf(t)}$ and $t^{1-a}g(t)$ have a Taylor expansion at $t = 0$. The one of $t^{1-a}g(t) = 1$ is trivial (observe that in this case $s = a - 1$) and:

$$e^{-zf_1(t)} = e^{\alpha z t}(1-t)^{b-a-1} = \sum_{n=0}^{\infty} a_n(z) t^n, \quad |t| < 1, \tag{7}$$

with

$$a_n(z) = \sum_{k=0}^{n} \frac{(a+1-b)_k (b-a-1)^{n-k}}{k!(n-k)!}.$$

Therefore, from (2) and (3) we have that, for large b and z with fixed $\alpha > 1$:

$$M(a, b; z) \sim \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} \frac{\Gamma(2n+a-1)\left[(2n+a-1)a_{2n}(z) + (b-a-z-1)a_{2n-1}(z)\right]}{(b-a-z-1)^{2n+a}}. \tag{8}$$

Table 2
All the columns represent the relative error in the approximation of $M(3/2, b; z)$ taking the first n terms of the expansion (8) for the given values of b and z.

b, z	n				
	0	1	2	3	4
50, 20	−0.11362533	0.01752476	−0.00241692	−0.00088089	0.00177587
100, 40	−0.05430195	0.00399357	−0.00027097	−0.00003898	0.00004085
200, 80	−0.02657879	0.00095620	−0.000032248	-2.0363×10^{-6}	1.0984×10^{-6}
300, 100	−0.01424644	0.00025788	-3.4181×10^{-6}	-3.0658×10^{-7}	6.2285×10^{-8}
1000,200	−0.00293981	9.5455×10^{-6}	-2.5216×10^{-9}	-1.0740×10^{-9}	3.4167×10^{-11}

The first few terms of the expansion are

$$M(a, b; z) \sim \frac{\Gamma(b)}{\Gamma(b-a)} \frac{1}{(b-a-z-1)^a} \left[1 - \frac{a(a+1)(a+1-b)}{2(b-a-z-1)^2} + \mathcal{O}\left(\frac{1}{b^2}\right)\right].$$

Observe that the integration interval $(0, 1)$ in (5) is contained in the disk of convergence of the expansion (7). As it is argued in [9], this means that (8) holds not only for $b - z - a - 1 > 0$, but also for $\Re(b - z - a - 1) > 0$.

Table 2 shows a numerical experiment about the accuracy of the above approximation.

3.3. Case III: $b = z + a + 1$

The functions $f(t)$ and $g(t)$ are the same than those in case II, but now $\alpha = 1$ and the unique critical point of $f(t)$ is $t = 1 - \alpha = 0$. Again, $f(t)$ attains its minimum on $[0, 1]$ at $t_0 = 0$ and $f(0) = 0$, but now $f'(0) = \alpha - 1 = 0$, $f''(0) = \alpha \neq 0$ and $f'''(0) \neq 0$. With the notation of Section 2 we have $m = 2$ and $p = 3$. Once again, the Taylor expansion at $t_0 = 0$ of $t^{1-a}g(t) = 1$ is trivial and:

$$e^{-zf(t)} = e^{(b-a-1)t}(1-t)^{b-a-1} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{(1+a-b)_k (b-a-1)^{n-k}}{k!(n-k)!}\right] t^n, \quad |t| < 1.$$

Then, the coefficients $a_n(z)$ read

$$a_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{n-2k} \frac{(a+1-b)_j (b-a-1)^{n-k-j}}{j!k!(n-2k-k)!2^k}.$$

Therefore, from (2) and (3) we have that, for large b and z and $\alpha = 1$:

$$\begin{aligned} M(a, b; z) \sim &\frac{\Gamma(b)}{2\Gamma(a)\Gamma(b-a)} \left(\frac{2}{b-a-1}\right)^{a/2} \\ &\times \left[\Gamma\left(\frac{a}{2}\right) + \sum_{k=0}^{1} a_{2k+1}(z) \Gamma\left(k + \frac{a+1}{2}\right) \left(\frac{2}{b-a-1}\right)^{k+1/2} \right. \\ &\left. + \sum_{n=2}^{\infty} \sum_{k=0}^{2} a_{3n+2k-4}(z) \Gamma\left(k + \frac{3n+a}{2} - 2\right) \left(\frac{2}{b-a-1}\right)^{k+3n/2-2} \right]. \end{aligned} \tag{9}$$

The first few terms of the expansion are

$$M(a, b; z) \sim \frac{\Gamma(b)}{2\Gamma(a)\Gamma(b-a)} \left(\frac{2}{b-a-1}\right)^{a/2} \left[\Gamma\left(\frac{a}{2}\right) - \frac{2}{3}\Gamma\left(\frac{a+3}{2}\right) \sqrt{\frac{2}{b-a-1}} + \mathcal{O}\left(\frac{1}{z}\right)\right].$$

As in the previous case, (9) holds not only for $b - z - a - 1 = 0$ but also for $\Re(b - z - a - 1) = 0$.

Table 3 shows a numerical experiment about the accuracy of the above approximation.

4. Asymptotic expansions $U(a, b; z)$

The starting point is the integral representation [[11], Equation 13.2.5]:

$$U(a, b; z) = \frac{1}{\Gamma(a)} \int_0^{\infty} t^{a-1} (1+t)^{b-a-1} e^{-zt} \, dt, \quad \Re z > 0, \ \Re a > 0. \tag{10}$$

The integral (10) is of the form (1) with $g(t) = t^{a-1}$ and $f(t) = t - \alpha \log(1+t)$. As in the previous section, we define $\alpha = (b - a - 1)/z$ and we consider b and z large and of the same order and then a and α are fixed. We consider again three different cases: $z + a + 1 - b > 0$, $z + a + 1 - b < 0$ and $z + a + 1 - b = 0$.

Table 3
All the columns represent the relative error in the approximation of $M(3/2, b; b-5/2)$ taking the first n terms of (9) for the given values of b.

b	n				
	0	1	2	3	4
20	−0.23709530	0.02068824	0.01951005	−0.03234603	0.02805755
50	−0.13678165	0.00699928	0.00244375	−0.00168864	0.00042447
100	−0.0932349680	0.00327724	0.00056032	−0.00011135	−0.00012957
200	−0.06445130	0.00157444	0.00013380	0.00001967	−0.00004709
500	−0.04003611	0.00061037	0.00002086	9.33063×10^{-6}	-6.85496×10^{-6}

Table 4
All the columns represent the relative error in the approximation of $U(3/2, b; z)$ taking the first n terms of the expansion (11) for the given values of b and z.

b, z	n			
	0	1	2	3
80, 30	−0.00729139	0.000288621	−0.000171287	−0.000150116
150, 50	−0.0037508	0.0000616996	−0.0000264384	−0.0000121119
500, 300	−0.00109959	0.0000228974	1.06293×10^{-7}	-6.2741×10^{-8}
1000, 600	−0.0.00055101	5.71282×10^{-6}	1.76091×10^{-8}	-3.41763×10^{-9}
1500, 800	−0.000387509	-1.77026×10^{-6}	-5.68313×10^{-9}	-8.70561×10^{-10}

4.1. Case I: $b > z + a + 1$

In this case we have $\alpha > 1$, the unique critical point of $f(t)$ is $t_0 = \alpha - 1$ and $f(t)$ attains its unique minimum on $[0, 1]$ at $t_0 = \alpha - 1$. Both functions $e^{-zf(t)}$ and $g(t)$ have a Taylor expansion at $t_0 = \alpha - 1$ (with $s = 0$):

$$\begin{aligned} e^{-zf(t)} &= e^{-zf(t_0)}e^{-z(t-t_0)}\left(1+\frac{t-t_0}{1+t_0}\right)^{\alpha x} \\ &= e^{-zf(t_0)}\sum_{n=0}^{\infty}\left[(-z)^n\sum_{j=0}^{n}\frac{(-\alpha z)_k}{k!(n-k)!(\alpha z)^k}\right](t-t_0)^n, \quad |t-(\alpha-1)|<\alpha, \\ g(t) &= \sum_{n=0}^{\infty}\frac{(-1)^n(1-a)_n(1-\alpha)^{a-1-n}}{n!}(t-t_0)^n, \quad |t-(\alpha-1)|<\alpha-1. \end{aligned}$$

We have $f(t_0) = \alpha - 1 - \alpha\ln\alpha$, $f''(t_0) = 1/\alpha$ and $f'''(t_0) \neq 0$. With the notation of Section 2 we have $m = 2$, $p = 3$ and

$$a_n(z) = \sum_{k=0}^{[n/2]}\sum_{j=0}^{n-2k}\sum_{l=0}^{j}\frac{(-1)^n z^{k+j-l}(-\alpha z)_l(1-a)_{n-2k-j}(\alpha-1)^{2k-n+j}}{k!(j-l)!l!(n-2k-j)!2^k\alpha^{k+l}}.$$

Therefore, from (2) and (3) we have that, for large b and z and fixed $\alpha > 1$:

$$U(a,b;z)\sim\frac{1}{\Gamma(a)}\left(\frac{\alpha}{e}\right)^{\alpha z}(\alpha-1)^{a-1}\sqrt{\frac{2\alpha\pi}{z}}\left[1+\sum_{n=1}^{\infty}\sum_{k=0}^{2}\frac{a_{6n+2k-4}(z)}{\sqrt{\pi}}\left(\frac{2\alpha}{z}\right)^{3n+k-2}\Gamma\left(3n+k-\frac{3}{2}\right)\right]. \tag{11}$$

The first few terms of the expansion are

$$U(a,b;z)\sim\frac{1}{\Gamma(a)}\left(\frac{\alpha}{e}\right)^{\alpha z}(\alpha-1)^{a-1}\sqrt{\frac{2\alpha\pi}{z}}\left[1+\left(\frac{(2-a\alpha)(1-a)}{2(1-\alpha)^2}+\frac{1}{12\alpha}\right)\frac{1}{z}+\mathcal{O}\left(\frac{1}{z^2}\right)\right].$$

Table 4 shows a numerical experiment about the accuracy of the above approximation.

4.2. Case II: $b < z + a + 1$

In this case $\alpha < 1$, the unique critical point of $f(t)$ is $t = \alpha - 1 \notin [0, 1]$ and $f(t)$ attains its minimum on $[0, 1]$ at $t_0 = 0$. We have $f(0) = 0$, $f'(0) = 1 - \alpha$ and $f''(0) \neq 0$. With the notation of Section 2 we have $m = 1$ and $p = 2$. Both functions $e^{-zf(t)}$ and $t^{1-a}g(t)$ have a Taylor expansion at $t = 0$. The one of $t^{1-a}g(t) = 1$ is trivial (observe that in this case $s = a - 1$) and:

$$e^{-zf_1(t)} = e^{-\alpha zt}(1+t)^{b-a-1} = \sum_{n=0}^{\infty}a_n(z)t^n, \quad |t|<1,$$

with

$$a_n(z) = \sum_{k=0}^{n}\frac{(a+1-b)_k(b-a-1)^{n-k}}{k!(n-k)!}.$$

Table 5
All the columns represent the relative error in the approximation of $U(3/2, b; z)$ taking the first n terms of the expansion (12) for the given values of b and z.

b, z	n 0	1	2	3	4
40, 100	0.017069	−0.00123825	0.000100331	-8.92728×10^{-6}	8.26084×10^{-7}
80, 200	0.00941109	−0.000363507	0.0000163158	-8.3974×10^{-7}	4.81173×10^{-8}
100, 300	0.00438706	−0.0000906646	1.99168×10^{-6}	-4.93653×10^{-8}	1.37619×10^{-9}
300, 700	0.00341076	−0.0000441363	7.05264×10^{-7}	-1.34366×10^{-8}	2.97661×10^{-10}
600, 1200	0.00306411	−0.0000315543	4.30402×10^{-7}	-7.22087×10^{-9}	1.25242×10^{-10}

Table 6
All the columns represent the relative error in the approximation of $U(3/2, b; b-5/2)$ taking the first n terms of the expansion (13) for the given values of b.

b	n 0	1	2	3	4
50	−0.117071	−0.00539794	−0.0018597	−0.0050693	−0.00671053
100	−0.0836316	−0.00273335	−0.00045599	−0.00101899	−0.00100372
300	−0.0489939	−0.000930897	−0.0000502492	−0.0000873915	−0.000052745
500	−0.0381539	−0.000563225	−0.0000180304	−0.0000286884	−0.0000137196
1000	−0.0271356	−0.000284228	-4.47042×10^{-6}	-6.47886×10^{-6}	-2.24977×10^{-6}

Therefore, from (2) and (3) we have that, for large b and z and fixed $\alpha < 1$:

$$U(a,b;z) = \frac{1}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(2n+a-1)\left[(2n+a-1)a_{2n}(z)+(z+a+1-b)a_{2n-1}(z)\right]}{(z+a+1-b)^{2n+a}}. \tag{12}$$

The first few terms of the expansion are

$$U(a,b;z) \sim \frac{1}{(z+a+1-b)^a}\left[1-\frac{(a)_1(b-a-1)}{2(z+a+1-b)^2}+\frac{(8z+(3a+1)(b-a-1))(b-a-1)(a)_2}{24(z+a+1-b)^4}+\mathcal{O}\left(\frac{1}{b^3}\right)\right].$$

Table 5 shows a numerical experiment about the accuracy of the above approximation.

4.3. *Case III*: $b = z + a + 1$

In this case $\alpha = 1$, the unique critical point of $f(t)$ is $t = \alpha - 1 = 0$ and again, $f(t)$ attains its minimum on $[0, 1]$ at $t_0 = 0$ and $f(0) = 0$, but now $f'(0) = 1 - \alpha = 0$, $f''(0) \neq 0$ and $f'''(0) \neq 0$. Then we have $m = 2$ and $p = 3$. Once again, the Taylor expansion at $t_0 = 0$ of $t^{1-a}g(t) = 1$ is trivial (with $s = a - 1$) and:

$$e^{-zf(t)} = e^{(b-a-1)t}(1+t)^{b-a-1} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{(-1)^n(1+a-b)_k(b-a-1)^{n-k}}{k!(n-k)!}\right]t^n, \quad |t|<1.$$

Then, the coefficients $a_n(z)$ read

$$a_n(z) = \sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{n-2k}\frac{(-1)^n(a+1-b)_j(b-a-1)^{n-k-j}}{j!k!(n-2k-j)!2^k}.$$

Therefore, from (2) and (3) we have that, for large b and z and $\alpha = 1$:

$$U(a,b;z) = \frac{1}{2\Gamma(a)}\left(\frac{2}{b-a-1}\right)^{a/2}\left[\Gamma\left(\frac{a}{2}\right)+\sum_{k=0}^{1}a_{2k+1}(z)\Gamma\left(k+\frac{a+1}{2}\right)\left(\frac{2}{b-a-1}\right)^{k+1/2}\right.$$
$$\left.+\sum_{n=2}^{\infty}\sum_{k=0}^{2}a_{3n+2k-4}(z)\Gamma\left(k+\frac{3n+a}{2}-2\right)\left(\frac{2}{b-a-1}\right)^{k+3n/2-2}\right]. \tag{13}$$

The first few terms of the expansion are

$$U(a,b;z) \sim \frac{1}{2\Gamma(a)}\left(\frac{2}{b-a-1}\right)^{a/2}\left[\Gamma\left(\frac{a}{2}\right)+\frac{2}{3}\Gamma\left(\frac{a+3}{2}\right)\sqrt{\frac{2}{b-a-1}}+\mathcal{O}\left(\frac{1}{b}\right)\right].$$

Table 6 shows a numerical experiment about the accuracy of the above approximation.

As a final remark, we want to mention that the accuracy of the above expansions is similar to the accuracy of the expansions given in [7], as well as the computational speed. The advantage of the expansions of this paper is analytical, as every one of the terms of the expansions are given by simple and explicit formulas.

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