A two-grid method based on Newton iteration for the Navier–Stokes equations

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Abstract

In this paper, we consider a two-grid method for resolving the nonlinearity in finite element approximations of the equilibrium Navier–Stokes equations. We prove the convergence rate of the approximation obtained by this method. The two-grid method involves solving one small, nonlinear coarse mesh system and two linear problems on the fine mesh which have the same stiffness matrix with only different right-hand side. The algorithm we study produces an approximate solution with the optimal asymptotic in $h$ and accuracy for any Reynolds number. Numerical example is given to show the convergence of the method.

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1. Introduction

We consider a two-grid method for the resolution of the nonlinear system arising from finite element discretizations of the equilibrium, incompressible Navier–Stokes equations:

\[
\begin{aligned}
&-\nu \Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in } \Omega, \\
&\nabla \cdot u = 0 \quad \text{in } \Omega, \\
&u = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

and

\[
\int_{\Omega} p \, dx = 0,
\]

where $u$ is the velocity, $p$ is the pressure, $f$ is the body force, $\nu$ is the viscosity of the fluid, and $\Omega$ is the flow domain with sufficiently smooth boundary $\partial \Omega$.

If (1) is discretized by the usual Galerkin finite element method, it results in a large system of nonlinear algebraic equations. A common choice for the solution of this nonlinear algebraic equations is linearization by Newton’s method

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Generally, we should iterate several times to get the solutions. In this way, we may get some trouble. Because we require reassembly of the linearized problem each time when the point of linearization changes. Especially, when the meshsize $h \to 0$, this may get much more difficult. So Layton takes a two level discretization method to solve the Navier–Stokes equations in [6] (This technique can also be found in [9].) Moreover, Xu [11,12] proposes a general two-grid method on nonlinear problems. This technique makes the coarse mesh extremely coarse (in contrast to fine mesh) which still maintains the optimal approximation. This means that solving a nonlinear equation (including the NS equations) is not much more difficult than solving one linear equation, and the work for solving a nonlinear problem on coarse mesh is relatively negligible. Now we simply state the general two-grid method as follows: first we solve one small, nonlinear system upon a coarse mesh, then solve one linearized problem on fine mesh based on Newton’s method and at last solve one linear correction problem on the coarse mesh. Compared with Xu’s method, Layton [6] has not used the coarse correction step. However, when Layton–Tobiska [8] add this coarse correction step to solve the Navier–Stokes equations, the benefits of this step seem to be marginal.

In this paper, we introduce another two-grid method for the Navier–Stokes equations. The first two steps are the same as before, and we do some improvements in the third step. In this method we do not solve a correction problem on coarse mesh in last step, but still solve the same linear problem with different loads on fine mesh, which has a great improvement both on errors in $H^2$ and $L^2$ norms.

We now introduce some notations. Let $H^1_0(\Omega)$ be the standard Sobolev spaces (see [1]) equipped with the usual norm $\| \cdot \|_{1, \Omega}$. We use $\| \cdot \|_1$ for its semi-norm, which is equivalent to the $\| \cdot \|_{1, \Omega}$ in $H^1_0(\Omega)$. Let $Y = X \times Q$, where $X = (H^1_0)^d$ and $Q$ is a subspace of $L^2(\Omega)$ defined by

$$ Q = L^2_0(\Omega) = \left\{ q : q \in L^2(\Omega), \int_{\Omega} q \, dx = 0 \right\}. $$

The scalar product and norm in $Q$ are denoted by the usual $L^2(\Omega)$ inner product and $\| \cdot \|_0$, respectively. Then the velocity–pressure variational formulation for (1) reads: find $(u, p) \in Y$ such that

$$ a(u, v) + b(u, u, v) - (p, \nabla \cdot v) + (q, \nabla \cdot u) = (f, v) \quad \forall (v, q) \in Y, $$

where the bilinear form $a(\cdot, \cdot)$ and the trilinear form $b(\cdot, \cdot, \cdot)$ are given by

$$ a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad b(u, w, v) = \int_{\Omega} u \cdot \nabla w \cdot v \, dx \quad \forall u, v, w \in X. $$

To present the two-grid method, we first consider the usual finite element approximation to (1), which can be described as follows. Assume that $X^h$ and $Q^h$ are the finite element subspaces of $X$ and $Q$. We set $Y^h = X^h \times Q^h$, which is clearly a subspace of the $Y$. Then the usual finite element approximation $(u_h, v_h)$ is calculated by solving the nonlinear system: find $(u_h, v_h) \in Y^h$ such that, for all $(v, q) \in Y^h$,

$$ a(u_h, v) + b(u_h, u_h, v) - (p_h, \nabla \cdot v) + (q, \nabla \cdot u_h) = (f, v). \quad (2) $$

The two-grid algorithm we study is as follows. Choose a coarse mesh width $H$ and a fine mesh width $h$ with $H \gg h > 0$, and construct associated conforming finite element spaces $(X^H, Q^H)$ and $(X^h, Q^h)$. Now find $(u^h, p^h)$ as follows:

**Algorithm 1: Two-grid algorithm, two linear problems on fine mesh based on Newton method**

**Step 1:** Solve nonlinear system on coarse mesh: Find $(u^H, p^H) \in (X^H, Q^H)$ such that for all $(v, q) \in (X^H, Q^H)$

$$ a(u^H, v) + b(u^H, u^H, v) - (p, \nabla \cdot u^H) + (q, \nabla \cdot v) = (f, v). $$

**Step 2:** Update on fine mesh with one Newton iteration: Find $(u^*_h, p^*_h) \in (X^h, Q^h)$ such that for all $(v, q) \in (X^h, Q^h)$

$$ a(u^*_h, v) + b(u^*_h, u^*_h, v) + b(u^H, u^*_h, v) - (p^*_h, \nabla \cdot v) + (q, \nabla \cdot u^*_h) = (f, v) + b(u^H, u^H, v). $$

**Step 3:** Update on fine mesh: Find $(u^h, p^h) \in (X^h, Q^h)$ such that for all $(v, q) \in (X^h, Q^h)$

$$ a(u^h, v) + b(u^h, u^H, v) + b(u^H, u^h, v) - (p^h, \nabla \cdot v) + (q, \nabla \cdot u^h) = (f, v) + b(u^H, u^*_h, v) + b(u^*_h, u^H - u^*_h, v). $$
Our analysis shows that \((u^h, p^h)\) attains the same asymptotic accuracy as \((u_h, p_h)\) of (2) for suitable choice of coarse meshsize \(H\). Furthermore, we improve the scaling for errors in energy norm and \(L^2\) norm, e.g., for energy norm it gets to \(h \sim H^{k-2}\) and for \(L^2\) norm it gets to \(h \sim H^2\), if \(k = 1\). In this process, the third step plays an important role in improving the scalings for errors. In comparing methods, the scalings of method we studied are almost but not quite as favorable as a full second Newton step on the fine mesh and more accurate than a third coarse mesh correction.

Although we need to solve two large linear problems, we can see from Algorithm 1 that they have the same stiffness matrix with only different right-hand side. Thus there is a little more difficult than solving one linear problem on fine mesh. In practical computing our method does not require reassembly of the linearized problem on Step 3, which saves a lot of CPU time. This means that the method we studied is much more efficient than two full Newton steps and a bit less efficient than the third coarse mesh correction. Because the third step also works on fine mesh, the Galerkin projection: \(Y \rightarrow Y^H\) is not required any more, which makes the analysis simpler.

The rest of the paper is organized as follows. In Section 2, we give some assumptions on the finite element subspace \(Y^h(Y^H)\) and show the error in the energy norm. The error in \(L^2\) norm will be showed in Section 3. At last, in Section 4 a numerical example is given to show our analysis.

2. The error estimate in the energy norm

This section gives the basic error analysis of Algorithm 1 in the natural energy norm for the Navier–Stokes equations, given by

\[
\|(v, q)\| := (|v|^2_1 + \|q\|^2_0)^{1/2}.
\]

The finite element spaces \(X^h, Q^h\) can be thought of as generated from \(X^H, Q^H\) by a mesh refinement process, e.g., [10], and therefore nested. It is neither algorithmically necessary nor needed for the results of our convergence theorems too hold. However, we shall assume them nested since it will simplify our analysis substantially, i.e., \((X^H, Q^H) \subset (X^h, Q^h)\). Further, we assume \((X^H, Q^H), \mu = h, H\), satisfies the usual \(\text{inf–sup}\) or LBB stability condition for compatibility of the velocity–pressure spaces:

\[
\inf_{q \in Q^\mu} \sup_{v \in X^\mu} \frac{(q, \nabla \cdot u)}{\|q\|_0 \|\nabla v\|_0} \geq \beta > 0, \quad \mu = H \text{ and } h.
\]

Moreover, we assume the spaces \((X^\mu, Q^\mu)(\mu = H \text{ and } h)\) satisfy the approximation properties typical of piecewise polynomials of degree \((k, k-1)\). Specifically, for all \((u, p) \in Y \cap (H^{k+1}(\Omega)^d \times H^k(\Omega))\),

\[
\inf_{(v^h, q^h) \in (X^h, Q^h)} \{h|u - u^h|_1 + \|u - u^h\|_0 + h\|p - q^h\|_0 \leq C h^{k+1}(|u|_{k+1} + |p|_k),
\]

and similarly for \((X^H, Q^H)\).

For \(\mu = H, h\), \(V\) and \(V^\mu\) will denote the space of vector functions in \(X \text{ and } X^\mu\), respectively, which are divergence-free and discretely divergence-free, respectively,

\[
V = \{v \in X : (q, \nabla \cdot v) = 0, \forall q \in Q\},
\]

\[
V^\mu = \{v \in X^\mu : (q, \nabla \cdot v) = 0, \forall q \in Q^\mu\}.
\]

It is known [4] that provided the \(\text{inf–sup}\) condition (3) holds, \(V^\mu\) is a well-defined and nontrivial subspace of \(X^\mu\), and that a function in \(V\) can be approximated in \(V^\mu\) with the same asymptotic accuracy as in \(X^\mu\).

The following optimal error estimates are well known (cf. [2,4]):

**Theorem 1.** Suppose \((u, p) \in Y \cap (H^{k+1}(\Omega)^d \times H^k(\Omega))\) is a nonsingular solution of (2) and the finite element subspaces \(Y^h\) satisfy the LBB condition (3) and the approximation assumption (4). Suppose the linearized dual problem (1) is \(H^2\) regular, and \(h\) is sufficiently small. Then there exists a solution \((u_h, p_h)\) to the Galerkin equation (2) and we have the following optimal error estimates for certain constant \(C\) independent of \(h\)

\[
h|u - u^h|_1 + \|u - u^h\|_0 + h\|p - p^h\|_0 \leq C h^{k+1}(|u|_{k+1} + |p|_k).
\]
Before we study the error of Algorithm 1 in detail, we formulate sufficient conditions which guarantee the unique solvability of the linear problems in Steps 2 and 3 and a set of inequalities for estimating the trilinear form $b$, respectively.

First note that Algorithm 1 can be reformulated. Indeed, introducing the continuous bilinear form $B^H : Y \times Y \to \mathbb{R}$ given by

$$B^H[(u, p); (v, q)] := a(u, v) + b(u^H, u, v) + b(u, u^H, v) - (p, \nabla \cdot v) + (q, \nabla \cdot u),$$

(5)

we can reformulate Steps 2 and 3 of Algorithm 1 in the product space to be:

**Step 2:** $B^H[(u^*_h, p^*_h); (v, q)] = (f, v) + b(u^H, u^H, v);$

**Step 3:** $B^H[(u^h, p^h); (v, q)] = (f, v) + b(u^H, u^*_h, v) + b(u^*_h, u^H - u^*_h, v).$

(6)

(7)

The bilinear form $B^H[\cdot, \cdot]$ are easily shown to be continuous on the product space $Y \times Y$ with continuity constant depending on $|u^H|_1$. Thus, provided $|u^H|_1$ is bounded uniformly in $H$, the bilinear form $B^H[\cdot, \cdot]$ is continuous on $Y \times Y$ uniformly in $H$. Uniform boundedness in $H$ of $|u^H|_1$ follows by the assumption that $H$ is small enough, implying that $u^H$ is close to $u$:

$$|u^H|_1 \leq |u^H - u|_1 + |u|_1 \leq H(|u|_2 + |p|_1) \leq C(u, p).$$

Thus, $B^H[\cdot, \cdot]$ is continuous on $Y \times Y$ uniformly in $H$.

In order to guarantee the solvability of the problems in Steps 2 and 3, we shall always assume that the following formulation holds:

$$\inf_{(u, p) \in Y^h} \sup_{(v, q) \in Y^h} \frac{B^H[(u, p); (v, q)]}{\|(u, p)|| \cdot ||(v, q)||} \geq \gamma > 0,$$

(8)

whose details can be found in [8]. Thus, both Steps 2 and 3 have a unique solution.

The next lemma gives the boundedness of the trilinear $b$, which is useful for the analysis of error estimations.

**Lemma 1.** Suppose the boundary of the domain $\Omega$ satisfies the strong Lipschitz condition of Adams [1]. Then the following estimates hold:

$$|b(u, v, w)| \leq C|u|_1|v|_1|w|_1 \text{ for all } u, v, w \in X;$$

$$|b(u, v, w)| \leq C\|u\|_0^{-\delta}|u|^s_1|v|_1|w|_1 \text{ for all } u, v, w \in X;$$

$$|b(u, v, w)| \leq C\|u\|_0|v|_1||w||_2 \text{ for all } u, v \in X, w \in H^2(\Omega);$$

$$|b(u, v, w)| \leq C|u|_1||v||_0||w||_2 \text{ for all } u, v \in X, w \in H^2(\Omega);$$

where $s = \varepsilon > 0$ is arbitrarily small for $d = 2$ and $s = \frac{1}{2}$ for $d = 3$.

**Proof.** See Layton and Tobiska [8]. □

With the above sufficient conditions, we now state the error estimation in the energy norm.

**Theorem 2.** Suppose $X^H \subset X^h \subset X, Q^H \subset Q^h \subset Q$, and each pair $(X^H, Q^H), (X^h, Q^h)$ satisfies the LBB condition (3) and the approximation assumption (4). Let $(u, p)$ be a nonsingular solution of the Navier–Stokes equations that is smooth enough, $(u, p) \in Y \cap (H^{k+1}(\Omega)^d \times H^k(\Omega))$. Let $(u^h, p^h)$ be a nonsingular solution of Algorithm 1. Suppose the linearized dual problem to (1) is $H^2$ regular, $0 < h < H$ and $H$ is sufficiently small. Let $s = \frac{1}{2}$ of $d = 3$ and $s = \varepsilon > 0$ if $d = 2$. Then, there is a constant $C = C(s, Re, u, p)$ such that

$$\|(u - u^h, p - p^h)\| \leq C(h^k + H^{3k+1-s}).$$

**Proof.** First, we consider briefly the error of the method after Step 2. The error $(u - u^*_h, p - p^*_h)$ satisfies the following approximate Galerkin orthogonality relation for all $(v, q) \in Y^h$:

$$B^H[(u - u^*_h, p - p^*_h); (v, q)] = b(u - u^H, u^H - u, v),$$

(9)
then, by means of the continuity of $B^H$, we have, for $(I^h u, J^h p) \in Y^h$ and interpolant of $(u, p)$ into $Y^h$,
\[
\|(u - u^*_h, p - p^*_h)\| \leq \|(u - I^h u, p - J^h p)\| + \|(I^h u - u^*_h, J^h p - p^*_h)\| \leq Ck^d \left(\|u\|_{k+1} + |p|_k\right) + \frac{1}{\gamma} \sup_{(v,q) \in Y^h} \frac{B^H[(I^h u - u^*_h, J^h p - p^*_h); (v,q)\]}{\|(v,q)\|} \leq Ck^d \left(\|u\|_{k+1} + |p|_k\right) + \frac{1}{\gamma} \sup_{(v,q) \in Y^h} \frac{B^H[(u - u^*_h, p - p^*_h); (v,q)\]}{\|(v,q)\|} \leq Ck^d \left(\|u\|_{k+1} + |p|_k\right) + \frac{1}{\gamma} \sup_{(v,q) \in Y^h} \frac{B^H[(u - u^*_h, p - p^*_h); (v,q)\]}{\|(v,q)\|}.
\]

Here, as in Lemma 1, $s$ is the dimension dependent parameter given by $s = \epsilon > 0$ arbitrary small if $d = 2$ and $s = \frac{1}{2}$ if $d = 3$. Thus, using the bounds of Lemma 1 we have:
\[
b(u - u^H, u^H - u, v) \leq C\|u - u^H\|_0^{1-s}\|u - u^H\|_1^{1+s}\|v\|_1 \leq CH^{2k+1-s}\|(v,q)\|.
\]

Then we get the error estimation after Step 2 (see also [8])
\[\|(u - u^*_h, p - p^*_h)\| \leq C(h^k + H^{2k+1-s}). \quad (10)\]

Next, we show that the error $(u - u^h, p - p^h)$ satisfies the following relation for all $(v,q) \in Y^h$:
\[
B^H[(u - u^h, p - p^h); (v,q)] = b(u - u^H, u^*_h - u, v) + b(u - u^*_h, u^H - u, v) + b(u^*_h - u, u^*_h - v, v).
\]

Now, the remainder of the proof of Theorem 2 follows by normal-type arguments applying (8). Indeed,
\[
\|(u - u^h, p - p^h)\| \leq \|(u - I^h u, p - J^h p)\| + \|(I^h u - u^h, J^h p - p^h)\| \leq Ck^d \left(\|u\|_{k+1} + |p|_k\right) + \frac{1}{\gamma} \sup_{(v,q) \in Y^h} \frac{B^H[(u - u^h, p - p^h); (v,q)\]}{\|(v,q)\|} \leq Ck^d \left(\|u\|_{k+1} + |p|_k\right) + \frac{1}{\gamma} \sup_{(v,q) \in Y^h} \frac{B^H[(u - u^h, p - p^h); (v,q)\]}{\|(v,q)\|} \leq Ck^d \left(\|u\|_{k+1} + |p|_k\right) + \frac{1}{\gamma} \sup_{(v,q) \in Y^h} \frac{b(u - u^H, u^*_h - u, v) + b(u - u^*_h, u^H - u, v) + b(u^*_h - u, u^*_h - u, v)}{\|(v,q)\|}. \quad (12)
\]

Thus, using the bounds of Lemma 1 and (10) repeatedly,
\[
b(u - u^H, u^*_h - u, v) \leq C\|u - u^H\|_0^{1-s}\|u - u^H\|_1^{1+s}\|v\|_1 \leq CH^{k+1-s}(h^k + H^{2k+1-s})\|(v,q)\|, \quad b(u - u^*_h, u^H - u, v) \leq C\|u - u^*_h\|_1\|u^H - u\|_1\|v\|_1 \leq CH^k\|u^H - u\|_1\|(v,q)\|, \quad b(u^*_h - u, u^*_h - u, v) \leq C\|u - u^*_h\|_1\|v\|_1 \leq CH^k\|u^*_h - u\|_1\|(v,q)\|.
\]
Inserting the above equations into (12) yields
\[ \|(u - u^h, p - p^h)\| \leq C(h^k + H^{3k+1-s}), \]
which is the main result of this section. □

Compared with the little improvement attained in applying the method of Xu [11,12] to Navier–Stokes equations, we can see from the proof of Theorem 2 that the third correction step makes great improvement, i.e., from \( h \sim H^{3-s} \) to \( h \sim H^{4-s} \), if \( k = 1 \). Steps 2 and 3 have the same stiffness matrix with only different right-hand side, which have just a little more difficult than solving one linear system on fine mesh. Thus, this two-grid algorithm is a effective method for solving the Navier–Stokes equations.

3. The \( L^2 \)-estimate

We shall now give an \( L^2 \)-error estimate for Algorithm 1. The \( L^2 \)-error analysis is based on first upon the assumptions needed to obtain the \( H_1 \) estimate of the previous section and second, the assumption that the following linearized adjoint problem is \( H^2(\Omega) \) regular.

Find \((\phi, \lambda) \in Y\) such that, for all \((w, r) \in Y\),
\[ a(w, \phi) + b(u, w, \phi) + b(w, u, \phi) - (r, \nabla \cdot \phi) + (\lambda, \nabla \cdot w) = (g, w). \] (13)

Since \( u \) is a nonsingular solution, \((\phi, \lambda) \) exists uniquely. The assumption that the linearized adjoint problem (13) is \( H^2(\Omega) \) regular means that, for any \( g \in (L^2(\Omega))^d \), there is a solution \((\phi, \lambda) \) belonging to \( Y \cap (H^{k+1}(\Omega)^d \times H^k(\Omega)) \) and the inequality
\[ \|\phi\|_2 + \|\lambda\|_1 \leq C\|g\|_0 \] (14)
holds. This \( H^2(\Omega) \)-regularity assumption is also implicit in the previous section as well, in the form of the assumed \( L^2 \)-optimality of the coarse mesh approximation.

**Theorem 3.** Suppose that the solution of (13) satisfies (14). Under the assumptions of Theorem 2, we have the error estimates in \( L^2 \) norm,
\[ \|u - u^h\|_0 \leq C(h^{k+1} + h^k H^{k+1} + H^{3k+2-s}). \]

**Proof.** First, the solution \((\phi, \lambda) \) of (13) satisfies
\[ B^H[w, r]; (\phi, \lambda) = (g, w) + b(u^H, u - u^h, \phi) + b(u^H - u, w, \phi) \]
for all \((w, r) \in Y\).

Let \( g := u - u^h \in (L^2(\Omega))^d \), and \((w, r) := (u - u^h, p - p^h)\), which gives
\[ \|u - u^h\|_0^2 = B^H[(u - u^h, p - p^h); (\phi, \lambda)] - b(u - u^h, u^H - u, \phi) - b(u^H - u, u - u^h, \phi), \]
and, taking into account the error equation (11), we have
\[ \|u - u^h\|_0^2 = B^H[(u - u^h, p - p^h); (\phi - \phi^h, \lambda - \lambda^h)] - b(u - u^h, u^H - u, \phi) \]
\[ - b(u^H - u, u - u^h, \phi) + b(u - u^H, u^h - u, \phi^h) \]
\[ + b(u - u^h, u^H - u, \phi^h) + b(u^h - u, u^h - u, \phi^h). \] (15)

Then we estimate the right-hand side term-by-term:
\[ B^H[(u - u^h, p - p^h); (\phi - \phi^h, \lambda - \lambda^h)] \leq \|(u - u^h, p - p^h)\| \|((\phi - \phi^h, \lambda - \lambda^h))\|
\leq C(h^k + H^{3k+1-s})h(\|\phi\|_2 + \|\lambda\|_1)
\leq C(h^{k+1} + h H^{3k+1-s})\|u - u^h\|_0, \]
|b(u - uh, uH - u, \phi)| \leq |u - uh|_1 \|uH - u\|_0 \|\phi\|_2 \\
\leq C(h^k H^{k+1} + H^{4k+2-s})\|u - uh\|_0,
|b(uH - u, u - uh, \phi)| \leq \|uH - u\|_0 \|u - uh\|_1 \|\phi\|_2 \\
\leq C(h^k H^{k+1} + H^{4k+2-s})\|u - uh\|_0,
|b(u - uH, uh - u, \phi)h)| \leq |b(u - uH, uh - u, \phi)| + |b(u - uH, uh - u, \phi)| \\
\leq |u - uH|_1 |uH - u|_1 |\phi - \phih|_1 + \|u - uH\|_0 |uH - u|_1 \|\phi\|_2 \\
\leq C(H^k (h^k + H^{2k+1-s})\|\phi\|_2 + H^{k+1} (h^k + H^{2k+1-s})\|\phi\|_2) \\
\leq C(h^k H^{k+1} + H^{3k+2-s})\|u - uh\|_0,
|b(u - uh, uH - u, \phi)| \leq |b(u - uh, uH - u, \phi)| + |b(u - uh, uH - u, \phi)| \\
\leq |uH - u|_1 |uH - u|_1 |\phi - \phih|_1 + |uH - u|_1 \|uH - u\|_0 \|\phi\|_2 \\
\leq C(h^k H^{k+1} + H^{3k+2-s})\|u - uh\|_0.

To estimate the last term \(|b(u^h - u, u^h - u, \phi^h)|\), we should have the \(L^2\)-error estimate of the method after Step 2. We state it here which can be found in [7,8],
\[\|u - u^h\|_0 \leq C(h^{k+1} + H^{2k+1}).\]

Using the above inequality we get
\[|b(u^h - u, u^h - u, \phi^h)| \leq |b(u^h - u, u^h - u, \phi)| + |b(u^h - u, u^h - u, \phi)| \\
\leq |u^h - u|_1 |u^h - u|_1 |\phi - \phi^h|_1 + |u^h - u|_1 \|u^h - u\|_0 \|\phi\|_2 \\
\leq C((h^k + H^{2k+1-s})^2 \|\phi\|_2 + (h^{k+1} + H^{2k+1})(h^k + H^{2k+1-s})\|\phi\|_2).

Putting all estimates of the right-hand side of (15) together, we have
\[\|u - uh\|_0 \leq C(h^{k+1} + h^k H^{k+1} + H^{3k+2-s}).\]

4. Numerical example

In this section, we present a simple numerical results to demonstrate efficiency of our proposed schemes. To make it simple, we assume the viscosity of the fluid \(\nu = 1\). With the \(\Omega = (0, 1) \times (0, 1) \in \mathbb{R}^2\), the right-hand side term \(f\) is chosen such that the exact solution \(u = (u_1, u_2)\) and \(p\) of (1) is given by
\[u_1 = \sin^2 3\pi x \sin 6\pi y, \quad u_2 = -\sin^2 3\pi y \sin 6\pi x\]
\[p = x^2 - y^2\]

The domain \(\Omega\) is divided into families \(\mathcal{T}_H\) and \(\mathcal{T}_h\) of quadrilaterals, and \(V_H, V_h \subset H^0(\Omega)\) are linear spaces of piecewise continuous bilinear functional defined on \(\mathcal{T}_H, \mathcal{T}_h\), respectively. In such a way, to obtain the asymptotically optimal accuracy, it suffices to take \(H = \mathcal{O}(h^{1/2})\) for both \(L^2(\Omega)\) and \(H^1(\Omega)\) norms (for \(H^1(\Omega)\) norm, it even suffices to take \(H = \mathcal{O}(h^{1/4-s})\)).

On the coarse grid level, we solve the nonlinear problem by the Newton iteration. Because of its small size, this nonlinear system takes very little time to solve compared with the larger linear systems. The fine grid-linearized equations are solved by the generalized minimum residual method. We take \(h = H^3\) with \(H = \frac{1}{4}\). The numerical results are shown in Table 1.
Table 1

<table>
<thead>
<tr>
<th></th>
<th>$L^2$-norm</th>
<th>$H^1$-norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u - u_H$</td>
<td>0.20253219780145</td>
<td>1.56882205667490</td>
</tr>
<tr>
<td>$u - u_h^*$</td>
<td>0.00544348997331</td>
<td>0.11952243058943</td>
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<tr>
<td>$u - u_h^*$</td>
<td>0.00253641648520</td>
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<tr>
<td>$u - u_h$</td>
<td>0.00479364303574</td>
<td>0.11275305679031</td>
</tr>
</tbody>
</table>

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References