Journal of Number Theory 129 (2009) 2369–2386

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Evaluations of multiple Dirichlet L-values via symmetric functions

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ARTICLE INFO

Article history:
Received 9 July 2008
Revised 3 April 2009
Available online 3 July 2009
Communicated by David Goss

MSC:
primary 11M41
secondary 11B68

Keywords:
Multiple L-values
Multiple q-L-values
Central limit values
Dirichlet L-functions
Bernoulli numbers
Euler numbers
Symmetric functions

ABSTRACT

We explicitly evaluate a special type of multiple Dirichlet L-values at positive integers in two different ways: One approach involves using of symmetric functions, while the other involves using of a generating function of the values. Equating these two expressions, we derive several summation formulae involving the Bernoulli and Euler numbers. Moreover, values at non-positive integers, called central limit values, are also studied.

1. Introduction

Let $d \in \mathbb{N}$, $s_1, \dots, s_d \in \mathbb{C}$ and $f_1, \dots, f_d$ be certain arithmetic functions such as Dirichlet characters. In this paper, for each $\omega \in \{\bullet, \star\}$, we study the multiple L-function

$$L_d^{\omega}(s_1, \dots, s_d; f_1, \dots, f_d) := \sum_{(m_1, \dots, m_d) \in I_d^{\omega}} \frac{f_1(m_1) \cdots f_d(m_d)}{m_1^{s_1} \cdots m_d^{s_d}},$$

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2009.04.011

where $I_d^{\omega} := \lim_{M\to+\infty} I_d^{\omega}(M)$ with

$$I_d^{\omega}(M) := \begin{cases} \{(m_1,\dots,m_d)\in\mathbb{N}^d \mid m_1 < \cdots < m_d \leqslant M\} & (\omega=\bullet), \\ \{(m_1,\dots,m_d)\in\mathbb{N}^d \mid m_1 \leqslant \cdots \leqslant m_d \leqslant M\} & (\omega=\star). \end{cases}$$

In particular, when $f_j = \mathbf{1}$ for all j where $\mathbf{1}$ is the trivial character, we write L_d^{ω} as ζ_d^{ω} and call it the multiple zeta function ($\zeta_d^{\star}$ is often called the multiple zeta-star function or the non-strict multiple zeta function). Moreover, when $d=1$, we write $L(s;f) = L_1^{\omega}(s;f)$, whence $L(s;\mathbf{1}) = \zeta(s)$ where $\zeta(s)$ is the Riemann zeta function.

Let $\{x\}^d$ be the d-times copy of the variable x. Though it is, in general, difficult to evaluate the value $\zeta_d^{\omega}(k_1,\dots,k_d)$ for given positive integers $k_1,\dots,k_d$, there are many explicit results concerning the values $\zeta_d^{\omega}(\{2k\}^d)$ (see [6,7,17,23,29,31] for the case $\omega=\bullet$ and [22,25,26] for $\omega=\star$). Let χ be a Dirichlet character and κ an integer with the same parity as that of χ. The purpose of the present paper is, as a generalization and unification of the above results, to obtain explicit descriptions of the values $L_d^{\omega}(\{\kappa\}^d;\{\chi\}^d)$ from a symmetric functions' viewpoint. That is, we evaluate the values $L_d^{\bullet}(\{\kappa\}^d;\{\chi\}^d)$ and $L_d^{\star}(\{\kappa\}^d;\{\chi\}^d)$ as specializations of the elementary and complete symmetric function, respectively.

When κ is positive, we do this in the following two distinct ways: In the first method, making use of the relations among the above symmetric functions and the power-sum symmetric function, we express the values $L_d^{\omega}(\{\kappa\}^d;\{\chi\}^d)$ in terms of a sum over partitions of d (Theorem 2.1). In the second method, we consider a generating function of $L_d^{\omega}(\{\kappa\}^d;\{\chi\}^d)$. From the product expressions of the generating function, we show that $L_d^{\omega}(\{\kappa\}^d;\{\chi\}^d)$ can be expressed in terms of $A_{\kappa d}^{\omega}(\kappa;\chi)$, which is defined by a finite sum involving the multinomial coefficients and roots of unity (Theorem 3.4). As an application, in Section 4, equating the two expressions, we give several summation formulae involving the Bernoulli numbers B_n and the Euler numbers E_n. It should be noted that some of these (formulae (4.4), (4.5) and (4.9)) can be regarded as solutions of a kind of "inverse problem", as previously studied in [15] (see also [12]), for the sequences B_{2k} and E_{2k}.

Moreover, we also study a special value, called a *central limit value* (see [2,3]), of the function $L_d^{\omega}(s_1,\dots,s_d;\{\chi\}^d)$ at $s_1=\cdots=s_d=-\kappa$ where κ is a non-negative integer. Note that such a value can be defined because $L_d^{\omega}(s_1,\dots,s_d;\{\chi\}^d)$ admits a meromorphic continuation to the whole space $\mathbb{C}^d$. We then obtain formulae that extend the well-known expressions concerning the value $L(-\kappa,\chi)$ of the usual Dirichlet L-function (Theorem 5.1).

In the final section, we discuss some generalizations of L_d^{ω}, and also give questions for the future study.

Note that, for completeness and the reader's convenience, we restate or briefly prove the corresponding results for $\chi=\mathbf{1}$.

Throughout the present paper, we use the following notation for Dirichlet characters: Let χ be a Dirichlet character modulo N. Denote by χ' the primitive character that induces χ, $N(\chi)$ the conductor, $e(\chi)\in\{0,1\}$ the parity, i.e., $\chi(-1)=(-1)^{e(\chi)}$ (we call χ even if $e(\chi)=0$ and odd otherwise), $\tau(\chi) := \sum_{a=1}^{N}\chi(a)e^{\frac{2\pi ia}{N}}$ the Gauss sum of χ and $B_{n,\chi}$ the generalized Bernoulli number associated with χ defined by $\sum_{a=1}^{N}\frac{\chi(a)te^{at}}{e^{Nt}-1} = \sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!}$. Note that $B_{n,\mathbf{1}}=B_n$ and $B_{n,\chi_{-4}} = -\frac{1}{2}nE_{n-1}$ where χ_{-4} is the primitive Dirichlet character modulo 4. Further, for simplification, we always write $\kappa=\kappa(k,\chi) := 2k+e(\chi)$ with $k\in\mathbb{Z}$.

2. Multiple L-values via the symmetric functions

2.1. Symmetric functions

Let $\lambda=(\lambda_1,\lambda_2,\dots)$ be a partition. We write $\lambda\vdash d$ if λ is a partition of d, that is, $|\lambda| := \sum_{j\geqslant1}\lambda_j = d$ and put $\ell(\lambda) := \max\{j \mid \lambda_j\neq0\}$ (the length of λ) and $m_i(\lambda) := \#\{j \mid \lambda_j = i\}$ (the multiplicity of i in λ) as usual. We denote by λ_{o} (resp. λ_{e}) the partition whose parts consist of all the odd (resp. even) ones of λ and set $\varepsilon_\lambda := (-1)^{|\lambda|-\ell(\lambda)}$ and $z_\lambda := \prod_{j\geqslant1} j^{m_j(\lambda)}m_j(\lambda)!$.

For a partition $\lambda=(\lambda_1,\lambda_2,\ldots)$, the elementary symmetric function e_λ, the complete symmetric function h_λ and the power-sum symmetric function p_λ in $\boldsymbol{x}=(x_1,x_2,\ldots)$ are respectively defined by $e_\lambda(\boldsymbol{x}):=\prod_{j\geqslant 1}e_{\lambda_j}(\boldsymbol{x})$, $h_\lambda(\boldsymbol{x}):=\prod_{j\geqslant 1}h_{\lambda_j}(\boldsymbol{x})$ and $p_\lambda(\boldsymbol{x}):=\prod_{j\geqslant 1}p_{\lambda_j}(\boldsymbol{x})$ where, for $r\in\mathbb{N}$,

$$e_r:=\sum_{n_1<\cdots<n_r}x_{n_1}\cdots x_{n_r},\qquad h_r:=\sum_{n_1\leqslant\cdots\leqslant n_r}x_{n_1}\cdots x_{n_r}\quad\text{and}\quad p_r:=\sum_{n=1}^{\infty}x_n^r.$$

Put $v_\lambda^{\bullet}:=e_\lambda$, $v_\lambda^{\star}:=h_\lambda$, $\varepsilon_\lambda^{\bullet}:=\varepsilon_\lambda$, $\varepsilon_\lambda^{\star}:=1$, $\varepsilon_\bullet:=-1$ and $\varepsilon_\star:=1$. Then, for each $\omega\in\{\bullet,\star\}$, we have

$$v_{(d)}^{\omega}=\sum_{\mu\vdash d}\frac{\varepsilon_\mu^{\omega}}{z_\mu}p_\mu. \tag{2.1}$$

This follows from the following identities of the generating function of $v_{(d)}^{\omega}(\boldsymbol{x})$ (for further details, see the proof of (2.14) and (2.14)′ in [19, p. 25]);

$$\sum_{d=0}^{\infty}\varepsilon_\omega^d v_{(d)}^{\omega}t^d=\prod_{n=1}^{\infty}(1-x_nt)^{-\varepsilon_\omega}=\exp\left(\varepsilon_\omega\sum_{n=1}^{\infty}\frac{1}{n}p_nt^n\right)=\sum_{d=0}^{\infty}\varepsilon_\omega^d\left\{\sum_{\mu\vdash d}\frac{\varepsilon_\mu^{\omega}}{z_\mu}p_\mu\right\}t^d. \tag{2.2}$$

2.2. Evaluation formula I

For a partition $\mu=(\mu_1,\mu_2,\ldots)$ and $a,b\in\mathbb{Z}$ satisfying $a\mu_{\ell(\mu)}+b\geqslant 0$, let $\chi_\mu^j:=(\chi^{\mu_j})'$ for $1\leqslant j\leqslant\ell(\mu)$ and set

$$N_\mu(\chi):=\prod_{j=1}^{\ell(\mu)}N\big(\chi_\mu^j\big)^{\mu_j},\qquad e_\mu(\chi):=\sum_{j=1}^{\ell(\mu)}e\big(\chi_\mu^j\big),\qquad \tau_\mu(\chi):=\prod_{j=1}^{\ell(\mu)}\tau\big(\chi_\mu^j\big)$$

and

$$\widetilde{B}_{a\mu+b,\chi}:=\prod_{j=1}^{\ell(\mu)}\frac{B_{a\mu_j+b,\chi_\mu^j}}{(a\mu_j+b)!}.$$

We similarly put $\widetilde{B}_{a\mu+b}:=\prod_{j=1}^{\ell(\mu)}\frac{B_{a\mu_j+b}}{(a\mu_j+b)!}$ and $\widetilde{E}_{a\mu+b}:=\prod_{j=1}^{\ell(\mu)}\frac{E_{a\mu_j+b}}{(a\mu_j+b)!}$. Moreover, set

$$\alpha_\chi(s):=\prod_{\substack{p:\text{prime}\\ p\mid N,\ p\nmid N(\chi')}}\big(1-\chi'(p)p^{-s}\big)\quad\text{and}\quad W_\mu(s;\chi):=\prod_{j=1}^{\ell(\mu)}\alpha_{\chi^{\mu_j}}(s\mu_j).$$

Using these definitions, we obtain the following theorem:

Theorem 2.1. *Let χ be a Dirichlet character modulo N and $\kappa=2k+e(\chi)\geqslant 1$. Then, we have*

$$L_d^{\omega}\big(\{\kappa\}^d;\{\chi\}^d\big)=(-1)^{\frac{\kappa d}{2}}2^{\kappa d}\left\{\sum_{\mu\vdash d}\frac{\varepsilon_\mu^{\omega}}{z_\mu}\frac{(-1)^{\ell(\mu)-e_\mu(\chi)}}{2^{\ell(\mu)}}\frac{W_\mu(\kappa;\chi)\tau_\mu(\chi)}{N_\mu(\chi)^{\kappa}}\widetilde{B}_{\kappa\mu,\overline{\chi}}\right\}\pi^{\kappa d}. \tag{2.3}$$

Proof. We first recall the case $d=1$ (see, e.g., [24]). If χ is primitive, then we have

$$L(\kappa;\chi)=(-1)^{k+1-\frac{e(\chi)}{2}}\frac{\tau(\chi)}{2}\left(\frac{2\pi}{N(\chi)}\right)^{\kappa}\frac{B_{\kappa,\overline{\chi}}}{\kappa!}. \tag{2.4}$$

Otherwise, from the Euler product expression, we have $L(\kappa;\chi)=\alpha_{\chi}(\kappa)L(\kappa;\chi')$. Hence, in this case, the claim (2.3) clearly holds.

Now, let $d\geqslant 2$. Specializing $x_n=\chi(n)n^{-s}$ with $\mathrm{Re}(s)>1$, we have

$$v^{\omega}_{(d)}(\boldsymbol{x})=L^{\omega}_{d}\bigl(\{s\}^d;\{\chi\}^d\bigr)\quad\text{and}\quad p_{(d)}(\boldsymbol{x})=L\bigl(ds;\chi^d\bigr). \tag{2.5}$$

Note that these series also converge for $s=1$ if χ is not principal (see [4]). Then, from expression (2.1), we have

$$L^{\omega}_{d}\bigl(\{s\}^d;\{\chi\}^d\bigr)=\sum_{\mu\vdash d}\frac{\varepsilon^{\omega}_{\mu}}{z_{\mu}}\prod_{j=1}^{\ell(\mu)}L\bigl(\mu_j s;\chi^{\mu_j}\bigr)=\sum_{\mu\vdash d}\frac{\varepsilon^{\omega}_{\mu}}{z_{\mu}}W_{\mu}(s,\chi)\prod_{j=1}^{\ell(\mu)}L\bigl(\mu_j s;\chi^{j}_{\mu}\bigr). \tag{2.6}$$

We here put $s=\kappa=\kappa(k,\chi)\geqslant 1$. Notice that $e(\chi^j_{\mu})\equiv e(\chi)\mu_j \pmod 2$ because $(-1)^{e(\chi^j_{\mu})}=\chi^j_{\mu}(-1)=\chi(-1)^{\mu_j}=(-1)^{e(\chi)\mu_j}$, and hence

$$\kappa\mu_j=\kappa(k,\chi)\mu_j=\begin{cases}\kappa(k\mu_j,\chi^j_{\mu}) & \text{if } e(\chi)=0,\\ \kappa(k\mu_j+\frac{\mu_j}{2},\chi^j_{\mu}) & \text{if } e(\chi)=1 \text{ and } \mu_j \text{ is even},\\ \kappa(k\mu_j+\frac{\mu_j-1}{2},\chi^j_{\mu}) & \text{if } e(\chi)=1 \text{ and } \mu_j \text{ is odd}.\end{cases} \tag{2.7}$$

Therefore, from Eqs. (2.4) and (2.7), we have

$$L\bigl(\kappa\mu_j;\chi^j_{\mu}\bigr)=(-1)^{\frac{\kappa\mu_j}{2}+1-e(\chi^j_{\mu})}\frac{\tau(\chi^j_{\mu})}{2}\left(\frac{2\pi}{N(\chi^j_{\mu})}\right)^{\kappa\mu_j}\frac{B_{\kappa\mu_j,\overline{\chi^j_{\mu}}}}{(\kappa\mu_j)!}. \tag{2.8}$$

Substituting expression (2.8) into Eq. (2.6), we obtain the desired formula. □

2.3. Examples

For a partition μ, we denote by $\prod^{\mu,\mathrm{o}}_{\mu_j\geqslant a}$ (resp. $\prod^{\mu,\mathrm{e}}_{\mu_j\geqslant a}$) the product over all $1\leqslant j\leqslant \ell(\mu)$ such that μ_j is odd (resp. even) and $\mu_j\geqslant a$. In particular, we omit the condition "$\mu_j\geqslant a$" if $a=1$.

2.3.1. Principal Dirichlet characters

Let $\chi^{(N)}_1$ be the principal Dirichlet character modulo N. Note that $\kappa=\kappa(k,\chi^{(N)}_1)=2k$.

Corollary 2.2. *It holds that*

$$L^{\omega}_{d}\bigl(\{2k\}^d;\bigl\{\chi^{(N)}_1\bigr\}^d\bigr)=(-1)^{kd}2^{2kd}\left\{\sum_{\mu\vdash d}\frac{\varepsilon^{\omega}_{\mu}}{z_{\mu}}\frac{(-1)^{\ell(\mu)}}{2^{\ell(\mu)}}\left(\prod_{\substack{p:\text{prime}\\ p|N}}\prod_{j=1}^{\ell(\mu)}\bigl(1-p^{-2k\mu_j}\bigr)\right)\widetilde{B}_{2k\mu}\right\}\pi^{2kd}. \tag{2.9}$$

In particular, we have $L^{\omega}_{d}(\{2k\}^d;\{\chi^{(N)}_1\}^d)\in\mathbb{Q}\pi^{2kd}$.

Proof. Since $(\chi_1^{(N)})_\mu^j = \mathbf{1}$ for all $1 \leqslant j \leqslant \ell(\mu)$, we have $N_\mu(\chi_1^{(N)}) = 1$, $e_\mu(\chi_1^{(N)}) = 0$, $\tau_\mu(\chi_1^{(N)}) = 1$ and $\widetilde{B}_{2k\mu,\chi_1^{(N)}} = \widetilde{B}_{2k\mu}$. Hence the claim follows immediately from expression (2.3). $\square$

Example 2.3 *(The case $\chi = \chi_1^{(1)} = \mathbf{1}$).* For $\kappa = 2k$ with $k \in \mathbb{N}$, we have

$$\zeta_d^\omega\big(\{2k\}^d\big) = L_d^\omega\big(\{2k\}^d; \{\mathbf{1}\}^d\big) = (-1)^{kd} 2^{2kd} \left\{ \sum_{\mu \vdash d} \frac{\varepsilon_\mu^\omega}{z_\mu} \frac{(-1)^{\ell(\mu)}}{2^{\ell(\mu)}} \widetilde{B}_{2k\mu} \right\} \pi^{2kd}. \tag{2.10}$$

The expression (2.10) for $\omega = \star$ gives the result obtained in [25].

Example 2.4 *(The case $\chi = \chi_1^{(2)}$).* For $\kappa = 2k$ with $k \in \mathbb{N}$, we have

$$L_d^\omega\big(\{2k\}^d; \big\{\chi_1^{(2)}\big\}^d\big) = (-1)^{kd} \left\{ \sum_{\mu \vdash d} \frac{\varepsilon_\mu^\omega}{z_\mu} \frac{(-1)^{\ell(\mu)} \prod_{j=1}^{\ell(\mu)} (2^{2k\mu_j} - 1)}{2^{\ell(\mu)}} \widetilde{B}_{2k\mu} \right\} \pi^{2kd}. \tag{2.11}$$

2.3.2. Primitive real Dirichlet characters

For a fundamental discriminant D, let χ_D be the associated primitive real Dirichlet character modulo $|D|$ defined by the Kronecker symbol $(\frac{D}{\cdot})$ (see [30]). Note that $e(\chi_D) = e(D) := \frac{1-\operatorname{sgn} D}{2}$ where $\operatorname{sgn} D$ is the signature of D, whence $\kappa = \kappa(k, \chi_D) = 2k + e(D)$.

Corollary 2.5. *It holds that*

$$\begin{aligned} &L_d^\omega\big(\{\kappa\}^d; \{\chi_D\}^d\big) \\ &\quad = (-1)^{\frac{\kappa d}{2}} 2^{\kappa d} \left\{ \sum_{\mu \vdash d} \frac{\varepsilon_\mu^\omega}{z_\mu} \frac{(-1)^{\ell(\mu) - \frac{1}{2} e(D) \ell(\mu_{\mathrm{o}})}}{2^{\ell(\mu)}} \frac{\prod_{\substack{p:\text{prime} \\ p \| D|}} \prod_j^{\mu,\mathrm{e}} (1 - p^{-\kappa \mu_j})}{|D|^{\kappa |\mu_{\mathrm{o}}| - \frac{1}{2} \ell(\mu_{\mathrm{o}})}} \widetilde{B}_{\kappa \mu_{\mathrm{o}}, \chi_D} \widetilde{B}_{\kappa \mu_{\mathrm{e}}} \right\} \pi^{\kappa d}. \end{aligned} \tag{2.12}$$

Proof. Since χ_D is real, we have $(\chi_D)_\mu^j = \mathbf{1}$ if μ_j is even and χ_D otherwise. One can therefore obtain the formula by using the well-known property $\tau(\chi_D) = i^{e(D)} |D|^{\frac{1}{2}}$. $\square$

Example 2.6 *(The case $\chi = \chi_{-4}$).* For $\kappa = 2k+1$ with $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$L_d^\omega\big(\{\kappa\}^d; \{\chi_{-4}\}^d\big) = (-1)^{\frac{\kappa d}{2}} \left\{ \sum_{\mu \vdash d} \frac{\varepsilon_\mu^\omega}{z_\mu} \frac{(-1)^{\ell(\mu) + \frac{1}{2}\ell(\mu_{\mathrm{o}})} \prod_j^{\mu,\mathrm{e}} (2^{\kappa \mu_j} - 1)}{2^{\ell(\mu) + \kappa |\mu_{\mathrm{o}}|}} \widetilde{E}_{\kappa \mu_{\mathrm{o}} - 1} \widetilde{B}_{\kappa \mu_{\mathrm{e}}} \right\} \pi^{\kappa d}. \tag{2.13}$$

Remark 2.7. For a non-principal Dirichlet character χ, it is known that $L(1,\chi) \neq 0$ (see, e.g., [30]). However, in general, it does not seem to be the case that $L_d^\omega(\{1\}^d; \{\chi\}^d) \neq 0$ for $d \geqslant 2$. In fact, when $\omega = \bullet$, we can obtain the following example of a pair of d and χ such that $L_d^\bullet(\{1\}^d; \{\chi\}^d) = 0$ (although we were not able to derive a similar result for the case $\omega = \star$):

$$\begin{aligned}
L_2^{\bullet}(\{1\}^2;\{\chi_{-8}\}^2) &= \frac{1}{1\cdot 3}+\frac{-1}{1\cdot 5}+\frac{-1}{1\cdot 7}+\frac{1}{1\cdot 9}+\frac{1}{1\cdot 11}+\frac{-1}{1\cdot 13}+\frac{-1}{1\cdot 15}+\frac{1}{1\cdot 17}+\cdots \\
&\quad +\frac{-1}{3\cdot 5}+\frac{-1}{3\cdot 7}+\frac{1}{3\cdot 9}+\frac{1}{3\cdot 11}+\frac{-1}{3\cdot 13}+\frac{-1}{3\cdot 15}+\frac{1}{3\cdot 17}+\cdots \\
&\quad +\frac{1}{5\cdot 7}+\frac{-1}{5\cdot 9}+\frac{-1}{5\cdot 11}+\frac{1}{5\cdot 13}+\frac{1}{5\cdot 15}+\frac{-1}{5\cdot 17}+\cdots \\
&\quad +\cdots \\
&= -\left\{\frac{3}{4}\frac{B_2}{2!}-\frac{1}{16}\left(\frac{B_{1,\chi_{-8}}}{1!}\right)^2\right\}\pi^2 = 0.
\end{aligned}$$

In the last equality, we have used the facts $B_2 = \frac{1}{6}$ and $B_{1,\chi_{-8}} = -1$.

2.4. *q-Analogues of the multiple L-functions*

Let $0 < q < 1$ and $[n]_q := \frac{1-q^n}{1-q}$. Defined a q-analogue $L_{q,d}^{\omega}$ of L_d^{ω} by

$$L_{q,d}^{\omega}(s_1,\ldots,s_d;f_1,\ldots,f_d) := \sum_{(m_1,\ldots,m_d)\in I_d^{\omega}} \frac{f_1(m_1)q^{m_1(s_1-1)}\cdots f_d(m_d)q^{m_d(s_d-1)}}{[m_1]_q^{s_1}\cdots[m_d]_q^{s_d}}.$$

Similarly to the case for L_d^{ω} (or $q=1$), we write $L_{q,d}^{\omega}$ as $\zeta_{q,d}^{\omega}$ if $f_j = \mathbf{1}$ for all j and $L_q(s;f) = L_{q,1}^{\omega}(s;f)$. The function $L_{q,d}^{\omega}$ is a natural extension of the q-analogue of the Riemann zeta function $\zeta_q(s) := L_q(s;\mathbf{1})$ studied in [16] (see also [18]). Many relations among the values $\zeta_{q,d}^{\omega}$ at positive integers have been studied; see, e.g., [9,10,28,33] for the case $\omega = \bullet$ and [27] for $\omega = \star$. Specializing $x_n = \chi(n)q^{n(s-1)}[n]_q^{-s}$ with $\mathrm{Re}(s) > 1$ in Eq. (2.1) and using

$$v_{(d)}^{\omega}(\boldsymbol{x}) = L_{q,d}^{\omega}(\{s\}^d;\{\chi\}^d) \quad \text{and} \quad p_{(r)}(\boldsymbol{x}) = \sum_{l=0}^{d-1}\binom{d-1}{l}(1-q)^l L_q(ds-l;\chi^d), \tag{2.14}$$

we obtain the following expression which gives a q-analogue of expression (2.6).

Proposition 2.8. *It holds that*

$$L_{q,d}^{\omega}(\{s\}^d;\{\chi\}^d) = \sum_{\mu\vdash d}\frac{\varepsilon_\mu^{\omega}}{z_\mu}\prod_{j=1}^{\ell(\mu)}\sum_{l_j=0}^{\mu_j-1}\binom{\mu_j-1}{l_j}(1-q)^{l_j}L_q(\mu_j s - l_j;\chi^{\mu_j}). \tag{2.15}$$

It may be noted that a recursive expression of $\zeta_{q,d}^{\bullet}(\{s\}^d)$ was obtained in [9, Theorem 1].

Remark 2.9. Let $(P,Q) = (L^{\bullet},L^{\star})$ or $(L_q^{\bullet},L_q^{\star})$. Then, for $d \geqslant 1$, it is straightforward to obtain from Eq. (2.2) that $\sum_{\substack{a,b\geqslant 0\\ a+b=d}}(-1)^a P_a(\{s\}^a;\{\chi\}^a)Q_b(\{s\}^b;\{\chi\}^b) = 0$ (here we understand that $(L^{\omega})_c = L_c^{\omega}$ and $(L_q^{\omega})_c = L_{q,c}^{\omega}$). Further, one can obtain more explicit relation between $P_a(\{s\}^a;\{\chi\}^a)$ and $Q_b(\{s\}^b;\{\chi\}^b)$. In fact, for any $(P,Q)\in\{(L^{\bullet},L^{\star}),(L^{\star},L^{\bullet}),(L_q^{\bullet},L_q^{\star}),(L_q^{\star},L_q^{\bullet})\}$, it holds that

$$P_d(\{s\}^d;\{\chi\}^d) = \sum_{\mu\vdash d}\varepsilon_\mu u_\mu\prod_{j=1}^{\ell(\mu)}Q_{\mu_j}(\{s\}^{\mu_j};\{\chi\}^{\mu_j}). \tag{2.16}$$

This follows from the relations $e_{(d)} = \sum_{\mu\vdash d} \varepsilon_\mu u_\mu h_\mu$ and $h_{(d)} = \sum_{\mu\vdash d} \varepsilon_\mu u_\mu e_\mu$ (see [19, Example 20, p. 33]) together with relations (2.5) and (2.14). Here, we put $u_\mu := \binom{\ell(\mu)}{m_1(\mu), m_2(\mu), \ldots}$ where $\binom{n}{a,b,\ldots}$ $(n = a + b + \cdots)$ denotes the multinomial coefficient.

3. Multiple *L*-values via generating functions

3.1. Product expressions of generating functions

The second method to evaluate the value $L_d^\omega(\{\kappa\}^d; \{\chi\}^d)$ is well known for the case $\chi = \mathbf{1}$. Namely, we start from the following identity which is obtained by specializing $x_n = \chi(n)n^{-\kappa}$ and replacing t by t^κ in the generating functions (2.2):

$$\sum_{d=0}^{\infty} \varepsilon_\omega^d L_d^\omega\big(\{\kappa\}^d; \{\chi\}^d\big) t^{\kappa d} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(n)t^\kappa}{n^\kappa}\right)^{-\varepsilon_\omega}. \tag{3.1}$$

The right-hand side of Eq. (3.1) can then be written as follows.

Proposition 3.1. *It holds that*

$$\prod_{n=1}^{\infty}\left(1 - \frac{\mathbf{1}(n)t^{2k}}{n^{2k}}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{t^{2k}}{n^{2k}}\right) = \prod_{l=1}^{k} \frac{\sin(\pi\zeta_{2k}^l t)}{\pi\zeta_{2k}^l t}, \tag{3.2}$$

$$\prod_{n=1}^{\infty}\left(1 - \frac{\chi_1^{(2)}(n)t^{2k}}{n^{2k}}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{t^{2k}}{(2n-1)^{2k}}\right) = \prod_{l=1}^{k} \cos\left(\frac{\pi\zeta_{2k}^l t}{2}\right) \tag{3.3}$$

for $k \in \mathbb{N}$ and, for a Dirichlet character χ modulo $N \geqslant 3$,

$$\prod_{n=1}^{\infty}\left(1 - \frac{\chi(n)t^\kappa}{n^\kappa}\right) = \big(N^{-1}2^{N-1}\big)^{\frac{1}{2}\kappa} \prod_{j=1}^{\bar{N}}\prod_{l=1}^{\kappa} \sin\frac{\pi(j - \chi(j)^{\frac{1}{\kappa}}\zeta_\kappa^l t)}{N}. \tag{3.4}$$

Here, $\zeta_m := e^{\frac{2\pi i}{m}}$ for $m \in \mathbb{N}$ and $\bar{N} := \lfloor\frac{N-1}{2}\rfloor$ with $\lfloor x\rfloor$ being the largest integer not exceeding x. In Eq. (3.4), we take the argument of $\chi(j)$ satisfying $\chi(j) \neq 0$ to be $0 \leqslant \arg\chi(j) < 2\pi$.

We need the following lemma (see [11, Example 11, Chapter VI, p. 115]).

Lemma 3.2. *For $a_i, b_i \in \mathbb{C}$ $(1 \leqslant i \leqslant l)$ satisfying $\sum_{i=1}^l a_i = \sum_{i=1}^l b_i$, we have*

$$\prod_{n=1}^{\infty} \frac{(n+a_1)\cdots(n+a_l)}{(n+b_1)\cdots(n+b_l)} = \frac{\Gamma(1+b_1)\cdots\Gamma(1+b_l)}{\Gamma(1+a_1)\cdots\Gamma(1+a_l)}. \tag{3.5}$$

Proof of Proposition 3.1. Let $N \geqslant 3$. By the periodicity of χ, the left-hand side of Eq. (3.4) equals

$$\begin{aligned}\prod_{n=1}^{\infty}\prod_{j=1}^{N-1}\left(1-\frac{\chi(nN-(N-j))t^{\kappa}}{(nN-(N-j))^{\kappa}}\right) &= \prod_{n=1}^{\infty}\prod_{j=1}^{N-1}\frac{(n-\frac{N-j}{N})^{\kappa}-\chi(j)(\frac{t}{N})^{\kappa}}{(n-\frac{N-j}{N})^{\kappa}}\\ &= \prod_{n=1}^{\infty}\frac{\prod_{j=1}^{N-1}\prod_{l=1}^{\kappa}(n-\frac{N-j+\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t}{N})}{\prod_{j=1}^{N-1}(n-\frac{N-j}{N})^{\kappa}}\\ &= \left(\prod_{j=1}^{N-1}\Gamma\left(\frac{j}{N}\right)\right)^{\kappa}\prod_{j=1}^{N-1}\prod_{l=1}^{\kappa}\Gamma\left(\frac{j-\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t}{N}\right)^{-1}. \qquad (3.6)\end{aligned}$$

In the last equality, we have used formula (3.5) (it is easy to check the condition in Lemma 3.2 from $\sum_{j=1}^{N-1}\chi(j)=0$ when $\kappa=1$ and $\sum_{l=1}^{\kappa}\zeta_{\kappa}^{l}=0$ otherwise). Note that $\prod_{j=1}^{N-1}\Gamma(\frac{j}{N})=(N^{-1}(2\pi)^{N-1})^{\frac{1}{2}}$ by the Gauss–Legendre formula $\Gamma(Na)(2\pi)^{\frac{N-1}{2}}=N^{Na-\frac{1}{2}}\prod_{j=0}^{N-1}\Gamma(a+\frac{j}{N})$ with $a=1$. Further, the double product in the final right-hand side of (3.6) can be written as

$$\prod_{l=1}^{\kappa}\Gamma\left(\frac{\frac{N}{2}-\chi(\frac{N}{2})^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t}{N}\right)^{-\delta_N}\prod_{j=1}^{\bar{N}}\Gamma\left(\frac{j-\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t}{N}\right)^{-1}\Gamma\left(\frac{N-j-\chi(N-j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t}{N}\right)^{-1},$$

where $\delta_N:=1$ if N is even and 0 otherwise. Notice also that $\chi(\frac{N}{2})=0$ for even $N\geqslant 3$ and $\chi(N-j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}=-\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l-k}$ if $0\leqslant\arg\chi(j)<\pi$ and $-\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l+k}$ otherwise. Therefore, using the formula $\Gamma(\frac{1}{2})=\sqrt{\pi}$ and replacing $l\pm k$ in $\zeta_{\kappa}^{l\pm k}$ by l, we see that the above expression is equal to

$$\begin{aligned}&\pi^{-\frac{\delta_N}{2}}\prod_{l=1}^{\kappa}\prod_{j=1}^{\bar{N}}\Gamma\left(\frac{j-\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t}{N}\right)^{-1}\Gamma\left(1-\frac{j-\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t}{N}\right)^{-1}\\ &=\pi^{-\frac{(N-1)}{2}\kappa}\prod_{l=1}^{\kappa}\prod_{j=1}^{\bar{N}}\sin\frac{\pi(j-\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t)}{N}.\end{aligned}$$

Here, we have used the reflection formula $\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}$ and the equality $-\frac{\delta_N}{2}\kappa-\bar{N}\kappa=-\frac{(N-1)}{2}\kappa$. Hence, substituting the above expression into (3.6), we obtain the desired formula.

Eqs. (3.2) and (3.3) are easily obtained from the infinite product expressions $\frac{\sin(\pi t)}{\pi t}=\prod_{n=1}^{\infty}(1-\frac{t^2}{n^2})$ and $\cos(\frac{\pi t}{2})=\prod_{n=1}^{\infty}(1-\frac{t^2}{(2n-1)^2})$, respectively (see [23] for (3.2)). This completes the proof. □

3.2. Evaluation formula II

Let $a\notin\pi\mathbb{Z}$. For $\omega\in\{\bullet,\star\}$, we define the sequences $\{T_n^{\omega}(a)\}_{n\geqslant 0}$ by the expansions

$$\sin(a+t)=\sum_{n=0}^{\infty}T_n^{\bullet}(a)\frac{t^n}{n!}\quad\text{and}\quad\operatorname{cosec}(a+t)=\sum_{n=0}^{\infty}T_n^{\star}(a)\frac{t^n}{n!}. \qquad (3.7)$$

It is clear that $T_n^{\bullet}(a)=(-1)^{\frac{1}{2}n(n-1)}\mathrm{tri}_n(a)$ where $\mathrm{tri}_n(a)=\sin(a)$ if n is even and $\cos(a)$ otherwise. On the other hand, $T_n^{\star}(a)$ can be written as $T_n^{\star}(a)=\operatorname{cosec}(a)\sum_{k=0}^{n}\binom{n}{k}i^kE_kh_{n-k}(\cot(a))$ where $h_l(\alpha)$ is the polynomial in α of degree l defined by $(1+\alpha\tan(t))^{-1}=\sum_{l=0}^{\infty}h_l(\alpha)\frac{t^l}{l!}$ (note that h_l is even if l is and odd otherwise).

For a Dirichlet character χ modulo N, define the sequence $\{A_n^{\omega}(\kappa,\chi)\}_{n\geqslant 0}$ as follows;
(i) The case $N=1$ (i.e., $\chi=\mathbf{1}$ and $\kappa=2k$): For $n\in\mathbb{Z}_{\geqslant 0}$, let $A_{2n+1}^{\omega}(2k,\mathbf{1})\equiv 0$ and

$$A_{2n}^{\omega}(2k,\mathbf{1}):=\begin{cases}\sum_{\substack{n_1,\dots,n_k\geqslant 0\\ n_1+\cdots+n_k=n}}\binom{2n+k}{2n_1+1,\dots,2n_k+1}\zeta_k^{\sum_{l=1}^{k} ln_l} & (\omega=\bullet),\\ \sum_{\substack{n_1,\dots,n_k\geqslant 0\\ n_1+\cdots+n_k=n}}\binom{2n}{2n_1,\dots,2n_k}\left(\prod_{l=1}^{k}(2^{2n_l}-2)B_{2n_l}\right)\zeta_k^{\sum_{l=1}^{k} ln_l} & (\omega=\star).\end{cases}\tag{3.8}$$

(ii) The case $N=2$ (i.e., $\chi=\chi_1^{(2)}$ and $\kappa=2k$): For $n\in\mathbb{Z}_{\geqslant 0}$, let $A_{2n+1}^{\omega}(2k,\chi_1^{(2)})\equiv 0$ and

$$A_{2n}^{\omega}\left(2k,\chi_1^{(2)}\right):=\begin{cases}\sum_{\substack{n_1,\dots,n_k\geqslant 0\\ n_1+\cdots+n_k=n}}\binom{2n}{2n_1,\dots,2n_k}\zeta_k^{\sum_{l=1}^{k} ln_l} & (\omega=\bullet),\\ \sum_{\substack{n_1,\dots,n_k\geqslant 0\\ n_1+\cdots+n_k=n}}\binom{2n}{2n_1,\dots,2n_k}\left(\prod_{l=1}^{k}E_{2n_l}\right)\zeta_k^{\sum_{l=1}^{k} ln_l} & (\omega=\star).\end{cases}\tag{3.9}$$

(iii) The case $N\geqslant 3$: Define

$$A_n^{\omega}(j;\kappa,\chi):=\sum_{\substack{n_1,\dots,n_\kappa\geqslant 0\\ n_1+\cdots+n_\kappa=n}}\binom{n}{n_1,\dots,n_\kappa}\left(\prod_{l=1}^{\kappa}T_{n_l}^{\omega}\left(\frac{\pi j}{N}\right)\right)\zeta_\kappa^{\sum_{l=1}^{\kappa} ln_l}\quad (1\leqslant j\leqslant \bar{N})\tag{3.10}$$

and let

$$A_n^{\omega}(\kappa,\chi):=\sum_{\substack{n_1,\dots,n_{\bar{N}}\geqslant 0\\ n_1+\cdots+n_{\bar{N}}=n}}\binom{n}{n_1,\dots,n_{\bar{N}}}\prod_{j=1}^{\bar{N}}A_{n_j}^{\omega}(j;\kappa,\chi)\chi(j)^{\frac{n_j}{\kappa}}.\tag{3.11}$$

Example 3.3 *(The case $\chi=\chi_{-4}$).* Let $E_n(x)$ be the Euler polynomial defined by $\frac{2e^{tx}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}$. Note that $E_n=2^nE_n(\frac{1}{2})$. Then, since

$$\operatorname{cosec}\left(\frac{\pi}{4}+t\right)=\frac{\sqrt{2}}{\cos(t)+\sin(t)}=\frac{\sqrt{2}(i+1)}{2}\frac{2e^{4it\cdot\frac{3}{4}}}{e^{4it}+1}-\frac{\sqrt{2}(i-1)}{2}\frac{2e^{-4it\cdot\frac{3}{4}}}{e^{-4it}+1},$$

we see that $T_n^{\star}(\frac{\pi}{4})=(-1)^{\frac{1}{2}n(n+1)}2^{\frac{4n+1}{2}}E_n(\frac{3}{4})$. While on the other hand, it is clear that $T_n^{\bullet}(\frac{\pi}{4})=2^{-\frac{1}{2}}(-1)^{\frac{1}{2}n(n-1)}$. Hence, from definitions (3.10) and (3.11), we have

$$\begin{aligned}A_n^{\omega}(\kappa,\chi_{-4})&=A_n^{\omega}(1;\kappa,\chi_{-4})\\&=\begin{cases}2^{-\frac{\kappa}{2}}\sum_{\substack{n_1,\dots,n_\kappa\geqslant 0\\ n_1+\cdots+n_\kappa=n}}\binom{n}{n_1,\dots,n_\kappa}(-1)^{\frac{1}{2}\sum_{l=1}^{\kappa}n_l(n_l-1)}\zeta_\kappa^{\sum_{l=1}^{\kappa} ln_l} & (\omega=\bullet),\\ 2^{\frac{\kappa(2d+1)}{2}}\sum_{\substack{n_1,\dots,n_\kappa\geqslant 0\\ n_1+\cdots+n_\kappa=n}}\binom{n}{n_1,\dots,n_\kappa}\left(\prod_{l=1}^{\kappa}E_{n_l}(\frac{3}{4})\right)(-1)^{\frac{1}{2}\sum_{l=1}^{\kappa}n_l(n_l+1)}\zeta_\kappa^{\sum_{l=1}^{\kappa} ln_l} & (\omega=\star).\end{cases}\end{aligned}\tag{3.12}$$

Put $\widetilde{\varepsilon}_{\omega}:=\frac{\varepsilon_{\omega}+1}{2}$, that is, $\widetilde{\varepsilon}_{\bullet}=0$ and $\widetilde{\varepsilon}_{\star}=1$. We then obtain the following theorem which gives extensions of the formulae for ζ_d^{ω} obtained in [6,23] for the case $\omega=\bullet$ and [22] for $\omega=\star$ (see also [29]).

Theorem 3.4. *Let χ be a Dirichlet character modulo N and $\kappa = 2k + e(\chi) \geqslant 1$. Let*

$$C_d^{\omega}(\kappa,\chi) := \begin{cases} \frac{\varepsilon_{\omega}^{d}(-1)^{k(d-\widetilde{\varepsilon}_{\omega})}}{((2d+1-\widetilde{\varepsilon}_{\omega})k)!} & (N=1,\ \kappa=2k), \\ \frac{\varepsilon_{\omega}^{d}(-1)^{kd}}{2^{2kd}(2kd)!} & (N=2,\ \kappa=2k), \\ \frac{\varepsilon_{\omega}^{d}(-1)^{\kappa d}}{(N^{2d-\varepsilon_{\omega}}2^{(N-1)\varepsilon_{\omega}})^{\frac{\kappa}{2}}(\kappa d)!} & (N \geqslant 3). \end{cases}$$

Then, we have $A_n^{\omega}(\kappa,\chi) = 0$ if $\kappa \nmid n$ and

$$L_d^{\omega}\big(\{\kappa\}^d;\{\chi\}^d\big) = C_d^{\omega}(\kappa,\chi)A_{\kappa d}^{\omega}(\kappa,\chi)\pi^{\kappa d}. \tag{3.13}$$

Proof. Let $N \geqslant 3$. It is straightforward from the definitions of $T_n^{\omega}(a)$ to see that

$$\prod_{j=1}^{\bar{N}}\prod_{l=1}^{\kappa}\left(\sin\frac{\pi(j-\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t)}{N}\right)^{-\varepsilon_{\omega}} = \sum_{n=0}^{\infty}A_n^{\omega}(\kappa,\chi)\frac{(-1)^n}{n!}\left(\frac{\pi t}{N}\right)^n. \tag{3.14}$$

In fact, when $\omega = \bullet$, the left-hand side of Eq. (3.14) is given by

$$\begin{aligned}\prod_{j=1}^{\bar{N}}\sum_{n_1,\dots,n_{\kappa}=0}^{\infty}\prod_{l=1}^{\kappa}T_{n_l}^{\bullet}\left(\frac{\pi j}{N}\right)\frac{1}{n_l!}\left(-\frac{\pi\chi(j)^{\frac{1}{\kappa}}\zeta_{\kappa}^{l}t}{N}\right)^{n_l} &= \prod_{j=1}^{\bar{N}}\sum_{n=0}^{\infty}A_n^{\bullet}(j;\kappa,\chi)\frac{(-1)^n}{n!}\left(\frac{\pi\chi(j)^{\frac{1}{\kappa}}t}{N}\right)^n \\ &= \sum_{n_1,\dots,n_{\bar{N}}=0}\prod_{j=1}^{\bar{N}}A_{n_j}^{\bullet}(j;\kappa,\chi)\frac{(-1)^n}{n!}\left(\frac{\pi\chi(j)^{\frac{1}{\kappa}}t}{N}\right)^{n_j} \\ &= \sum_{n=0}^{\infty}A_n^{\bullet}(\kappa,\chi)\frac{(-1)^n}{n!}\left(\frac{\pi t}{N}\right)^n.\end{aligned}$$

The case when $\omega = \star$ is similar. Therefore, from Eq. (3.4), we have

$$\prod_{n=1}^{\infty}\left(1-\frac{\chi(n)t^{\kappa}}{n^{\kappa}}\right)^{-\varepsilon_{\omega}} = \sum_{n=0}^{\infty}\left\{\big(N^{-1}2^{N-1}\big)^{-\frac{1}{2}\varepsilon_{\omega}\kappa}A_n^{\omega}(\kappa,\chi)\frac{(-1)^n}{n!}\left(\frac{\pi}{N}\right)^n\right\}t^n. \tag{3.15}$$

Hence, comparing the coefficients of $t^{\kappa d}$ in expressions (3.1) and (3.15), we see that $A_n^{\omega}(\kappa,\chi) = 0$ if $\kappa \nmid n$ and obtain formula (3.13) for $N \geqslant 3$.

One can similarly obtain formulae (3.13) for the other cases, that is, $N = 1$ or $N = 2$, by using the expansions

$$\begin{aligned}\sin(t) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}t^{2n+1}, & \operatorname{cosec}(t) &= \sum_{n=0}^{\infty}\frac{(-1)^{n-1}(2^{2n}-2)B_{2n}}{(2n)!}t^{2n-1}, \\ \cos(t) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}t^{2n}, & \sec(t) &= \sum_{n=0}^{\infty}\frac{(-1)^nE_{2n}}{(2n)!}t^{2n}.\end{aligned} \tag{3.16}$$

Actually, for example let $N = 1$. Then, from (3.2), we have

$$\sum_{d=0}^{\infty} \varepsilon_\omega^d L_d^\omega(\{2k\}^d;\{\mathbf{1}\}^d)t^{2kd} = \prod_{n=1}^{\infty}\left(1-\frac{\mathbf{1}(n)t^{2k}}{n^{2k}}\right)^{-\varepsilon_\omega} = \prod_{l=1}^{k}\left(\frac{\sin(\pi\zeta_{2k}^l t)}{\pi\zeta_{2k}^l t}\right)^{-\varepsilon_\omega}$$
$$= \begin{cases} \sum_{n=0}^{\infty} A_{2n}^{\bullet}(2k,\mathbf{1})\frac{(-1)^n\pi^{2n}}{(2n+k)!}t^{2n} & (\omega=\bullet), \\ \sum_{n=0}^{\infty} A_{2n}^{\star}(2k,\mathbf{1})\frac{(-1)^{n-k}\pi^{2n}}{(2n)!}t^{2n} & (\omega=\star). \end{cases} \quad (3.17)$$

In the last equality with $\omega=\bullet$ (resp. $\omega=\star$), we have used the expansion of $\sin(t)$ (resp. $\operatorname{cosec}(t)$) in (3.16). Now comparing the coefficients of t^{2kd} on both sides of (3.17), we have

$$L_d^\omega(\{2k\}^d;\{\mathbf{1}\}^d)\left(=\zeta_d^\omega(\{2k\}^d)\right) = \begin{cases} \frac{\varepsilon_\bullet^d(-1)^{kd}}{(2kd+k)!}A_{2kd}^{\bullet}(2k,\mathbf{1})\pi^{2kd} & (\omega=\bullet), \\ \frac{\varepsilon_\star^d(-1)^{kd-k}}{(2kd)!}A_{2kd}^{\star}(2k,\mathbf{1})\pi^{2kd} & (\omega=\star), \end{cases}$$

whence the desired formula follows. This completes the proof of the theorem. □

Example 3.5 *(The case $\chi=\mathbf{1}$).* The following results are well known (see [6,22]):

$\kappa=2k$	2	4	6
$\zeta_d^\bullet(\{\kappa\}^d)$	$\frac{1}{(2d+1)!}\pi^{2d}$	$\frac{2^{2d+1}}{(4d+2)!}\pi^{4d}$	$\frac{3\cdot 2^{6d+1}}{(6d+3)!}\pi^{6d}$
$\zeta_d^\star(\{\kappa\}^d)$	$\frac{(-1)^{d-1}(2^{2d}-2)B_{2d}}{(2d)!}\pi^{2d}$	–	–

Notice that, from expression (3.13), we have $\zeta_d^\star(\{4\}^d)=((2^{4d}+4)S_{2d}(-1)-4S_{2d}(-4))\pi^{4d}/(4d)!$ where $S_k(t):=\sum_{n=0}^{k}\binom{2k}{2n}t^n B_{2n}B_{2k-2n}$. However, we do not know whether the expression can be reduced to a "simpler" (or "closed") form, because it would appear to be difficult to simplify the sum $S_k(t)$ for a general $t\in\mathbb{C}$ (see [12] for the comment on $S_k(-1)$). We remark that, on the other hand, it is well known that $S_k(1)=-(2k-1)B_{2k}$ for $k\neq 1$ and $S_1(1)=\frac{1}{3}$.

Example 3.6 *(The case $\chi=\chi_1^{(2)}$).* We obtain the following results:

$\kappa=2k$	2	4	6
$L_d^\bullet(\{\kappa\}^d;\{\chi_1^{(2)}\}^d)$	$\frac{1}{2^{2d}(2d)!}\pi^{2d}$	$\frac{1}{2^{2d}(4d)!}\pi^{4d}$	$\frac{3}{4(6d)!}\pi^{6d}$
$L_d^\star(\{\kappa\}^d;\{\chi_1^{(2)}\}^d)$	$\frac{(-1)^d E_{2d}}{2^{2d}(2d)!}\pi^{2d}$	–	–

From expression (3.13), it can be expressed as $L_d^\star(\{4\}^d;\{\chi_1^{(2)}\}^d)=T_{2d}(-1)\pi^{4d}/(2^{4d}(4d)!)$ where $T_k(t):=\sum_{n=0}^{k}\binom{2k}{2n}t^n E_{2n}E_{2k-2n}$. We also note that $T_k(1)=2^{2k+1}E_{2k+1}(1)$ for $k\geqslant 0$ (see [12]).

Example 3.7 *(The case $\chi=\chi_{-4}$).* We obtain the following results:

$\kappa=2k+1$	1	3	5
$L_d^\bullet(\{\kappa\}^d;\{\chi_{-4}\}^d)$	$\frac{(-1)^{\frac{1}{2}d(d-1)}}{2^{2d}d!}\pi^d$	$\frac{(-1)^{\frac{1}{2}d(d-1)}\cdot 3}{2^{3d+1}(3d)!}\pi^{3d}$	$\frac{(-1)^{\frac{1}{2}d(d-1)}\cdot 5(L_{5d}-1)}{2^{5d+2}(5d)!}\pi^{5d}$
$L_d^\star(\{\kappa\}^d;\{\chi_{-4}\}^d)$	$\frac{(-1)^{\frac{1}{2}d(d-1)}E_d(\frac{3}{4})}{d!}\pi^d$	–	–

Here, L_d is the Lucas number defined by the recursion equation $L_1=1$, $L_2=3$ and $L_d=L_{d-1}+L_{d-2}$ for $d\geqslant 3$ (it can be shown that $L_d=\alpha^d+\beta^d$ where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$). Note that it is obtained in [6] that $\zeta_d^\bullet(\{10\}^d)=2^{10d+1}\cdot 5(L_{10d+5}+1)\pi^{10d}/(10d+5)!$, and can be similarly shown that $L_d^\bullet(\{10\}^d;\{\chi_1^{(2)}\}^d)=5(L_{10d}+1)\pi^{10d}/(2^4(10d)!)$.

4. Summation formulae for the Bernoulli and Euler numbers

From Theorems 2.1 and 3.4, we immediately obtain the following corollary:

Corollary 4.1. *It holds that*

$$\sum_{\mu\vdash d}\frac{\varepsilon_\mu^\omega}{z_\mu}\frac{(-1)^{\ell(\mu)-e_\mu(\chi)}}{2^{\ell(\mu)}}\frac{W_\mu(\kappa;\chi)\tau_\mu(\chi)}{N_\mu(\chi)^\kappa}\widetilde{B}_{\kappa\mu,\overline{\chi}}=(-1)^{\frac{\kappa d}{2}}2^{-\kappa d}C_d^\omega(\kappa,\chi)A_{\kappa d}^\omega(\kappa,\chi). \tag{4.1}$$

The above equation expresses many (non-trivial) relations among the generalized Bernoulli numbers. We here give several formulae involving the Bernoulli and Euler numbers.

Example 4.2. Let $\chi=\mathbf{1}$ and $\chi=\chi_1^{(2)}$. Then, from expression (4.1), we have respectively

$$\sum_{\mu\vdash d}\frac{\varepsilon_\mu^\omega}{z_\mu}\frac{(-1)^{\ell(\mu)}}{2^{\ell(\mu)}}\widetilde{B}_{2k\mu}=\frac{\varepsilon_\omega^d(-1)^{k\widetilde{\varepsilon}_\omega}A_{2kd}^\omega(2k,\mathbf{1})}{2^{2kd}((2d+1-\widetilde{\varepsilon}_\omega)k)!}, \tag{4.2}$$

$$\sum_{\mu\vdash d}\frac{\varepsilon_\mu^\omega}{z_\mu}\frac{(-1)^{\ell(\mu)}\prod_{j=1}^{\ell(\mu)}(2^{2k\mu_j}-1)}{2^{\ell(\mu)}}\widetilde{B}_{2k\mu}=\frac{\varepsilon_\omega^d A_{2kd}^\omega(2k,\chi_1^{(2)})}{2^{2kd}(2kd)!}. \tag{4.3}$$

These expressions show that both finite sums $A_{2kd}^\omega(2k,\mathbf{1})$ and $A_{2kd}^\omega(2k,\chi_1^{(2)})$ can be written as sums of products of the Bernoulli numbers. On the other hand, putting $d=1$ and $\omega=\bullet$ in expressions (4.2) and (4.3), one can also see from definitions (3.8) and (3.9) that B_{2k} ($k\geqslant 1$) has the following expressions in terms of the multinomial coefficients and the k-th root of unity:

$$B_{2k}=\frac{(2k)!}{2^{2k-1}(3k)!}\sum_{\substack{n_1,\dots,n_k\geqslant 0\\ n_1+\cdots+n_k=k}}\binom{3k}{2n_1+1,\dots,2n_k+1}\zeta_k^{\sum_{l=1}^k ln_l}, \tag{4.4}$$

$$B_{2k}=\frac{1}{2^{2k-1}(2^{2k}-1)}\sum_{\substack{n_1,\dots,n_k\geqslant 0\\ n_1+\cdots+n_k=k}}\binom{2k}{2n_1,\dots,2n_k}\zeta_k^{\sum_{l=1}^k ln_l}. \tag{4.5}$$

Formula (4.4) was essentially obtained by Nakamura [23] (one can easily check that expression (4.4) is equivalent to (2.2) in [23]). Furthermore, putting $k=1$ and $\omega=\star$ in expression (4.2) (resp. (4.3)) and noting $A_{2d}^\bullet(2,\chi_1^{(2)})=(2^{2d}-2)B_{2d}$ (resp. $A_{2d}^\star(2,\chi_1^{(2)})=E_{2d}$), we obtain an expression for B_{2d} (resp. E_{2d}) ($d\geqslant 1$) in terms of a sum of products of the Bernoulli numbers:

$$B_{2d}=-\frac{2^{2d}(2d)!}{(2^{2d}-2)}\sum_{\mu\vdash d}\frac{1}{z_\mu}\frac{(-1)^{\ell(\mu)}}{2^{\ell(\mu)}}\prod_{j=1}^{\ell(\mu)}\frac{B_{2\mu_j}}{(2\mu_j)!}, \tag{4.6}$$

$$E_{2d}=2^{2d}(2d)!\sum_{\mu\vdash d}\frac{1}{z_\mu}\frac{(-1)^{\ell(\mu)}}{2^{\ell(\mu)}}\prod_{j=1}^{\ell(\mu)}(2^{2\mu_j}-1)\prod_{j=1}^{\ell(\mu)}\frac{B_{2\mu_j}}{(2\mu_j)!}. \tag{4.7}$$

Formula (4.6) was obtained by Ohno in [26].

Example 4.3. Let $\chi=\chi_{-4}$. Then, formula (4.1) yields

$$\sum_{\mu\vdash d}\frac{\varepsilon_\mu^\omega}{z_\mu}\frac{(-1)^{\ell(\mu)+\frac{1}{2}\ell(\mu_{\mathrm{o}})}\prod_j^{\mu,\mathrm{e}}(2^{\kappa\mu_j}-1)}{2^{\ell(\mu)+\kappa|\mu_{\mathrm{o}}|}}\widetilde{E}_{\kappa\mu_{\mathrm{o}}-1}\widetilde{B}_{\kappa\mu_{\mathrm{e}}}=\frac{\varepsilon_\omega^{\kappa d}(-1)^{\frac{\kappa d}{2}}A_{\kappa d}^\omega(\kappa,\chi_{-4})}{2^{2\kappa d(1-\widetilde{\varepsilon}_\omega)}(\kappa d)!}. \tag{4.8}$$

Similarly to expressions (4.4) and (4.5), putting $d=1$ and $\omega=\bullet$ and writing $\kappa=2k+1$, we see from expression (3.12) that E_{2k} ($k\geqslant 0$) can be written as

$$E_{2k}=\frac{(-1)^k}{(2k+1)2^{2k}}\sum_{\substack{n_1,\dots,n_{2k+1}\geqslant 0\\ n_1+\dots+n_{2k+1}=2k+1}}\binom{2k+1}{n_1,\dots,n_{2k+1}}(-1)^{\frac{1}{2}\sum_{l=1}^{2k+1}n_l(n_l-1)}\zeta_{2k+1}^{\sum_{l=1}^{2k+1}ln_l}. \tag{4.9}$$

5. Special values at non-positive integers

It has been shown that, in some special cases, $L_d^\bullet(s_1,\dots,s_d;\chi_1,\dots,\chi_d)$ can be continued meromorphically to the whole space $\mathbb{C}^d$ and has possible singularities on $s_d=1$ and $\sum_{k=1}^j s_{d-k+1}\in\mathbb{Z}_{\leqslant j}$ for $j=2,3,\dots,d$. (See [3,21,32] when $\chi_j=\mathbf{1}$ for all j, and [1] when $\chi_1,\dots,\chi_d$ are of the same conductor. One can also obtain more precise information about the singularities from these references.) Hence, in such cases, it is easy to see that $L_d^\star(s_1,\dots,s_d;\chi_1,\dots,\chi_d)$ also admits a meromorphic continuation to $\mathbb{C}^d$ with the same possible singularities since $L_d^\star$ can be expressed in terms of $L_{d'}^\bullet$ for $1\leqslant d'\leqslant d$. In this section, we study a special value, called a *central limit value*, of $L_d^\omega(s_1,\dots,s_d;\{\chi\}^d)$ at $s_1=\dots=s_d=-\kappa$ where $\kappa=2k+e(\chi)$ is a non-negative integer. Since it is seen that such a point is a point of indeterminacy of the function, to define the value, we have to choose the limiting process of $(s_1,\dots,s_d)\to(-\kappa,\dots,-\kappa)$ (for more precise details, see [2,3]). Here, the central limit values $(L_d^\omega)^C(s_1,\dots,s_d;\chi_1,\dots,\chi_d)$ are defined by

$$\left(L_d^\omega\right)^C(s_1,\dots,s_d;\chi_1,\dots,\chi_d):=\lim_{\delta\to 0}L_d^\omega(s_1+\delta,\dots,s_d+\delta;\chi_1,\dots,\chi_d).$$

We then obtain the following results, which give generalizations of the formulae for $\zeta_d^\bullet$ obtained in [17, Corollary 2(ii)].

Theorem 5.1.

(i) *For $N\geqslant 1$, it holds that*

$$\left(L_d^\omega\right)^C\left(\{0\}^d;\left\{\chi_1^{(N)}\right\}^d\right)=\begin{cases}\varepsilon_\omega^d(-1)^d\binom{\frac{\varepsilon_\omega}{2}}{d} & (N=1),\\ 0 & (N\geqslant 2),\end{cases} \tag{5.1}$$

$$\left(L_d^\omega\right)^C\left(\{-2k\}^d;\left\{\chi_1^{(N)}\right\}^d\right)=0\quad (k\in\mathbb{N}). \tag{5.2}$$

(ii) *If χ is non-principal, then, for $\kappa=2k+e(\chi)\geqslant 0$, we have $(L_d^\omega)^C(\{-\kappa\}^d;\{\chi\}^d)=0$.*

Proof. We start from the equation $\sum_{d=0}^\infty\varepsilon_\omega^d L_d^\omega(\{s\}^d;\{\chi\}^d)t^d=\exp(\varepsilon_\omega\sum_{n\geqslant 1}\frac{1}{n}L(ns;\chi^n)t^n)$, which is obtained from expressions (2.2) and (2.5). Note that this is valid for any $s\in\mathbb{C}$ unless $(s,\dots,s)$ is a singularity of L_d^ω. Hence, using the formula $L(-\kappa,\chi)=-\frac{B_{\kappa+1,\chi}}{\kappa+1}$, we have

$$\sum_{d=0}^\infty\varepsilon_\omega^d\left(L_d^\omega\right)^C\left(\{-\kappa\}^d;\{\chi\}^d\right)t^d=\exp\left(-\varepsilon_\omega\sum_{n=1}^\infty\frac{B_{n\kappa+1,\chi^n}}{n(n\kappa+1)}t^n\right). \tag{5.3}$$

We first assume that χ is not principal. Then, noting $(n\kappa+1)+e(\chi^n)=2n(k+e(\chi))+1\equiv 1 \pmod 2$ (here we have used the identity $e(\chi^n)\equiv ne(\chi) \pmod 2$), we have $B_{n\kappa+1,\chi^n}=0$ for all $n\geqslant 1$ because $B_{m,\chi}=0$ if $m+e(\chi)\equiv 1 \pmod 2$. This implies that the right-hand side of Eq. (5.3) is identically equal to $e^0=1$, whence the claim (ii) follows.

We next let $\chi=\chi_1^{(N)}$. Note that

$$B_{m,(\chi_1^{(N)})^n}=B_{m,\chi_1^{(N)}}=\prod_{\substack{p:\text{prime}\\ p\mid N}}\left(1-p^{m-1}\right)\cdot B_m \quad (m\in\mathbb{Z}_{\geqslant 0}). \tag{5.4}$$

Therefore, writing $\kappa=2k$ for $k\in\mathbb{N}$ in Eq. (5.3) and using $B_{2nk+1}=0$, one similarly obtains Eq. (5.2). To prove Eq. (5.1), we further set $\kappa=2k=0$ in Eq. (5.3). When $N\geqslant 2$, since $B_{1,(\chi_1^{(N)})^n}=0$ by Eq. (5.4), we obtain the claim. On the other hand when $N=1$, since $B_1=\frac{1}{2}$, the right-hand side of Eq. (5.3) can be written as

$$\exp\left(-\frac{\varepsilon_\omega}{2}\sum_{n\geqslant 1}\frac{1}{n}t^n\right)=\exp\left(\frac{\varepsilon_\omega}{2}\log(1-t)\right)=(1-t)^{\frac{\varepsilon_\omega}{2}}=\sum_{d=0}^{\infty}(-1)^d\binom{\frac{\varepsilon_\omega}{2}}{d}t^d.$$

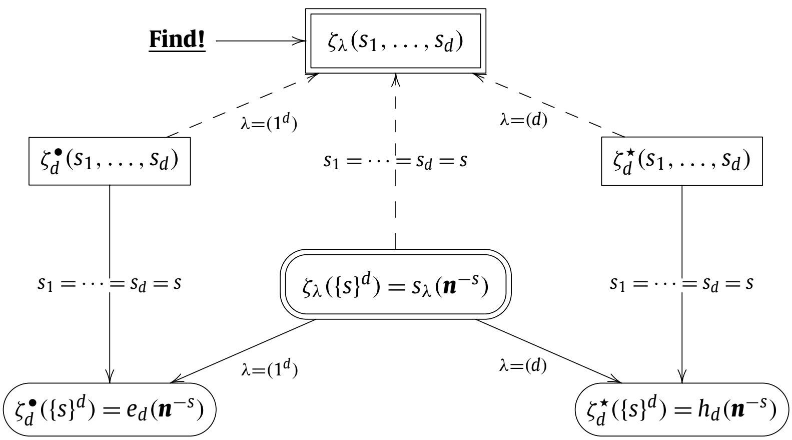
This shows the claim. □

Remark 5.2. By employing a "renormalization procedures" in quantum fields theory, one can also define values of multiple zeta functions at non-positive integers. For further details, see [5,13,14,20] and references therein.

6. Concluding remarks

6.1. Multiple zeta functions attached to the Schur functions

It may be interesting to find a function $\zeta_\lambda(s_1,\ldots,s_d)$ for a partition $\lambda\vdash d$ such that $\zeta_{(1^d)}=\zeta_d^{\bullet}$ and $\zeta_{(d)}=\zeta_d^{\star}$. Recall that $\zeta_d^{\bullet}(\{s\}^d)=e_d(\boldsymbol{n}^{-s})$ and $\zeta_d^{\star}(\{s\}^d)=h_d(\boldsymbol{n}^{-s})$ where $\boldsymbol{n}^{-s}=(1^{-s},2^{-s},\ldots)$. Therefore, since $s_{(1^d)}=e_d$ and $s_{(d)}=h_d$ where s_λ is the Schur function attached to λ (see [19]), such a function ζ_λ can be regarded as a "multiple zeta function attached to s_λ":



Question 6.1. For a fixed λ, are there any relations (such as a sum formula or a duality) among the values $\zeta_\lambda(k_1,\ldots,k_d)$, which interpolate the relations for $\zeta_d^{\omega}(k_1,\ldots,k_d)$?

6.2. Alternating multiple zeta functions

Let

$$\zeta_d^{\omega}(\overline{s_1},\ldots,\overline{s_d}) := \sum_{(m_1,\ldots,m_d)\in I_d^{\omega}} \frac{(-1)^{m_1}\cdots(-1)^{m_d}}{m_1^{s_1}\cdots m_d^{s_d}} = L_d^{\omega}\bigl(s_1,\ldots,s_d;\ \{\varphi_2^1\}^d\bigr),$$

where $\varphi_N^a(m) := \zeta_N^{am}$ for $1 \leqslant a \leqslant N$. This is an alternating analogue of $\zeta_d^{\omega}(s_1,\ldots,s_d)$. When $\omega = \bullet$, the values at positive integers and their relations have been studied in [6,8]. Similarly to the proof of Theorem 2.1, using expression (2.1) with the specialization $x_n = (-1)^n n^{-s}$, we get the following expressions of $\zeta_d^{\omega}(\{\bar{s}\}^d)$ in terms of a sum over partitions of d (note that this series also converges for $s = 1$):

$$\zeta_d^{\omega}\bigl(\{\bar{1}\}^d\bigr) = \sum_{\mu\vdash d}\left\{\frac{\varepsilon_\mu^{\omega}}{z_\mu}\frac{(-1)^{\ell(\mu)+\frac{|\mu_{\mathrm{e}}|}{2}}\prod_{\mu_j\geqslant 3}^{\mu,\mathrm{o}}(2^{\mu_j-1}-1)}{2^{\ell(\mu_{\mathrm{e}})-\ell(\mu_{\mathrm{o}})-(|\mu_{\mathrm{e}}|-|\mu_{\mathrm{o}}|)}}\widetilde{B}_{\mu_{\mathrm{e}}}\right\}(\log 2)^{m_1(\mu)}\cdot\prod_{\mu_j\geqslant 3}^{\mu,\mathrm{o}}\zeta(\mu_j)\cdot\pi^{|\mu_{\mathrm{e}}|},$$

$$\zeta_d^{\omega}\bigl(\{\overline{2k}\}^d\bigr) = (-1)^{kd}2^{2kd}\left\{\sum_{\mu\vdash d}\frac{\varepsilon_\mu^{\omega}}{z_\mu}\frac{(-1)^{\ell(\mu_{\mathrm{e}})}\prod_j^{\mu,\mathrm{o}}(2^{2k\mu_j-1}-1)}{2^{\ell(\mu_{\mathrm{e}})+2k|\mu_{\mathrm{o}}|}}\widetilde{B}_{2k\mu}\right\}\pi^{2kd} \in \mathbb{Q}\pi^{2kd}\quad (k\in\mathbb{N}).$$

These follow from the identities $L(1;\varphi_2^1) = -\log 2$ and

$$L\bigl(2k\mu_j;\ (\varphi_2^1)^{\mu_j}\bigr) = \begin{cases} -\frac{2^{2k\mu_j-1}-1}{2^{2k\mu_j-1}}\zeta(2k\mu_j) & \text{if } \mu_j \text{ is odd},\\ \zeta(2k\mu_j) & \text{otherwise}.\end{cases}$$

Furthermore, since $\prod_{n=1}^{\infty}(1-\frac{\varphi_2^1(n)t^{2k}}{n^{2k}})^{-\varepsilon_\omega} = \prod_{n=1}^{\infty}(1-\frac{(\zeta_{4k}t)^{2k}}{(2n-1)^{2k}})^{-\varepsilon_\omega}\prod_{n=1}^{\infty}(1-\frac{1}{n^{2k}}(\frac{t}{2})^{2k})^{-\varepsilon_\omega}$, then, upon using formulae (3.2), (3.3) and (3.16), we have

$$\zeta_d^{\omega}\bigl(\{\overline{2k}\}^d\bigr) = \frac{\varepsilon_\omega^d(-1)^{k(d-\widetilde{\varepsilon}_\omega)}A_{2kd}^{\omega}(2k,\varphi_2^1)}{2^{2kd}((2d+1-\widetilde{\varepsilon}_\omega)k)!}\pi^{2kd}, \tag{6.1}$$

where $\{A_n^{\omega}(2k,\varphi_2^1)\}_{n\geqslant 0}$ is defined by $A_{2n+1}^{\omega}(2k,\varphi_2^1)\equiv 0$ and

$$A_{2n}^{\omega}\bigl(2k,\varphi_2^1\bigr) := \sum_{\substack{p,q\geqslant 0\\ p+q=n}}\binom{2n+(1-\widetilde{\varepsilon}_\omega)k}{2p,\ 2q+(1-\widetilde{\varepsilon}_\omega)k}\zeta_{2k}^{p}A_{2p}^{\omega}\bigl(2k,\chi_1^{(2)}\bigr)A_{2q}^{\omega}(2k,\mathbf{1}).$$

Formula (6.1) can be straightforwardly obtained from (35) in [6].

6.3. A higher-rank generalization

Let $s_{ij}\in\mathbb{C}$ and f_{ij} be arithmetic functions for $1\leqslant i\leqslant r$ and $1\leqslant j\leqslant d$. Write $S=(s_{ij})$ and $F=(f_{ij})$ ($r\times d$ matrices). We introduce the following "higher-rank" multiple L-function:

$${}_rL_d^{\omega}(S;F) := \sum_{(m^1,\ldots,m^d)\in{}_rI_d^{\omega}}\prod_{j=1}^{d}\frac{f_{1j}(m_{1j})\cdots f_{rj}(m_{rj})}{m_{1j}^{s_{1j}}\cdots m_{rj}^{s_{rj}}},$$

where $m^j=(m_{1j},\dots,m_{rj})\in\mathbb{N}^r$ for $1\leqslant j\leqslant d$ and ${}_rI_d^{\omega}\subset(\mathbb{N}^r)^d$ is defined by

$$ {}_rI_d^{\omega}:=\begin{cases}\{(m^1,\dots,m^d)\in(\mathbb{N}^r)^d \mid m^1<\cdots<m^d\} & (\omega=\bullet),\\ \{(m^1,\dots,m^d)\in(\mathbb{N}^r)^d \mid m^1\leqslant\cdots\leqslant m^d\} & (\omega=\star).\end{cases} $$

Here, we are employing the lexicographic order on $\mathbb{N}^r$. Namely, $(m_1,\dots,m_r)>(n_1,\dots,n_r)$ if $m_1>n_1$, or $m_1=n_1$ and $m_2>n_2$, or $m_1=n_1$, $m_2=n_2$ and $m_3>n_3$, and so on. Clearly, ${}_1I_d^{\omega}=I_d^{\omega}$ and ${}_1L_d^{\omega}=L_d^{\omega}$. As for the case $r=1$, we write ${}_r\zeta_d^{\omega}(S)$ as ${}_rL_d^{\omega}(S;(\mathbf{1}))$.

By induction on d, it is shown that the function ${}_rL_d^{\omega}(S;F)$ can be expressed as a polynomial in $L_{d'}^{\omega}(t_1,\dots,t_{d'};g_1,\dots,g_{d'})$ for $1\leqslant d'\leqslant d$ where t_l is a sum of s_{ij} and g_l is a product of f_{ij}. For example, we see that

$$ {}_rL_1^{\omega}\big((s_{i1});(f_{i1})\big)=\prod_{i=1}^{r}L(s_{i1};f_{i1}), $$

$$ {}_rL_2^{\omega}\big((s_{ij});(f_{ij})\big)=\sum_{p=1}^{r}{}'\prod_{i=1}^{p-1}L(s_{i1}+s_{i2};f_{i1}f_{i2})L_2^{\bullet}(s_{p1},s_{p2};f_{p1},f_{p2})\prod_{i=p+1}^{r}L(s_{i1};f_{i1})L(s_{i2};f_{i2}). $$

Here, the prime $'$ means that we replace $\bullet$ by ω in the summand for $p=r$. Hence, if f_{ij} are bounded for all i,j, then we see from the expression of ${}_rL_d^{\omega}$ in terms of $L_{d'}^{\omega}$ that the series ${}_rL_d^{\omega}(S;F)$ converges absolutely for $\mathrm{Re}(s_{1j})\geqslant 1$ for $1\leqslant j\leqslant d-1$ and $\mathrm{Re}(s_{ij})>1$ otherwise.

Question 6.2. Are there any relations among the values ${}_r\zeta_d^{\omega}((k_{ij}))$, which are generalizations of the relations for $r=1$?

Let us denote by $\{x_1,\dots,x_r\}^d$ the $r\times d$ matrix (x_{ij}) such that $x_{i1}=\cdots=x_{id}=x_i$ for $1\leqslant i\leqslant r$. Let χ_i be a Dirichlet character modulo N_i and $\kappa_i=\kappa_i(k_i,\chi)=2k_i+e(\chi_i)$ for $1\leqslant i\leqslant r$. Though the function ${}_rL_d^{\omega}$ is not new in the above sense, one can calculate the special values ${}_rL_d^{\omega}(\{\kappa_1,\dots,\kappa_r\}^d;\{\chi_1,\dots,\chi_r\}^d)$ by our first method. In fact, we finally can obtain the following proposition:

Proposition 6.3. *Let $|\kappa|:=\kappa_1+\cdots+\kappa_r$. Then, we have*

$$ \begin{aligned} &{}_rL_d^{\omega}\big(\{\kappa_1,\dots,\kappa_r\}^d;\{\chi_1,\dots,\chi_r\}^d\big)\\ &\quad=(-1)^{\frac{|\kappa|d}{2}}2^{|\kappa|d}\left\{\sum_{\mu\vdash d}\frac{\varepsilon_\mu^{\omega}}{z_\mu}\frac{(-1)^{r\ell(\mu)-\sum_{i=1}^r e_\mu(\chi_i)}}{2^{r\ell(\mu)}}\prod_{i=1}^{r}\frac{W_\mu(\kappa_i;\chi_i)\tau_\mu(\chi_i)}{N_\mu(\chi_i)^{\kappa_i}}\widetilde{B}_{\kappa_i\mu,\overline{\chi_i}}\right\}\pi^{|\kappa|d}. \end{aligned} \tag{6.2} $$

In particular, for $k_1,\dots,k_r\in\mathbb{N}$ with $|k|:=k_1+\cdots+k_r$, it holds that

$$ {}_r\zeta_d^{\omega}\big(\{2k_1,\dots,2k_r\}^d\big)=(-1)^{|k|d}2^{2|k|d}\left\{\sum_{\mu\vdash d}\frac{\varepsilon_\mu^{\omega}}{z_\mu}\frac{(-1)^{r\ell(\mu)}}{2^{r\ell(\mu)}}\prod_{i=1}^{r}\widetilde{B}_{2k_i\mu}\right\}\pi^{2|k|d}\in\mathbb{Q}\pi^{2|k|d}. $$

Proof. Replacing the variable x_n by $x_{n_1}^1\cdots x_{n_r}^r$ in (2.2), we have

$$ \sum_{d=0}^{\infty}\varepsilon_\omega^d v_{(d)}^{\omega}(\boldsymbol{y})t^d=\prod_{n_1,\dots,n_r=1}^{\infty}\big(1-x_{n_1}^1\cdots x_{n_r}^r t\big)^{-\varepsilon_\omega}=\sum_{d=0}^{\infty}\varepsilon_\omega^d\left\{\sum_{\mu\vdash d}\frac{\varepsilon_\mu^{\omega}}{z_\mu}\prod_{i=1}^{r}p_\mu\big(\boldsymbol{x}^i\big)\right\}t^d, \tag{6.3} $$

where $\boldsymbol{y} = (x^1_{n_1} \cdots x^r_{n_r})_{n_1,\ldots,n_r \geqslant 1}$ (we also arrange the variables in $\boldsymbol{y}$ in the lexicographic order with respect to $(n_1, \ldots, n_r)$) and $\boldsymbol{x}^i = (x^i_{n_i})_{n_i \geqslant 1}$. Then, specializing $x^i_{n_i} = \chi_i(n_i) n_i^{-\kappa_i}$ in (6.3), we see that the left-hand side of Eq. (6.2) is equal to $\sum_{\mu \vdash d} \frac{\varepsilon^\omega_\mu}{z_\mu} \prod_{i=1}^r \prod_{j=1}^{\ell(\mu)} L(\kappa_i \mu_j; \chi_i^{\mu_j})$. Therefore, one can obtain Eq. (6.2) by following the same approach as in the proof of Theorem 2.1. □

Remark 6.4. From expression (6.2), we see that

$$ {}_rL^\omega_d\big(\{\kappa_{\sigma(1)}, \ldots, \kappa_{\sigma(r)}\}^d; \{\chi_{\sigma(1)}, \ldots, \chi_{\sigma(r)}\}^d\big) = {}_rL^\omega_d\big(\{\kappa_1, \ldots, \kappa_r\}^d; \{\chi_1, \ldots, \chi_r\}^d\big) $$

for any $\sigma \in \mathfrak{S}_r$ where $\mathfrak{S}_r$ is the symmetric group of degree r.

Remark 6.5. It appears to be difficult to evaluate the values ${}_rL^\omega_d(\{\kappa_1, \ldots, \kappa_r\}^d; \{\chi_1, \ldots, \chi_r\}^d)$ for $r \geqslant 2$ by using the second method of the present study (that is, by using the expansions (3.16) of trigonometric functions). This is because the generating function obtained from (6.3) for these values is a "multiple" infinite product.

Acknowledgment

The author would like to thank Professors Sho Matsumoto and Kazufumi Kimoto for valuable comments. He is also thankful to the referee for his/her careful reading and useful comments.

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