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The variety of Boolean semirings

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Abstract

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The variety of Boolean semirings $\mathcal{BSR}$ is the variety generated by the two 2-element semirings. We find a complete set of laws for this variety, and show that it is equivalent to the category of partially Stone spaces. We get a detailed description of the finitely generated free Boolean semirings and a normal form for their elements. From this, a formula is obtained for the free spectrum. This formula relates the free spectra of $\mathcal{BSR}$ and $\mathcal{DL}$, the variety of bounded distributive lattices. Finally, we show that the relational degree of $\mathcal{BSR}$ is 3.

Introduction

Semirings were introduced by Vandiver in the middle 1930s by weakening the additive laws that define a ring. There are several definitions of semiring in the literature. They go from very weak ones, two semigroup operations related by distributive laws (see, e.g., [5]), to the one most commonly used in formal languages and automata theory (see [4] or [7]). The latter is the one we will use; the only thing missing (in order to be a ring) is the existence of additive inverses, and 0 is required to be absorptive for multiplication.

There are a couple of papers in the literature that refer to Boolean semirings. The first one, [11], deals with a very different notion of semiring. It takes the ring laws, deletes one distributivity and adds a weak form of commutativity; it is not even close to any of the usual definitions of semiring. The second one, [5], calls Boolean any semiring (two semigroup operations related by distributive laws) that satisfies the idempotent law $x^2 = x$. In this paper we look at a different connection between Boolean rings and Boolean algebras (from which the former get their

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name). The connection is through the algebras that generate the variety, and not through the laws. To parallel this connection, we consider the semiring structures on the 2-element set, and call the variety generated by them, $\mathscr{BSR}$, the variety of Boolean semirings. In Section 1 we find a complete set of laws for $\mathscr{BSR}$, see Theorem 1.5, that includes the idempotent law plus a law for addition. In the process we determine the subdirectly irreducible Boolean semirings, and find a 3-element semiring S that generates the variety; see Theorem 1.7. This semiring S plays a crucial role in Sections 3 and 4.

In Section 2 it is shown that the categories of Boolean semirings and partially complemented distributive lattices are equivalent. Partially complemented distributive lattices are bounded distributive lattices with a principal order filter which is complemented; see Definition 2.3 and Corollary 2.5. We also get a Stone type representation theorem showing that the category of Boolean semirings is dual to the category of partially Stone spaces; see Definition 2.6 and Corollary 2.8.

In Section 3 a description of free Boolean semirings is given, Theorem 3.7, and we then use this to get a formula for the free spectrum of the variety; see Corollary 3.8. This formula relates the free spectra of $\mathscr{BSR}$ and $\mathscr{DL}$, the variety of bounded distributive lattices. In Section 4 we get a normal form for the elements of the free Boolean semiring; see Theorem 4.2. We then show that the relational degree of S is 3; see Theorem 4.6.

1. Boolean semirings

Definition 1.1. A *semiring* is a type $\langle 0, 0, 2, 2\rangle$ algebra $(A; 0, 1, +, \cdot)$ that satisfies:

(SR1) $x + (y + z) = (x + y) + z$,
(SR2) $x + 0 = x$,
(SR3) $x + y = y + x$,
(SR4) $x(yz) = (xy)z$,
(SR5) $x1 = x = 1x$,
(SR6) $x(y + z) = xy + xz$; $(x + y)z = xz + yz$,
(SR7) $x0 = 0 = 0x$.

In other words, if a semiring fails to be a ring, it is by the absence of additive inverses. The smallest nontrivial $(1 \neq 0)$ semiring must have at least 2 elements; in fact, on $\{0, 1\}$ there are two semiring structures. We must have

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

and the two structures correspond to $1 + 1 = 0$ and $1 + 1 = 1$. The first is the

Boolean ring $\mathbb{Z}_2$ and the second is the bounded distributive lattice reduct of the 2-element Boolean algebra, that we will denote by **B**; this semiring is called *the Boolean semiring* in [7]. We will denote the variety generated by the two 2-element semirings $\mathbb{Z}_2$ and **B** by $\mathcal{BSR}$, and call it the *variety of Boolean semirings*. Observe that a Boolean semiring satisfies:

(BSR1) $1 + x + x = 1$,

(BSR2) $x^2 = x$.

Our first goal is to show that (SR1–6),(BSR1–2) is an equational basis for the variety $\mathcal{BSR}$ of Boolean semirings. We exclude (SR7) from this list since it is a consequence of (BSR1–2); see Lemma 1.2(a),(d). Let $\mathcal{V} = V(\text{SR1–6,BSR1–2})$ denote the variety of algebras that satisfy (SR1–6) and (BSR1–2). When using (SR1–6) to prove facts about algebras in $\mathcal{V}$, we will not mention these laws explicitly.

Lemma 1.2. *Let $A \in \mathcal{V}$ and $x, y \in A$.*

(a) $x0 = 0$,

(b) $x + x + x = x$,

(c) $y + xy + xy = y = y + yx + yx$,

(d) $xy = yx$.

Proof. (a) Multiply (BSR1) by 0 on the right to get $x0 + x0 = 0$. Now multiply by x on the left and use (BSR2) to get $x0 + x0 = x0$. Therefore, $x0 = 0$.

(b) Multiply (BSR1) by x (on either side) and use (BSR2).

(c) Multiply (BSR1) by y.

(d) Using (BSR2) we have $x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + y + xy + yx$. Substituting xy (resp. yx) for y we obtain, using (BSR2) and (c) to simplify, $x + xy = x + xy + xxy + xyx = x + xyx$ and $x + yx = x + yx + xyx + yxx = x + xyx$ and therefore $x + xy = x + yx$. Now substituting xyx for x, adding xyx on both sides, and using (BSR2) and (c) to simplify, yields $xy = yx$. □

Given $A \in \mathcal{V}$ and $a \in A$, we define two binary relations on A as follows:

$$x \sim_a y \quad \text{if and only if} \quad ax = ay\,,$$

$$x \sim^a y \quad \text{if and only if} \quad a + x + ax = a + y + ay\,.$$

The identity relation on A will be denoted by Δ_A, or simply by Δ.

Proposition 1.3. *Let $A \in \mathcal{V}$ and $a \in A$.*

(a) *$\sim_a$ and $\sim^a$ are congruences on A,*

(b) *$\sim_a = \Delta_A$ if and only if $a = 1$,*

(c) *$\sim^a = \Delta_A$ if and only if $a = 0$.*

Proof. (a) Clearly both $\sim_a$ and $\sim^a$ are equivalence relations. Let us verify that they preserve the operations. If $x \sim_a y$ and $u \sim_a v$, then $ax = ay$ and $au = av$. Adding these equations we get $a(x + u) = a(y + v)$ so $x + u \sim_a y + v$. Multiplying we get $axau = ayav$ and using Lemma 1.2(d) and (BSR2) gives $axu = ayv$, so $xu \sim_a yv$.

If $x \sim^a y$ and $u \sim^a v$, then $a + x + ax = a + y + ay$ and $a + u + au = a + v + av$. Adding these two equations, adding a on both sides, and using Lemma 1.2(b) we get $a + (x + u) + a(x + u) = a + (y + v) + a(y + v)$, so $x + u \sim^a y + v$. Now if we multiply the first equation by u, add $a + au$ on both sides and use Lemma 1.2(c), we get $a + xu + axu = a + yu + ayu$. Similarly, from the second equation we get $a + uy + auy = a + vy + avy$, and using Lemma 1.2(d) we get $a + xu + axu = a + vy + avy$, so $xu \sim^a yv$.

(b) From (BSR2) we have $a \sim_a 1$, so $\sim_a = \Delta$ implies $a = 1$. The converse is trivial.

(c) From (BSR2) and Lemma 1.2(a),(b) we have $a \sim^a 0$, so $\sim^a = \Delta$ implies $a = 0$. The converse is again trivial. □

Proposition 1.4. *$\mathbb{Z}_2$ and* **B** *are the only subdirectly irreducible semirings in $\mathcal{V}$.*

Proof. Clearly $\mathbb{Z}_2$ and **B** are subdirectly irreducible. Let $A \in \mathcal{V}$ be subdirectly irreducible. Let $a \neq b$ be elements of A such that for every congruence $\sim \neq \Delta$ we have $a \sim b$. If we had $a,b \neq 1$, then by Lemma 1.3(b), $\sim_a, \sim_b \neq \Delta$ and therefore $a \sim_a b$ and $a \sim_b b$, i.e. $a^2 = ab$ and $ba = b^2$. Using (BSR2) and Lemma 1.2(d) yields $a = b$, a contradiction.

Without loss of generality assume $b = 1$.

If we had $a \neq 0$, then by Proposition 1.3(c) $\sim^a \neq \Delta$ and therefore $a \sim^a 1$, i.e. $a + a + a^2 = a + 1 + a$. Using (BSR2), Lemma 1.2(b) and (BSR1) we get $a = 1$, a contradiction, so we must have $a = 0$.

Suppose now that $|A| > 2$, and let $c \in A$, $c \neq 0,1$. By Proposition 1.3(b) $\sim_c \neq \Delta$, so $0 \sim_c 1$, i.e. $0 = c$, a contradiction. So A is a 2-element semiring, i.e. $\mathbb{Z}_2$ or **B**. □

Based on Birkhoff's theorem [1, p. 14], that a variety is generated by its subdirectly irreducibles we get the following consequences of the proposition.

Theorem 1.5. (SR1–6) *and* (BSR–2) *form an equational basis for the variety $\mathcal{BSR}$ of Boolean semirings.* □

Theorem 1.6. *The variety $\mathcal{BSR}$ of Boolean semirings is a minimal cover of the variety of Boolean rings, and also of the variety of bounded distributive lattices.* □

Boolean rings are characterized among rings by the law (BSR2), and (BSR1) is in fact a consequence of it. For semirings the laws (BSR1) and (BSR2) are independent. The semiring $\{0, h, 1\}$ with operations

$+$	0	h	1
0	0	h	1
h	h	h	1
1	1	1	1

$\cdot$	0	h	1
0	0	0	0
h	0	0	h
1	0	h	1

satisfies (BSR1), but does not satisfy (BSR2). The semiring $\{0, h, 1\}$ with operations

$+$	0	h	1
0	0	h	1
h	h	h	h
1	1	h	h

$\cdot$	0	h	1
0	0	0	0
h	0	h	h
1	0	h	1

satisfies (BSR2), but does not satisfy (BSR1).

The variety $\mathcal{BSR}$ is generated by the 4-element Boolean semiring $\mathbb{Z}_2 \times \mathbf{B}$. If $F(n)$ denotes the free Boolean semiring on n generators, then $|F(n)| \leq 4^{4^n}$. The next theorem yields a better bound.

Theorem 1.7. *The variety $\mathcal{BSR}$ of Boolean semirings is generated by the 3-element semiring $S = \{0, h, 1\}$ with operations*

$+$	0	h	1
0	0	h	1
h	h	h	1
1	1	1	h

$\cdot$	0	h	1
0	0	0	0
h	0	h	h
1	0	h	1

Proof. On one hand S is isomorphic to the subsemiring $\{(0,0), (0,1), (1,1)\}$ of the product $\mathbb{Z}_2 \times \mathbf{B}$, so S is a Boolean semiring. On the other hand $S/\theta_{h,1}$ is isomorphic to $\mathbf{B}$, and $S/\theta_{0,h}$ is isomorphic to $\mathbb{Z}_2$. □

Corollary 1.8. $|F(n)| \leq 3^{3^n}$. □

In Section 3 we will get a better bound; see Corollary 3.3.

It also follows from Proposition 1.4 that a Boolean semiring is the subdirect product of a bounded distributive lattice and Boolean ring. We can give a precise description of this situation as follows: let $(A; 0, 1, +, \cdot)$ be a Boolean semiring. Let $h = 1 + 1$, and for any $a \in A$ let $\tilde{a} = ha = a + a$ and $\hat{a} = h + a = 1 + 1 + a$. Let $\tilde{A} = \{\tilde{a} \mid a \in A\}$ and $\hat{A} = \{\hat{a} \mid a \in A\}$.

Proposition 1.9. *Let $(A; 0, 1, +, \cdot)$ be a Boolean semiring.*

(a) *$(\tilde{A}; 0, h, +, \cdot)$ is a bounded distributive lattice and the map $\tilde{\ }: A \to \tilde{A}$ given by $a \mapsto \tilde{a}$ is a surjective homomorphism.*

(b) $(\hat{A}; h, 1, +, \cdot)$ *is a Boolean ring and the map* $\hat{}: A \to \hat{A}$ *given by* $a \mapsto \hat{a}$ *is a surjective homomorphism.*

(c) *The map* $A \to \tilde{A} \times \hat{A}$ *given by* $a \mapsto (\tilde{a}, \hat{a})$ *is one-to-one.*

(d) *The* 3 *maps in* (a), (b) *and* (c) *give the only decomposition of* A *as a subdirect product of a bounded distributive lattice and a Boolean ring.*

Proof. (a) $\tilde{0} = h0 = 0$, $\tilde{1} = h1 = h$, $\widetilde{a+b} = h(a+b) = ha + hb = \tilde{a} + \tilde{b}$, and $\tilde{a}\tilde{b} = hahb = hab = \widetilde{ab}$, so $\tilde{}$ is a surjective homomorphism, and $\tilde{A}$ is a Boolean semiring. To see that $\tilde{A}$ is a bounded distributive lattice we must verify the idempotent law for $+$, and the two absorption laws.

$$\tilde{a} + \tilde{a} = \tilde{\tilde{a}} = hha = ha = \tilde{a}\,,$$

$$\begin{aligned} \tilde{a}(\tilde{a} + \tilde{b}) &= \tilde{a} + \tilde{a}\tilde{b} = a + a + ab + ab \\ &= a + a = \tilde{a} \quad \text{by Lemma 1.2(c)}\,. \end{aligned}$$

(b) $\hat{0} = h + 0 = h$, $\hat{1} = h + 1 = 1$ by Lemma 1.2(b), $\hat{a} + \hat{b} = h + a + h + b = h + a + b = \widehat{a+b}$ since $h + h = h$ by Lemma 1.2(b), and $\hat{a}\hat{b} = (h+a)(h+b) = h + ha + hb + ab = h + ab = \widehat{ab}$ since $h + ha = h$ by (BSR1). So $\hat{}$ is a surjective homomorphism, and $\hat{A}$ is a Boolean semiring. To see that $\hat{A}$ is a Boolean ring it suffices to show the existence of additive inverses.

$$\hat{a} + \hat{a} = h + a + h + a = h + a + a = h = \hat{0} \quad \text{by (BSR1)}\,.$$

(c) If $\tilde{a} = \tilde{b}$ and $\hat{a} = \hat{b}$, i.e. $ha = hb$ and $h + a = h + b$, then $a = a + a + a = ha + a = (h + a)a = (h + b)a = ha + ba$ and similarly $b = hb + ab$. Therefore, $a = b$.

(d) Parts (a), (b) and (c) show that A is a subdirect product of $\tilde{A}$ and $\hat{A}$. Suppose now that L is a bounded distributive lattice, R a Boolean ring and $\phi: A \to L \times R$ a one-to-one homomorphism of semirings, such that $\phi_1 = \pi_1 \circ \phi: A \to L$ and $\phi_2 = \pi_2 \circ \phi: A \to R$ are surjective. Observe first that $\ker(\tilde{}) = \theta_{h,1}$ and $\ker(\hat{}) = \theta_{0,h}$. In both cases the inclusion '$\supseteq$' is clear since $\tilde{h} = \tilde{1}$ and $\hat{0} = \hat{h}$. Let us show the other inclusion. If $\tilde{a} = \tilde{b}$, i.e. $ha = hb$, then $a = a + ha = 1a + hb$, and $b = ha + 1b$, so $(a, b) \in \theta_{h,1}$. If $\hat{a} = \hat{b}$, i.e. $h + a = h + b$, then $a = a + ha = a(a + h) = a(b + h) = ab + ha + 0b$ and $b = ab + 0a + hb$, so $(a, b) \in \theta_{0,h}$.

Now $\phi_1(h) = \phi_1(1) + \phi_1(1) = \phi_1(1)$ by idempotency in L, and we have $\theta_{h,1} \subseteq \ker(\phi_1)$. Therefore, there is a homomorphism $\psi_1: \tilde{A} \to L$ such that $\phi_1 = \psi_1 \circ \tilde{}$. Similarly $\phi_2(h) = \phi_2(1) + \phi_2(1) = 0 = \phi_2(0)$ since R has characteristic 2, and $\theta_{0,h} \subseteq \ker(\phi_2)$. So there is a homomorphism $\psi_2: \hat{A} \to R$ such that $\phi_2 = \psi_2 \circ \hat{}$. We now claim that ψ_1 and ψ_2 are isomorphisms. They are surjective since ϕ_1 and ϕ_2 are so. Suppose $\psi_1(\tilde{a}) = \psi_1(\tilde{b})$. Then $\phi_1(\tilde{a}) = \psi_1(\tilde{\tilde{a}}) = \psi_1(\tilde{a}) = \psi_1(\tilde{b}) = \psi_1(\tilde{\tilde{b}}) = \phi_1(\tilde{b})$, $\phi_2(\tilde{a}) = \psi_2(\hat{\tilde{a}}) = \psi_2(h + ha) = \psi_2(h) = \psi_2(\hat{0}) = \phi_2(0)$, and similarly $\phi_2(\tilde{b}) =$

$\phi_2(0)$, so $\phi(\tilde{a}) = \phi(\tilde{b})$, and we must have $\tilde{a} = \tilde{b}$, i.e. ψ_1 is one-to-one. Suppose $\psi_2(\hat{a}) = \psi_2(\hat{b})$. Then $\phi_2(\hat{a}) = \psi_2(\hat{a}) = \psi_2(\hat{a}) = \psi_2(\hat{b}) = \psi_2(\hat{b}) = \phi_2(\hat{b})$, $\phi_1(\hat{a}) = \psi_1(\tilde{a}) = \psi_1(h(h+a)) = \psi_1(h) = \psi_1(\tilde{1}) = \phi_1(1)$, and similarly $\phi_1(\hat{b}) = \phi_1(1)$, so $\phi(\hat{a}) = \phi(\hat{b})$, and we must have $\hat{a} = \hat{b}$, i.e. ψ_2 is one-to-one. □

Observe that $\tilde{A}$ and $\hat{A}$ are not subsemirings of A. Everything works except that the 'one' of $\tilde{A}$ is not 1 but h, and the 'zero' of $\hat{A}$ is not 0 but h.

In the language of [5] we get the following corollary:

Corollary 1.10. *Let $A \in \mathcal{BSR}$. Then A is a bounded distributive lattice D of Boolean rings $(A_\alpha \mid \alpha \in D)$ without unit.*

Proof. Let D be a bounded distributive lattice and B a Boolean ring such that A is a subdirect product of D and B. For each $\alpha \in D$, let $A_\alpha = \{(\alpha, b) \in A\}$. It is clear that A_α is a sub-$(+, \cdot)$-algebra of A, isomorphic to a subring (possibly without unit) of B. The rest of the statement is easy to check. See [5] for definitions. □

2. Partially complemented distributive lattices

The equivalence between the categories of Boolean rings and Boolean algebras, gives a characterization of Boolean rings as distributive lattices with some additional structure. In view of Propositions 1.4 and 1.9, it is natural to expect a similar result for Boolean semirings. We will obtain two such results in Theorems 2.1 and 2.4. In the first, the additional structure is a unary operation; in the second it is a constant. From these theorems we easily obtain a Stone type representation theorem which says that the category of Boolean semirings is dual to a category of spaces that we call partially Stone.

We will not mention bounded distributive lattice laws explicitly in the proofs.

Theorem 2.1. (a) *Let $(A; 0, 1, +, \cdot)$ be a Boolean semiring. Define operations*

$$x \wedge y = xy ,$$

$$x \vee y = x + y + xy ,$$

$$\bar{x} = 1 + x .$$

Then $(A; 0, 1, \wedge, \vee)$ is a bounded distributive lattice and $^-$ satisfies:

(L1) $x \vee \bar{x} = 1$,

(L2) $x \wedge \bar{x} = x \wedge \bar{1}$.

Moreover, we can recover the original operations by: $x + y = (x \vee y) \wedge (\bar{x} \vee \bar{y})$.

(b) *Let* $(A; 0, 1, \wedge, \vee)$ *be a bounded distributive lattice and* $^-$ *a unary operation on* A *satisfying* (L1–2). *Define operations*

$$x + y = (x \vee y) \wedge (\bar{x} \vee \bar{y}) ,$$

$$xy = x \wedge y .$$

Then $(A; 0, 1, +, \cdot)$ *is a Boolean semiring and* $\bar{x} = 1 + x$, $x \vee y = x + y + xy$.

Proof. (a) By Theorem 1.7 it suffices to check for the semiring S. Checking this is a trivial exercise. It should be noted that the lattice of S is given by $0 < h < 1$.

(b) Before we prove this part let us establish some properties of $^-$.

Lemma 2.2. *Let* A *be a bounded distributive lattice with a unary operation* $^-$ *that satisfies* (L1–2). *Let* $x, y \in A$.

(a) $\bar{1} \leq \bar{x}$.
(b) *If* $x \leq y$, *then* $\bar{y} \leq \bar{x}$.
(c) $\bar{x} \wedge \bar{\bar{x}} = \bar{1}$.
(d) $x \wedge (\bar{x} \vee y) = x \wedge (\bar{1} \vee y)$.
(e) $\bar{\bar{x}} = x \vee \bar{1}$.
(f) $\bar{\bar{\bar{x}}} = \bar{x}$.
(g) $\bar{x} \vee \bar{\bar{y}} = \bar{x} \vee y$.
(h) $\overline{x \wedge y} = \bar{x} \vee \bar{y}$.
(i) $\overline{x \vee y} = \bar{x} \wedge \bar{y}$.

Proof. (a) Using (L1–2) and lattice laws we have $\bar{x} = \bar{x} \vee (x \wedge \bar{x}) = \bar{x} \vee (x \wedge \bar{1}) = (\bar{x} \vee x) \wedge (\bar{x} \vee \bar{1}) = \bar{x} \vee \bar{1}$.

(b) If $x \leq y$, then from (L1) $1 = x \vee \bar{x} = (x \wedge y) \vee \bar{x} = y \vee \bar{x}$. From (L2) we get $\bar{x} \vee (y \wedge \bar{y}) = \bar{x} \vee (y \wedge \bar{1})$. Distributing and using $y \vee \bar{x} = 1$ and (a) we get $\bar{x} \vee \bar{y} = \bar{x}$.

(c) Follows at once from (L2) and (a).

(d) Follows from (L2).

(e) Using (a), (d) and (L1) we have $x \wedge \bar{\bar{x}} = x \wedge (\bar{\bar{x}} \vee \bar{1}) = x \wedge (\bar{\bar{x}} \vee \bar{x}) = x$, so $x \leq \bar{\bar{x}}$. Using now this, (c) and (L1) we get $x \vee \bar{1} = x \vee (\bar{x} \wedge \bar{\bar{x}}) = x \vee \bar{\bar{x}} = \bar{\bar{x}}$.

(f), (g) follow at once from (e) and (a).

(h) From (b) we get $\bar{x} \leq \overline{x \wedge y}$ and $\bar{y} \leq \overline{x \wedge y}$ so $\bar{x} \vee \bar{y} \leq \overline{x \wedge y}$. From (L1) we get $(x \wedge y) \vee (\bar{x} \vee \bar{y}) = 1$. Using these two facts, (a) and (L1–2), $\overline{x \wedge y} = \overline{x \wedge y} \wedge ((x \wedge y) \vee (\bar{x} \vee \bar{y})) = (x \wedge y \wedge \bar{1}) \vee (\bar{x} \vee \bar{y}) = \bar{1} \vee \bar{x} \vee \bar{y} = \bar{x} \vee \bar{y}$.

(i) Like in (h), from (b) we get $\overline{x \vee y} \leq \bar{x} \wedge \bar{y}$. From (L2) we get $(x \vee y) \wedge (\bar{x} \wedge \bar{y}) = (x \vee y) \wedge \bar{1}$. Using these two facts, (a) and (L1), $\overline{x \vee y} = \overline{x \vee y} \vee ((x \vee y) \wedge (\bar{x} \wedge \bar{y})) = \bar{x} \wedge \bar{y}$. □

Proof of Theorem 2.1(b). By (L1), $\bar{0} = 1$, so $x + 0 = (x \vee 0) \wedge (\bar{x} \vee \bar{0}) = x \wedge 1 = x$. Clearly $+$ is commutative. For associativity of $+$ we have

$$
\begin{aligned}
&(x+y)+z \\
&\quad = (((x \vee y) \wedge (\bar{x} \vee \bar{y})) \vee z) \\
&\qquad \wedge (((\bar{x} \wedge \bar{y}) \vee (\bar{\bar{x}} \wedge \bar{\bar{y}})) \vee \bar{z}) \quad \text{(Lemma 2.2(h),(i))} \\
&\quad = (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee z) \wedge (\bar{x} \vee \bar{\bar{x}} \vee \bar{z}) \\
&\qquad \wedge (\bar{x} \vee \bar{\bar{y}} \vee \bar{z}) \wedge (\bar{y} \vee \bar{\bar{x}} \vee \bar{z}) \wedge (\bar{y} \vee \bar{\bar{y}} \vee \bar{z}) \\
&\quad = (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee z) \wedge (\bar{x} \vee \bar{\bar{y}} \vee \bar{z}) \wedge (\bar{y} \vee \bar{\bar{x}} \vee \bar{z}) \quad \text{(L1)} \\
&\quad = (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee z) \\
&\qquad \wedge (\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee \bar{z}), \quad \text{(Lemma 2.2(g))} \\
&x+(y+z) \\
&\quad = (y+z)+x \\
&\quad = (y \vee z \vee x) \wedge (\bar{y} \vee \bar{z} \vee x) \wedge (\bar{y} \vee z \vee \bar{x}) \wedge (y \vee \bar{z} \vee \bar{x}).
\end{aligned}
$$

For distributivity

$$
\begin{aligned}
&xy + xz \\
&\quad = ((x \wedge y) \vee (x \wedge z)) \wedge (\bar{x} \vee \bar{y} \vee \bar{x} \vee \bar{z}) && \text{(Lemma 2.2(h))} \\
&\quad = x \wedge (y \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z}) \\
&\quad = x \wedge (y \vee z) \wedge (\bar{1} \vee \bar{y} \vee \bar{z}) && \text{(Lemma 2.2(d))} \\
&\quad = x \wedge (y \vee z) \wedge (\bar{y} \vee \bar{z}) && \text{(Lemma 2.2(a))} \\
&\quad = x(y+z).
\end{aligned}
$$

Finally, using Lemma 2.2(a), $1 + x = (1 \vee x) \wedge (\bar{1} \vee \bar{x}) = 1 \wedge \bar{x} = \bar{x}$, and

$$
\begin{aligned}
&x + y + xy \\
&\quad = x(1+y) + y \\
&\quad = x\bar{y} + y \\
&\quad = ((x \wedge \bar{y}) \vee y) \wedge (\bar{x} \vee \bar{\bar{y}} \vee \bar{y}) && \text{(Lemma 2.2(h))} \\
&\quad = (x \wedge \bar{y}) \vee y && \text{(L1)} \\
&\quad = x \vee y. && \text{(L1)} \quad \square
\end{aligned}
$$

The fact that $\bar{0} = 1$, together with Lemma 2.2(h),(i), almost makes $^{-}$ an Ockham operation in the sense of [10]. It fails because in general $\bar{1} \neq 0$. Also note that if $x \vee y = 1$, then using Lemma 2.2(e),(i) and (L1) we have $\bar{\bar{y}} = y \vee \bar{1} = y \vee$

$(\bar{x} \wedge \bar{y}) = (y \vee \bar{x}) \wedge (y \vee \bar{y}) = y \vee \bar{x}$ and therefore $\bar{x} \leq \bar{\bar{y}}$. This fact together with (L1) almost makes $^-$ a dual pseudo-complement. These failures of $^-$ come from looking at it in the wrong setting. In fact, when restricted to an appropriate subset of A, $^-$ is a complement.

Definition 2.3. A *partially complemented distributive lattice* is an algebra $(A; 0, 1, h, \vee, \wedge)$ such that $(A; 0, 1, \vee, \wedge)$ is a bounded distributive lattice, $h \in A$, and $([h, 1]; h, 1, \vee, \wedge)$ is a complemented distributive lattice, i.e. a Boolean algebra where $[h, 1] = \{a \in A \mid h \leq a\}$.

Theorem 2.4. (a) *Let* $(A; 0, 1, \vee, \wedge)$ *be a bounded distributive lattice and* $^-$ *a unary operation satisfying* (L1–2) *in Theorem* 2.1(a). *Then* $(A; 0, 1, \bar{1}, \vee, \wedge)$ *is a partially complemented distributive lattice, and* $\bar{x} = (x \vee \bar{1})'$, *where* $'$ *denotes the complement in* $[\bar{1}, 1]$.

(b) *Let* $(A; 0, 1, h, \vee, \wedge)$ *be a partially complemented distributive lattice and define* $\bar{x} = (x \vee h)'$, *where* $'$ *denotes the complement in* $[h, 1]$. *Then* $^-$ *satisfies* (L1–2), *and* $h = \bar{1}$.

Proof. (a) Let $x \in [\bar{1}, 1]$, then by (L1) $x \vee \bar{x} = 1$ and by (L2) $x \wedge \bar{x} = x \wedge \bar{1} = \bar{1}$, so $\bar{x}$ is the complement of x in $[\bar{1}, 1]$. Therefore, $(A; 0, 1, \bar{1}, \vee, \wedge)$ is a partially complemented distributive lattice. Now for any $x \in A$, $x \vee \bar{1} \in [\bar{1}, 1]$, so using Lemma 2.2(i),(e) $(x \vee \bar{1})' = \overline{x \vee \bar{1}} = \bar{x} \wedge \bar{\bar{1}} = \bar{x} \wedge 1 = \bar{x}$.

(b) For any $x \in A$, $\bar{x} \geq h$ so we have $1 = (x \vee h) \vee (x \vee h)' = x \vee h \vee \bar{x} = x \vee \bar{x}$. Clearly $\bar{1} = 1' = h$, so $\bar{1} = (x \vee h) \wedge (x \vee h)' = (x \vee h) \wedge \bar{x} = (x \wedge \bar{x}) \vee (h \wedge \bar{x}) = (x \wedge \bar{x}) \vee \bar{1}$, and we get $x \wedge \bar{x} \leq \bar{1}$; since $x \wedge \bar{x} \leq x$, we get $x \wedge \bar{x} \leq x \wedge \bar{1}$. $x \wedge \bar{1} \leq x \wedge \bar{x}$ follows from $\bar{1} \leq \bar{x}$. $\square$

Corollary 2.5. *The category of Boolean semirings is equivalent to the category of partially complemented distributive lattices.*

Proof. Theorems 2.1 and 2.4 establish the bijection between objects. But it is clear that a function $f : A \to B$ is a homomorphism of Boolean semirings if and only if it is a homomorphism of partially complemented distributive lattices. $\square$

Recall that a Stone space is a coherent, Hausdorff space. The Stone representation theorem gives a 1–1 correspondence between distributive lattices and coherent spaces, with Boolean algebras corresponding to Stone spaces. We refer the reader to [6] for notation, definitions and basic facts on coherent and Stone spaces. However we need to mention explicitly the following: if A is a distributive lattice, $\mathrm{Idl}(A)$ denotes the lattice of ideals of A, and $X = \mathrm{pt}(\mathrm{Idl}(A))$ the set of prime filters of A. If $I \in \mathrm{Idl}(A)$, let $\varphi(I) = \{p \in X \mid p \cap I \neq \emptyset\}$. Then $\mathcal{T} = \{\varphi(I) \mid I \in \mathrm{Idl}(A)\}$ is a topology on X which makes it into a coherent space, and φ is an isomorphism of lattices between $\mathrm{Idl}(A)$ and $\mathcal{T}$ (see [6, Chapter II]).

Definition 2.6. We say that (X, Y) is a *partially Stone space* if X is a coherent space, Y a compact open subspace of X, and $X - Y$ is Hausdorff. Given (X_1, Y_1) and (X_2, Y_2) partially Stone spaces, and a function $f : X_1 \to X_2$, we say that f is a *map of partially Stone spaces* if f is continuous, $f^{-1}\langle Y_2 \rangle = Y_1$, and $f|_{Y_1} : Y_1 \to Y_2$ is coherent.

Theorem 2.7. (a) *Let A be a partially complemented distributive lattice. Let $X = \mathrm{pt}(\mathrm{Idl}(A))$, and $Y = \{p \in X \mid h \in p\} = \varphi(\downarrow\{h\})$. Then (X, Y) is a partially Stone space.*

(b) *Let (X, Y) be a partially Stone space. Let $A = K\Omega(X)$ be the lattice of compact open subspaces of X. Then $(A; \emptyset, X, Y, \cup, \cap)$ is a partially complemented distributive lattice.*

Proof. (a) By the Stone representation theorem, X is a coherent space and $Y = \varphi(\downarrow\{h\})$ is a compact open subset. Let $p, q \in X - Y$ with $p \neq q$. Suppose $p \not\subseteq q$ and choose $a \in A$ such that $a \in p$ and $a \not\in q$. Since $a \vee \bar{a} = 1 \in q$ and q is prime, we must have $\bar{a} \in q$. Let $U = \varphi(\downarrow\{a\})$, $V = \varphi(\downarrow\{\bar{a}\})$. Then U and V are open neighborhoods of p and q respectively, and $U \cap V = \varphi(\downarrow\{a\}) \cap \varphi(\downarrow\{\bar{a}\}) = \varphi(\downarrow\{a\} \cap \downarrow\{\bar{a}\}) = \varphi(\downarrow\{a \wedge \bar{a}\}) = \varphi(\downarrow\{a \wedge h\}) \subseteq \varphi(\downarrow\{h\}) = Y$, so $U - Y$ and $V - Y$ are disjoint open neighborhoods of p and q in $X - Y$, and $X - Y$ is Hausdorff.

(b) By the Stone representation theorem $K\Omega(X)$ is a distributive lattice. For $U \in K\Omega(X)$ let $\bar{U} = (X - U) \cup Y$. Then $\bar{U}$ is compact and $U \cup \bar{U} = X$, $U \cap \bar{U} = U \cap Y$, so it remains to be seen that $\bar{U}$ is open. Let $x \in \bar{U}$; if $x \in Y$, then Y is an open neighborhood of x contained in $\bar{U}$. But if $x \not\in Y$, then for each $y \in U - Y$ let U_y and V_y be open neighborhoods of x and y respectively, such that $(U_y - Y)$ and $(V_y - Y)$ are disjoint. Then $U - Y \subseteq \bigcup_{y \in U - Y} V_y$ and since $U - Y$ is compact, there are $y_1, \ldots, y_n$ such that $U - Y \subseteq V_{y_1} \cup \cdots \cup V_{y_n}$. Let $W = U_{y_1} \cap \cdots \cap U_{y_n}$. Then W is an open neighborhood of x and for any $w \in W$, if $w \not\in Y$, then $w \in U_{y_i} - Y$, so $w \not\in V_{y_i} - Y$ and $w \not\in V_{y_i}$. It then follows that $w \not\in U$, and therefore $W \subseteq \bar{U}$. $\square$

Corollary 2.8. *The category of Boolean semirings is dual to the category of partially Stone spaces.*

Proof. If $f : A \to B$ is a homomorphism of partially complemented distributive lattices, then by the Stone representation theorem $f^* : \mathrm{pt}(\mathrm{Idl}(B)) \to \mathrm{pt}(\mathrm{Idl}(A))$, $q \mapsto f^{-1}\langle q \rangle$ is a coherent map. Moreover, $(f^*)^{-1}\langle \varphi(\downarrow\{h_A\}) \rangle = (f^*)^{-1}\langle \{p \mid h_A \in p\} \rangle = \{q \mid h_A \in f^*(q)\} = \{q \mid f(h_A) \in q\} = \{q \mid h_B \in q\} = \varphi(\downarrow\{h_B\})$, so f^* is a map of partially Stone spaces from $(\mathrm{pt}(\mathrm{Idl}(B)), \varphi(\downarrow\{h_B\}))$ to $(\mathrm{pt}(\mathrm{Idl}(A)), \varphi(\downarrow\{h_A\}))$.

Conversely, let $f : X_1 \to X_2$ be a map of partially Stone spaces from (X_1, Y_1) to (X_2, Y_2). If U is compact open in X_2, then $U - Y_2$ is compact open in $X_2 - Y_2$ and $U \cap Y_2$ is compact open in Y_2. $f^{-1}\langle U \rangle$ is open and $f^{-1}\langle U \rangle = f^{-1}\langle U - Y_2 \rangle \cup$

$f^{-1}\langle U \cap Y_2 \rangle$. Since $f|_{Y_1}$ is coherent, $f^{-1}\langle U \cap Y_2 \rangle$ is compact, and since $f|_{X_1 - Y_1} : X_1 - Y_1 \to X_2 - Y_2$ is a map of Hausdorff spaces it is coherent so $f^{-1}\langle U - Y_2 \rangle$ is compact. Therefore, $f^{-1}\langle U \rangle$ is compact, and f is coherent. By the Stone representation theorem $f^* : K\Omega(X_2) \to K\Omega(X_1)$, $U \mapsto f^{-1}\langle U \rangle$ is a homomorphism of distributive lattices. Moreover, $f^*(Y_2) = f^{-1}\langle Y_2 \rangle = Y_1$, so f^* is a homomorphism of partially complemented distributive lattices from $(K\Omega(X_2); \emptyset, X_2, Y_2, \cup, \cap)$ to $(K\Omega(X_1); \emptyset, X_1, Y_1, \cup, \cap)$.

Finally, under the isomorphism $\alpha : A \to K\Omega(\mathrm{pt}(\mathrm{Idl}(A)))$, $a \mapsto \varphi(\downarrow\{a\})$ we have $\alpha(h) = \varphi(\downarrow\{h\})$, so α is an isomorphism of partially complemented distributive lattices from $(A; 0, 1, h, \vee, \wedge)$ to $(K\Omega(\mathrm{pt}(\mathrm{Idl}(A))); \emptyset, \varphi(A), \varphi(\downarrow\{h\}), \cup, \cap)$. Under the homeomorphism $\beta : X \to \mathrm{pt}(\mathrm{Idl}(K\Omega(X)))$, $x \mapsto \{U \mid x \in U\}$ we have $x \in Y$ iff $Y \in \beta(x)$ iff $\beta(x) \cap \downarrow\{Y\} \neq \emptyset$ iff $\beta(x) \in \varphi(\downarrow\{Y\})$. So $\beta\langle Y \rangle = \varphi(\downarrow\{Y\})$, and β is an isomorphism of partially Stone spaces from (X, Y) to $(\mathrm{pt}(\mathrm{Idl}(K\Omega(X))), \varphi(\downarrow\{Y\}))$. $\square$

Given a coherent space X, and subspaces Y_1, Y_2 such that (X, Y_1) and (X, Y_2) are partially Stone spaces, it is clear that $(X, Y_1 \cap Y_2)$ is also a partially Stone space. However there is not necessarily a smallest Y such that (X, Y) is a partially Stone space. Consider an infinite set X with the cofinite topology. For any cofinite $Y \subseteq X$, (X, Y) is a partially Stone space.

3. Free spectrum

In this section we will give a description of the free Boolean semirings, and obtain some bounds for the size of the finitely generated free Boolean semirings. In Corollary 1.8 we already have an upper bound.

Let X be a set, S the Boolean semiring $\{0, h, 1\}$ with the operations given in Theorem 1.7, and B the sublattice $\{0, 1\}$ of S. Let S^X (resp. B^X) denote the set of all functions from X to S (resp. B). Let $[S^X \to S]$ (resp. $[B^X \to S]$) denote the semiring of all functions from S^X (resp. B^X) to S under pointwise operations. We embed X in $[S^X \to S]$ as follows: if $x \in X$, let x also denote the projection map onto the xth coordinate, i.e. if $\sigma \in S^X$, then $x(\sigma) = \sigma(x)$. We can identify the free Boolean semiring generated by X, $F_{\mathscr{BSR}}(X)$ with the subsemiring of $[S^X \to S]$ generated by X.

Lemma 3.1. *Let X be a set, $x \in X$, $f \in F_{\mathscr{BSR}}(X)$ and let $\sigma_1, \sigma_2, \sigma_3 \in S^X$ be such that $\sigma_1(x) = 0$, $\sigma_2(x) = h$, $\sigma_3(x) = 1$, and for $y \neq x$, $\sigma_1(y) = \sigma_2(y) = \sigma_3(y)$. Then*

$$(*) \qquad f(\sigma_2) = f(\sigma_1) + f(\sigma_3) + f(\sigma_3)\,.$$

Proof. By induction on f.

If $f = x$ we have $h = 0 + 1 + 1$. If $f = y \in X - \{x\}$, or if f is the constant function 0 or 1, it follows from Lemma 1.2(b).

If f, g satisfy (*), so do $f + g$ and fg. □

For $f \in [S^X \to S]$ let $f|_{B^X}$ be the restriction of f to B^X.

Proposition 3.2. *Let X be a set. The map $F_{\mathcal{BSR}}(X) \to [B^X \to S]$, $f \mapsto f|_{B^X}$ is one-to-one.*

Proof. Let $f, g \in F_{\mathcal{BSR}}(X)$, be such that $f|_{B^X} = g|_{B^X}$. It is clear that there is a finite $Y \subseteq X$ such that $f, g \in F_{\mathcal{BSR}}(Y)$, so without loss of generality, we may assume that X is finite. Let $\sigma \in S^X$, $k = |\sigma^{-1}\langle\{h\}\rangle|$, and $m = 3^k$. By Lemma 3.1 there are $\sigma_1, \ldots, \sigma_m \in B^X$ such that $f(\sigma) = f(\sigma_1) + \cdots + f(\sigma_m)$ and $g(\sigma) = g(\sigma_1) + \cdots + g(\sigma_m)$. Since $f(\sigma_i) = f|_{B^X}(\sigma_i) = g|_{B^X}(\sigma_i) = g(\sigma_i)$, $f(\sigma) = g(\sigma)$. □

Corollary 3.3. $2^{2^n} \le |F_{\mathcal{BSR}}(n)| \le 3^{2^n}$.

Proof. The first inequality follows from Theorem 1.6 since the free Boolean ring on n generators has cardinality 2^{2^n}. The second inequality follows from Proposition 3.2. □

From this point on, we will identify $F_{\mathcal{BSR}}(X)$ with the subsemiring of $[B^X \to S]$ which is the image of the map in Lemma 3.2.

Let X be a set and $I \subseteq B^X$. We say that I is a *lower subset* of B^X if $\sigma \in I$ and $\tau \le \sigma$ imply $\tau \in I$.

When X is finite, the free bounded distributive lattice generated by X, $F_{\mathcal{DL}}(X)$ is the sublattice of $[B^X \to B]$ generated by X when embedded via the projection maps, and it is isomorphic to the lattice of lower subsets of B^X [3, p. 61]. This isomorphism maps each lower subset of B^X to the characteristic function of its complement. In other words:

Theorem 3.4. *Let X be a finite set, and $f : B^X \to B$. $f \in F_{\mathcal{DL}}(X)$ iff $f^{-1}\langle\{0\}\rangle$ is a lower subset of B^X.* □

By enlarging the codomain, we can view any $f \in F_{\mathcal{DL}}(X)$ as a map $f : B^X \to S$.

Proposition 3.5. *Let X be a set. $F_{\mathcal{DL}}(X) \subseteq F_{\mathcal{BSR}}(X)$ when both are viewed as subsets of $[B^X \to S]$.*

Proof. $F_{\mathcal{DL}}(X)$ is the bounded sublattice of $[B^X \to S]$ generated by X. By Theorem 2.1 and Corollary 3.2, $F_{\mathcal{BSR}}(X)$ is the bounded sublattice with $^-$ of $[B^X \to S]$ generated by X. □

From Theorem 3.4 and Proposition 3.5 we get the following corollary.

Corollary 3.6. *Let X be a finite set and $f : B^X \to S$. If $f^{-1}\langle\{0\}\rangle$ is a lower subset of B^X and $f^{-1}\langle\{h\}\rangle = \emptyset$, then $f \in F_{\mathcal{BSR}}(X)$.*

We now obtain a necessary and sufficient condition for $f : B^X \to S$ to belong to $F_{\mathcal{BSR}}(X)$.

Theorem 3.7. *Let X be a finite set and $f \in [B^X \to S]$. $f \in F_{\mathcal{BSR}}(X)$ iff $f^{-1}\langle\{0\}\rangle$ is a lower subset of B^X.*

Proof. ($\Rightarrow$) By induction on f.

If f is one of the constants 0 or 1, then $f^{-1}\langle\{0\}\rangle$ is either empty or B^X which are lower subsets of B^X. If $f = x \in X$, and $\tau \leq \sigma \in f^{-1}\langle\{0\}\rangle$, then $f(\tau) = \tau(x) \leq \sigma(x) = f(\sigma) = 0$, so $f(\tau) = 0$ and $\tau \in f^{-1}\langle\{0\}\rangle$. Let $f, g \in F_{\mathcal{BSR}}(X)$ and suppose $f^{-1}\langle\{0\}\rangle$, and $g^{-1}\langle\{0\}\rangle$ are lower subsets of B^X. From the operation tables of S in Theorem 1.7 we see that $(f+g)^{-1}\langle\{0\}\rangle = f^{-1}\langle\{0\}\rangle \cap g^{-1}\langle\{0\}\rangle$, and $(fg)^{-1}\langle\{0\}\rangle = f^{-1}\langle\{0\}\rangle \cup g^{-1}\langle\{0\}\rangle$, which are again lower subsets of B^X.

($\Leftarrow$) By induction on $|f^{-1}\langle\{h\}\rangle|$.

If $|f^{-1}\langle\{h\}\rangle| = 0$, then by Corollary 3.6 $f \in \mathcal{F}_{\mathcal{BSR}}(X)$. If $|f^{-1}\langle\{h\}\rangle| > 0$, let $\sigma \in f^{-1}\langle\{h\}\rangle$, and let $g : B^X \to S$ be given by $g(\tau) = f(\tau)$ if $\tau \neq \sigma$, and $g(\sigma) = 1$. Since $|g^{-1}\langle\{h\}\rangle| < |f^{-1}\langle\{h\}\rangle|$, and $g^{-1}\langle\{0\}\rangle = f^{-1}\langle\{0\}\rangle$, by induction $g \in F_{\mathcal{BSR}}(X)$. Moreover, by Corollary 3.6, the maps $u, v : B^X \to S$ given by

$$u(\tau) = \begin{cases} 1 & \text{if } \tau \geq \sigma\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad v(\tau) = \begin{cases} 1 & \text{if } \tau > \sigma\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

are also in $F_{\mathcal{BSR}}(X)$. We claim that $f = g + u + v$.

In fact, $g(\sigma) + u(\sigma) + v(\sigma) = 1 + 1 + 0 = h = f(\sigma)$. If $\tau > \sigma$, since $f^{-1}\langle\{0\}\rangle$ is a lower subset of B^X and $f(\sigma) = h \neq 0$, then $f(\tau) \neq 0$. Therefore, $g(\tau) + u(\tau) + v(\tau) = f(\tau) + 1 + 1 = f(\tau) + h = f(\tau)$. But if $\tau \not\geq \sigma$, then $g(\tau) + u(\tau) + v(\tau) = f(\tau) + 0 + 0 = f(\tau)$. Therefore, $f = g + u + v \in F_{\mathcal{BSR}}(X)$. $\square$

Corollary 3.8. *When we identify $F_{\mathcal{SS}}(n)$ with the set of lower subsets of B^n,*

$$|F_{\mathcal{BSR}}(n)| = \sum_{I \in F_{\mathcal{SS}}(n)} 2^{|I|}\,.$$

Proof. Let I be a lower subset of B^n and let I' denote its complement. There are $2^{|I'|}$ elements $f \in F_{\mathcal{BSR}}(n)$ such that $f^{-1}\langle\{0\}\rangle = I$. $\square$

Let $l_{n,k}$ denote the number of lower subsets of B^n of size k; these numbers are denoted by $L_k(n)$ in [8]. Let $p_n(x) = \sum_{k=1}^{2^n} l_{n,k} x^k$. Then $p_n(1) = |F_{\mathcal{SS}}(n)|$ and $p_n(2) = |F_{\mathcal{BSR}}(n)|$. It should be noted that the problem of determining $|F_{\mathcal{SS}}(n)|$

Table 1

n	$F_{\mathscr{L}\mathscr{S}}(n)$	$F_{\mathscr{B}\mathscr{S}\mathscr{R}}(n)$
0	2	3
1	3	7
2	6	35
3	20	775
4	168	319107
5	7581	42122976711
6	7828354	$5.1496265927 \times 10^{20}$

(Dedekind's problem) is a difficult one, and the values are known for $n \leq 7$ [2]. The previous remarks suggest that the problem for $|F_{\mathscr{B}\mathscr{S}\mathscr{R}}(n)|$ is about as difficult as Dedekind's problem. We have computed the values for $n \leq 6$, see Table 1.

Corollary 3.9. $\log_2|F_{\mathscr{B}\mathscr{S}\mathscr{R}}(n)| \sim 2^n$.

Proof. From Corollaries 3.3 and 3.8 we have $2^{2^n} \leq |F_{\mathscr{B}\mathscr{S}\mathscr{R}}(n)| \leq 2^{2^n}|F_{\mathscr{L}\mathscr{S}}(n)|$ and therefore $2^n \leq \log_2|F_{\mathscr{B}\mathscr{S}\mathscr{R}}(n)| \leq 2^n + \log_2|F_{\mathscr{L}\mathscr{S}}(n)|$. But from [12] for any $\delta > 0$, $\log_2|F_{\mathscr{L}\mathscr{S}}(n)| \leq 2^n n^{-1/2+\delta}$ for n large enough. $\square$

4. Normal form and relational degree

In this section we obtain a normal form for the elements of $F_{\mathscr{B}\mathscr{S}\mathscr{R}}(X)$.

Definition 4.1. Let X be a set, and $F_{\mathscr{B}\mathscr{S}\mathscr{R}}(X)$ the free Boolean semiring generated by X.

(a) A *monomial* is a product of elements of X. We denote by M the set of all monomials. Observe that $1 \in M$; it is the empty product.

(b) A *polynomial* is a sum of monomials. Observe that 0 is a polynomial, the empty sum.

By (SR6), any element of $F_{\mathscr{B}\mathscr{S}\mathscr{R}}(X)$ can be written as a polynomial. By (BSR2) and Lemma 1.2(d) any monomial can be written without repeated factors. In fact, M can be identified with the set of finite subsets of X, by mapping each finite subset of X to the product of its elements. This correspondence is order reversing, i.e. $m \leq m'$ iff $m' \subseteq m$, for any $m,m' \in M$. By Lemma 1.2(b) a polynomial can be written without repeating any monomial more than twice, and $m + m = hm$ where $h = 1 + 1$. Moreover, if the monomial m appears in a polynomial t and $m' \in M$ is such that $m' \leq m$, then by Lemma 1.2(c) $m + m' + m' = m$ and t can be written so that m' does not appear more than once.

Theorem 4.2. *Any $t \in F_{\mathcal{BSR}}(X)$ can be written in a unique way as*

$$t = \sum_{m \in M} t_m m ,$$

where $t_m \in \{0, h, 1\}$ satisfies:

(i) *$t_m \neq 0$ for finitely many m's, and*

(ii) *$t_{m'} = h$, $m' < m \Rightarrow t_m = 0$.*

Proof. Condition (i) simply says that the right-hand side is a polynomial. Observe that (ii) is equivalent to $t_m \neq 0$, $m' < m \Rightarrow t_{m'} \neq h$, and we have just shown that any $t \in F_{\mathcal{BSR}}(X)$ can be written as a polynomial satisfying this condition.

For uniqueness let $t = \sum t_m m$ and $s = \sum s_m m$ where $\{t_m\}$ and $\{s_m\}$ satisfy (i) and (ii), and at least some $t_m \neq s_m$. The set of finite subsets of X has the descending chain condition, so M has the ascending chain condition. Let m_0 be maximal in $\{m \mid t_m \neq s_m\}$. Without loss of generality we can assume $t_{m_0} \neq 0$. Let $\sigma : X \to S$ be given by

$$\sigma(x) = \begin{cases} 1 & \text{if } m_0 \leq x , \\ 0 & \text{otherwise} . \end{cases}$$

Observe that for any $m \in M$, $m(\sigma) = 1$ if $m_0 \leq m$, and $m(\sigma) = 0$ otherwise. So we have

$$t(\sigma) = \sum_{m \in M} t_m m(\sigma) = \sum_{m \geq m_0} t_m m(\sigma) = \sum_{m \geq m_0} t_m = t_{m_0} + \sum_{m > m_0} t_m$$

and by the choice of m_0,

$$s(\sigma) = s_{m_0} + \sum_{m > m_0} s_m = s_{m_0} + \sum_{m > m_0} t_m .$$

If $t_{m_0} = 1$, then $s_{m_0} \neq 1$ and for any $a \in S$, $t_{m_0} + a \neq s_{m_0} + a$. In particular, $t(\sigma) \neq s(\sigma)$.

If $t_{m_0} = h$, then by (ii) $t_m = 0$ for any $m > m_0$ and $t(\sigma) = t_{m_0} \neq s_{m_0} = s(\sigma)$. Therefore, $t \neq s$. $\square$

If we identify $t \in F_{\mathcal{BSR}}(X)$ with the function $M \to S$, $m \mapsto t_m$, then condition (ii) says that $f^{-1}(\{h\})$ is an antichain and the lower subset under it is mapped to 0. For X finite, condition (i) always holds, and we get another proof of Corollary 3.8.

Definition 4.3. Let A be an algebra, l a cardinal, ρ an l-ary relation on A and $f : A^n \to A$ an n-ary operation.

(a) We say that *f preserves ρ* if $a_1, \ldots, a_n \in \rho$ imply $f(a_1, \ldots, a_n) \in \rho$, where f is applied componentwise on the l-tuples, i.e. $f(a_1, \ldots, a_n)_j = f(a_{1,j}, \ldots, a_{n,j})$.

(b) We say that ρ is a *classifying relation* for A if $\{f \mid f \text{ preserves } \rho\}$ is the clone of polynomials over A, $\mathrm{cl}(A)$.

(c) The least l such that there is an l-ary classifying relation for A is called the *relational degree of* A.

(d) If $(\sigma_1, \sigma_2, \ldots)$ is an l-tuple of n-tuples of A, then $(\sigma_1, \sigma_2, \ldots)^{\mathrm{t}}$ denotes the n-tuple of l-tuples of A given by $(\sigma_1, \sigma_2, \ldots)^{\mathrm{t}}_{i,j} = (\sigma_1, \sigma_2, \ldots)_{j,i}$. Therefore, $f(\sigma_1, \sigma_2, \ldots)^{\mathrm{t}} = (f(\sigma_1), f(\sigma_2), \ldots)$.

The main theorem in [9] shows that the relational degree is well defined for any algebra. For a finite algebra it is at most $\aleph_0$, and a 3-element algebra is more likely to have infinite relational degree. We will show that S has relational degree 3.

From Lemma 3.1 and Theorem 3.7 one easily gets the following corollary:

Corollary 4.4. *$f : S^n \to S$ is in* $\mathrm{cl}(S)$ *iff f satisfies* $(*)$ *in Lemma* 3.1 *and* $f^{-1}\langle\{0\}\rangle$ *is a lower subset of S^n.* □

Proposition 4.5. *The* 3*-ary relation* $\rho = \{(0,0,0), (0,h,h), (h,h,h), (1,1,h), (0,h,1), (h,h,1), (1,1,1)\}$ *is a classifying relation for S.*

Proof. $\mathrm{cl}(S)$ preserves ρ since it is a subsemiring of S^3 (it is in fact $F_{\mathcal{BSR}}(1)$).

Suppose now that $f : S^n \to S$ preserves ρ. If $\sigma_1, \sigma_2, \sigma_3 \in S^n$ are like in $(*)$ in Lemma 3.1, then $(\sigma_1, \sigma_2, \sigma_3)^{\mathrm{t}} \in \rho^n$ and therefore $(f(\sigma_1), f(\sigma_2), f(\sigma_3)) \in \rho$. It is now easy to verify that $f(\sigma_2) = f(\sigma_1) + f(\sigma_3) + f(\sigma_3)$.

If $\tau \leq \sigma \in S^n$ are such that $f(\sigma) = 0$, let $\mu = \tau + \sigma + \sigma$. It is easy to see that $(\tau, \mu, \sigma)^{\mathrm{t}} \in \rho^n$ and therefore $(f(\tau), f(\mu), f(\sigma)) \in \rho$. Since $f(\sigma) = 0$ we must have $f(\tau) = 0$. So $f^{-1}\langle\{0\}\rangle$ is a lower subset of S^n. By Corollary 4.4, f is in $\mathrm{cl}(S)$. □

Theorem 4.6. *The relational degree of S is* 3.

Proof. By Proposition 4.5 we have that relational degree of $S \leq 3$.

Suppose $\rho' \subseteq S^2$ is a classifying relation for S. Then ρ' is a subsemiring of S^2. We will denote any $f : S \to S$ by the triple $(f(0), f(h), f(1))$. We will show that there is $f : S \to S$ which preserves ρ' but is not in $\mathrm{cl}(S)$. Consider two cases:

(i) If $(1,h) \in \rho'$, then $(h,1) = (1,h) + (1,1) \in \rho'$ and $(1,h,1)$ preserves ρ'. But $(1,h,1) \not\in \rho = F_{\mathcal{BSR}}(1)$, and therefore $(1,h,1) \not\in \mathrm{cl}(S)$.

(ii) If $(1,h) \not\in \rho'$, then we must have $\rho' \subseteq \{(0,0), (h,h), (1,1), (0,h), (h,0)\}$ and $(0,0,1)$ preserves ρ' but $(0,0,1) \not\in \mathrm{cl}(S)$. □

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