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Compactly supported solutions to stationary degenerate diffusion equations

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ABSTRACT

We consider non-negative solutions of the semilinear elliptic equation in $\mathbb{R}^n$ with $n \geqslant 3$:

$$-\Delta u = a(x)u^q + b(x)u^p,$$

where $0 < q < 1$, $p > q$, $a(x)$ sign-changing, $a = a^+ - a^-$ and $b(x) \leqslant 0$ is non-positive. Under appropriate growth assumption on a^- at infinity, we prove that all solutions in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ are compactly supported and their support is contained in a large ball with radius determined by a. When $\Omega^{0+} = \{x \in \mathbb{R}^n \mid a(x) \geqslant 0\}$ has several compact connected components, we give conditions under which there may or may not exist solutions which vanish identically on one or more of the components of Ω^{0+}. For instance, we introduce a positive parameter λ and replace a by $\lambda a^+ - a^-$. We then show that for λ small, all solutions have compact support and there exist solutions with supports in any combination of these connected components of Ω^{0+}. For λ large and $p \leqslant 1$ the solution is unique and supported in all of Ω^{0+}. We also prove the existence of the limit $\lambda \to \infty$ of this solution, which solves $-\Delta w = a^+ w^q$ and $\lim_{|x|\to\infty} w(x) = 0$. The analysis is based on comparison arguments and *a priori* bounds.

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1. Introduction

In this paper, we study the semilinear elliptic problem in $\mathbb{R}^n$ with $n \geqslant 3$:

$$\begin{cases} -\Delta u = a(x)u^q + b(x)u^p & \text{in } \mathbb{R}^n,\ 0 < q < 1,\ p > q, \\ u \geqslant 0 & \text{in } \mathbb{R}^n,\ u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $a(x)$ and $b(x)$ are locally Hölder continuous, and $b(x) \leqslant 0$. By $\mathcal{D}^{1,2}(\mathbb{R}^n)$ we mean the completion of $C_0^\infty(\mathbb{R}^n)$ under the Dirichlet seminorm, $(\int_{\mathbb{R}^n} |\nabla u|^2\,dx)^{1/2}$. The important feature of this equation is that it combines a non-Lipschitz nonlinearity with a sign-changing coefficient $a(x)$, and it was originally observed by Schatzman [22] that solutions could vanish on large sets and in fact that, under appropriate hypotheses on $a(x)$, there exist solutions with compact support. Our main goal in this paper is to give general conditions on the coefficients $a(x)$ and $b(x)$ which guarantee that solutions must have compact support in $\mathbb{R}^n$, and more generally to study a phenomenon related to “dead cores”, regions in $\mathbb{R}^n$ in which solutions vanish identically (see [2,4,5]).

Equations of this type (1.1) arise as stationary solutions to the degenerate reaction–diffusion equations of the form,

$$w_t - \Delta\big(w^m\big) = w\big(a(x) + b(x)w^s\big), \quad m > 1,\ s > 0,$$

with nonlinearity of “logistic” type. Assuming time-independence and making the change of variable $u = w^m$, we arrive at (1.1) with $q = 1/m$ and $p = (s+1)/m > q$. If $w(x,t)$ represents a population density, then $a(x)$ represents a sort of growth rate, and the region where $a(x) > 0$ is favorable to population growth, whereas the region where $a(x) < 0$ is hostile to the species. It is therefore natural to assume that

$$\Omega^+ = \big\{x \in \mathbb{R}^n \;\big|\; a(x) > 0\big\} \text{ is not empty and bounded.} \tag{1.2}$$

Under this hypothesis alone we show that a solution exists:

Theorem 1.1. *Assume* (1.2)*, there exists a classical solution* $U \in \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *of* (1.1)*. Moreover* $U \leqslant \omega$*, where* ω *is the unique positive solution to*

$$-\Delta\omega = a^+\omega^q \quad \text{in } \mathbb{R}^n, \qquad \lim_{|x|\to\infty} \omega = 0. \tag{1.3}$$

Here we denote as usual, $a^+(x) = \max(0, a(x))$ and $a^-(x) = \max(0, -a(x))$. The existence and uniqueness of ω follows from Theorem 2′ in [8].

In order for solutions to have compact support, we require more information about $a(x)$. Let $\Omega^{0+} = \{x \in \mathbb{R}^n \mid a(x) \geqslant 0\}$. In case Ω^{0+} is unbounded, the strong maximum principle leads us to expect that in general solutions will not have compact support. So to obtain compactly supported solutions we will assume

$$\Omega^{0+} \text{ is bounded and } \Omega^+ \text{ is non-empty.} \tag{1.4}$$

However, this hypothesis is not sufficient, and in addition we must impose some conditions on the decay of the negative part $a^-(x)$. We prove:

Theorem 1.2. *Assume* (1.4)*. There exist* $\eta > 0$ *and* $\rho > 0$ *so that if*

$$\liminf_{|x|\to\infty} a^-|x|^{(n-2)(1-q)} > \eta, \tag{1.5}$$

every weak solution u *of* (1.1) *is classical and compactly supported, with* $\operatorname{supp}(u)$ *contained in* $B(0,\rho)$.

To understand the interdependence of the constants in Theorem 1.2, think of $a^+(x)$ as being fixed and consider how the asymptotic behavior of $a^-(x)$ affects the support of the solution. By the hypothesis (1.4) we can choose $\rho_1 > 0$ so that $\Omega^{0+} \Subset B(0, \rho_1)$. Then, the constant η will depend on q, p, n, ρ_1 and $\|a^+\|_{L^\infty(\mathbb{R}^n)}$, and the constant ρ will depend on q, p, n, ρ_1, $\|a^+\|_{L^\infty(\mathbb{R}^n)}$ and the asymptotic behavior of a^- near infinity. If a^- is decaying too rapidly to zero, then we may not have compact supported solutions (see Remark 2.12 at the end of Section 2). Nevertheless, we cannot claim that condition (1.5) is sharp, as we will see in Theorem 1.9 below.

In case $a(x)$ is bounded away from zero at infinity the result is somehow simpler:

Corollary 1.3. *Assume* (1.4)*, and suppose that there exist $\alpha > 0$ and $\rho_1 > 0$ such that*

$$a^-(x) \geqslant \alpha \quad \text{for all } |x| \geqslant \rho_1. \tag{1.6}$$

Then, there exists $R > 0$ so that every weak solution u has support $\operatorname{supp}(u) \subset B(0, R)$*. Moreover, R depends only on q, p, n, ρ_1, and $\|a^+\|_{L^\infty(\mathbb{R}^n)}$.*

Note that we cannot expect the result of Theorem 1.2 to hold in $\mathbb{R}^1$ or $\mathbb{R}^2$. Solutions to (1.1) in $\mathbb{R}^1$ or $\mathbb{R}^2$ will be compactly supported under the hypothesis (1.6), but we cannot prove the uniform control on the support in terms of the coefficients as in Corollary 1.3. Note also that if $q \geqslant 1$, by the strong maximum principle there could not be any compactly supported solution at all, so the sublinearity of q is crucial for the existence of solution with compact support.

We also study the structure of the solution set of (1.1) in case the favorable domain Ω^+ has several components. We make the following assumption on Ω^+:

$$\begin{cases} \text{(1.4) holds, } \ \Omega^+ \text{ has } k < \infty \text{ connected components with } \Omega^+ = \bigcup_{i=1}^k \Omega_i^+, \\ \text{and each connected component } \Omega_i^+ \text{ satisfies an interior ball condition.} \end{cases} \tag{1.7}$$

Set $M = \{1, 2, 3, \dots, k\}$. Under (1.7), for any solution $u(x)$ of (1.1), by the strong maximum principle it is easy to see that solution $u(x)$ is either positive in Ω_i^+ or completely vanishes in Ω_i^+ for any $i \in M$ (see Lemma 3.11). We make the following definition:

Definition 1.4. (1) For any non-empty subset $I \subset M = \{1, 2, \dots, k\}$, denote by S_I the class of solutions of (1.1) which are positive in $\Omega_I^+ = \bigcup_{i \in I} \Omega_i^+$.

(2) N_I denotes the set $\{u \in S_I \mid u \equiv 0 \text{ in } \Omega^+ - \Omega_I^+\}$.

For any non-empty $I \subset M$, a solution in N_I vanishes identically in $\Omega^+ - \Omega_I^+$, part of the favorable set Ω^+. We will call such solutions *dead core solutions*, in analogy with the classical use of the term [2,5,19]. This is a slight abuse of this term: in previous usage, it refers to solutions of boundary-value problems which vanish on compact subdomains, while our solutions will vanish on much larger, non-compact regions. Nevertheless, the phenomena are clearly related via the failure of the strong maximum principle for degenerate equations or non-Lipschitz nonlinearities.

If dead core solutions (as defined above) do exist, the class S_I can contain many elements: see the following proposition and Theorem 1.9. However, in many cases the class N_I can have at most one solution. Following the idea in [3], we present a generalization of the uniqueness result of Spruck [23]:

Theorem 1.5. *Assume* (1.7)*, if $p \geqslant 1$, then the number of elements in N_I is at most* 1 *for any non-empty I. In particular if $k = 1$, then the solution to* (1.1) *is unique and its support is connected.*

We may also prove uniqueness of solutions in class N_I with $q < p < 1$ under some additional hypotheses on $b(x)$; see Theorem 3.14.

When Theorem 1.5 applies, the solution space of (1.1) is completely characterized by the support properties of the solutions. Consider the following example: let Ω_i^+, $i = 1, \dots, k$, be any smooth,

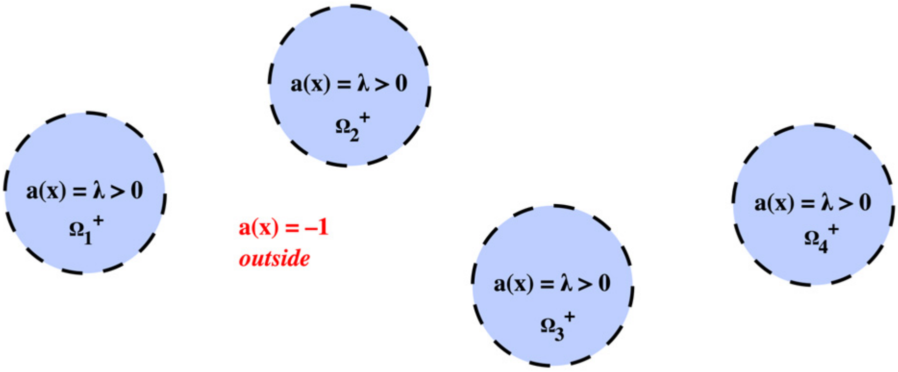

Fig. 1. $a(x)$ as in the example (1.8).

compact and connected open sets in $\mathbb{R}^n$, with $\min_{i\neq j}\operatorname{dist}(\Omega_i^+,\Omega_j^+)>0$. Let $b(x)\equiv -1$ and define $a(x)=a_\lambda(x)$ by

$$a(x)=\begin{cases}\lambda, & \text{if } x\in\Omega_i^+,\ i=1,\dots,k,\\ -1, & \text{if } x\notin\bigcup_{i=1}^k\Omega_i^+,\end{cases}\tag{1.8}$$

where $\lambda>0$ is a fixed constant. (See Fig. 1.) Combining the results of Corollary 1.3 and Theorem 1.5, we have:

Proposition 1.6. *Assume $a(x)$ is defined as in* (1.8) *with fixed $\lambda>0$, $b(x)\equiv -1$, and $p\geqslant 1$.*

1. *There exists $\delta^*=\delta^*(\Omega^+,\lambda)>0$ so that if $\min_{i\neq j}\operatorname{dist}(\Omega_i^+,\Omega_j^+)\geqslant\delta^*$, then N_I contains* exactly one *solution for each $I\neq\emptyset$.*
2. *There exists $\delta_*=\delta_*(\Omega^+,\lambda)>0$ so that if $\max_{i\neq j}\operatorname{dist}(\Omega_i^+,\Omega_j^+)\leqslant\delta_*$, then* (1.1) *admits* exactly one *solution in $\mathbb{R}^n$. This solution is positive on the set Ω^{0+}.*

Note that in case (1), Eq. (1.1) admits exactly 2^k-1 nontrivial solutions in all.

In the general case, we can assert the existence of a minimal and maximal element in S_I for each I:

Theorem 1.7. (a) *Under the hypothesis* (1.7), *S_I has a minimal element $u_I\in\mathcal{D}^{1,2}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ for any non-empty $I\subset M$.*

(b) *Assuming* (1.2), (1.1) *always has a maximal solution U in $\mathcal{D}^{1,2}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, which is positive in Ω^+ and bounded above by ω from* (1.3). *In particular under* (1.7), *S_I has the same maximal element U for any non-empty $I\subset M$.*

There is a connection between the maximal solution of (1.1) and the minimal energy solution of the functional $E:\mathcal{D}^{1,2}(\mathbb{R}^n)\cap L^{q+1}(\mathbb{R}^n)\to\mathbb{R}$, defined by

$$E(v)=\int_{\mathbb{R}^n}|\nabla v|^2\,dx-\frac{1}{q+1}\int_{\mathbb{R}^n}a(v^+)^{q+1}\,dx-\frac{1}{p+1}\int_{\mathbb{R}^n}b(v^+)^{p+1}\,dx.$$

If we assume $a(x)$ and $b(x)$ are uniformly bounded in $\mathbb{R}^n$, then E is smooth (see [12]). The interesting fact is that $\inf_{v\in\mathcal{D}^{1,2}(\mathbb{R}^n)\cap L^{q+1}(\mathbb{R}^n)}E(v)$ is achieved at a non-negative function U, which is a solution of (1.1). By minimization it is easy to see that $U>0$ in all connected components of Ω^+, that is $U\in S_M$.

By Theorem 1.5, when $p \geqslant 1$ this minimal energy solution is the (unique) maximal solution in S_M. On the other hand, as pointed out in [1], if there do exist dead core solutions (in some class N_I, $I \neq \emptyset$) then these solutions cannot be energy minimizers even modulo any finite dimensional subspace. As a result, our study of dead core solutions to (1.1) will rely on comparison arguments, monotone iteration, and *a priori* estimates.

Another way to influence the support properties of the solutions is by varying the relative strengths of the positive and negative parts of $a(x)$. We introduce a parameter $\lambda > 0$, and set

$$a_\lambda(x) = \lambda a^+(x) - a^-(x).$$

Clearly this does not affect the geometry of the favorable and unfavorable regions, and Ω^+, Ω^{0+} and Ω^- remain the same for the whole family of a_λ. Intuitively, we expect that the size of the support of the solutions of

$$-\Delta u = a_\lambda(x)u^q + bu^p \quad \text{in } \mathbb{R}^n,\ u \geqslant 0 \text{ and } u \in \mathcal{D}^{1,2}\big(\mathbb{R}^n\big), \tag{1.9}$$

should grow with increasing λ. We will study the asymptotic behavior of solutions for large and small λ, but we first require a further condition on the geometry of the set Ω^{0+}:

Definition 1.8. We say $a(x)$ is *admissible* if (1.7) holds, and:

1. Ω^{0+} also has exactly k connected components with $\Omega^{0+} = \bigcup_{i=1}^k \Omega_i^{0+}$,
2. $\Omega_i^+ \subset \Omega_i^{0+}$ for $i \in M$,
3. $\mathrm{dist}(\Omega_i^{0+}, \Omega_j^{0+}) > 0$ for $i \neq j$.

For admissible a we have the following theorem:

Theorem 1.9. *Assume $a(x)$ is admissible, there exists $\lambda_* > 0$ so that for all $\lambda < \lambda_*$, all solutions of* (1.9) *are compactly supported and $N_I^q \neq \emptyset$ for any non-empty collection $I \subset M$.*

Note that for $\lambda < \lambda_*$ the compact support of the solutions follows even in the absence of asymptotic conditions on $a(x)$, showing that condition (1.5) cannot be sharp. To emphasize the dependence on λ, Eq. (1.9) is often referred as $(1.9)_\lambda$ (the subscript λ is omitted if no confusion arises). For λ big, we have the following theorem in case $q < p \leqslant 1$.

Theorem 1.10. *Assume* (1.7), *if $q < p \leqslant 1$, there exists $\lambda_1^* > 0$ so that Eq. $(1.9)_\lambda$ has a unique solution u_λ, which is positive in $\overline{\Omega^+}$ for all $\lambda > \lambda_1^*$. Moreover,*

$$\begin{cases} \textit{if } p < 1, & \lim\limits_{\lambda\to\infty} \lambda^{\frac{1}{q-1}} u_\lambda = \omega, \quad \textit{where } \omega \textit{ is from } (1.3), \\ \textit{if } p = 1\ , & \lim\limits_{\lambda\to\infty} \lambda^{\frac{1}{q-1}} u_\lambda = \overline{\omega}, \quad \textit{where } \overline{\omega} \textit{ is in } (4.3). \end{cases}$$

In particular, combining Theorem 1.2 and the above theorem we conclude that as $\lambda \to \infty$ the support must grow.

Corollary 1.11. *Assume* (1.7), *if $q < p \leqslant 1$ and $\liminf_{|x|\to\infty} a^-|x|^{(n-2)(1-q)} = \infty$, then there exists $\lambda_2^* > 0$ so that problem* (1.9) *has a unique compactly supported solution u_λ with $u_\lambda > 0$ in Ω^+ for all $\lambda \geqslant \lambda_2^*$. Moreover,* $\mathrm{supp}(u_\lambda)$ *expands to infinity as λ goes to infinity, that is*

$$\bigcup_{\lambda > \lambda_2^*} \mathrm{supp}(u_\lambda) = \mathbb{R}^n.$$

For the case $p > 1$, the asymptotic behavior is more complicated, and depends strongly on the form of $b(x)$. Some specific results are proven in Theorem 4.7 in Section 4.

To illustrate our results on the parametrized problem (1.9), we return to the previous piecewise constant $a(x)$ from our example (1.8). For simplicity, assume $b(x) \equiv 0$. For λ small enough, Theorem 1.9 applies and we conclude that (1.9) admits a unique solution in N_I for all $I \neq \emptyset$, so (1.9) admits exactly $2^k - 1$ solutions in all, and all but one has dead cores. For λ sufficiently large, by Theorem 1.10 the equation has exactly one solution which is positive in all of Ω^+.

The existence of compactly supported solutions for equations on $\mathbb{R}^n$ with non-Lipschitz nonlinearities was originally proven by Schatzman [22], using Puel's existence theorem [20]. Spruck [23] imposed a monotonicity condition, $x \cdot \nabla a(x) < 0$, and considered (1.1) with $b(x) \equiv 0$. Spruck proved uniqueness of compactly supported solutions by means of a special version of Hopf's boundary lemma. He also proved that, under the same hypotheses, the support of the solution is star-shaped with Lipschitz boundary. In [3,4] C. Bandle, M.A. Pozio and A. Tesei studied dead core solutions for this problem in a bounded domain with both Dirichlet and Neumann boundary condition, and S. Alama [1] used a bifurcation analysis for dead cores in the Neumann problem for similar equations. Most other previous work on equations of the form (1.1) has been for bounded domains or for constant coefficient equations in the whole space $\mathbb{R}^n$: see [6,9,11,13,15,17,18] and the references therein.

Our method for proving compact support is inspired by the paper of Cortázar, Elgueta and Felmer [11] on the constant coefficient equation $-\Delta u = u^p - u^q$ in $\mathbb{R}^n$. In this paper, they prove that all H^1 solutions of the equation must have compact support. (See Gui [15] for an alternative approach to this problem.) In this paper we follow [11] to prove Theorem 1.2, by deriving *a priori* estimates and applying comparison theorems. This is the content of Section 2. In Section 3 we deal with existence and uniqueness questions and prove Theorems 1.5 and 1.7 and Proposition 1.6. Section 4 concerns the parametrized problems (1.9). Finally, in Section 5 we discuss solutions which are not in the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and present some examples related to our results.

Finally, we thank the reviewer for the care and patience taken in reading this paper and for the many helpful suggestions.

2. Compact support

In this section we prove Theorem 1.2. The method we use is derived from the approach of Cortázar, Elgueta and Felmer [11] on the constant coefficient equation $-\Delta u = u^p - u^q$.

First we develop a few useful lemmas. Assume (1.4), and pick $\rho_1 > 0$ such that $\Omega^{0+} \Subset B(0, \rho_1)$.

Lemma 2.1. *Assume* (1.4). *Then, any weak solution u of* (1.1) *is a classical solution and* $\lim_{|x|\to\infty} u(x) = 0$.

Proof. The regularity of u follows from standard bootstrap arguments; see Appendix B in Struwe [24] or Theorem 0 in Brezis [7]. Since $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, then $u \in L^{2^*}(\mathbb{R}^n)$. Hence for any $\epsilon > 0$, there exists $R(\epsilon) > \rho_1$ such that

$$\|u\|_{L^{2^*}(\mathbb{R}^n - B(0,R))} < \epsilon \quad \text{for all } R > R(\epsilon).$$

So for any $x \in \mathbb{R}^n - B(0, R(\epsilon) + 2)$, we have $B(x, 1) \Subset \mathbb{R}^n - B(0, R(\epsilon))$ and $-\Delta u(y) \leqslant 0$ in $B(x, 1)$. Therefore, by the mean value property of subharmonic function, we conclude:

$$0 \leqslant u(x) \leqslant \frac{1}{|B(x,1)|} \int\limits_{B(x,1)} u(y)\, dy \leqslant C\|u\|_{L^{2^*}(B(x,1))} \leqslant C\epsilon,$$

that is $\lim_{|x|\to\infty} u(x) = 0$. $\square$

Remark 2.2. Note that we only need to assume the minimal hypothesis (1.2) and $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$. On the other hand, obtaining $\lim_{|x|\to\infty} u(x) = 0$ is very crucial to the succeeding arguments. In Section 5, we will discuss solutions whose Dirichlet energy is not assumed to be finite.

The next lemma shows that u not only uniformly tends to zero, but also goes to zero with certain speed, as $|x|$ goes to infinity.

Lemma 2.3. *Assume* (1.4)*, then* $u(x) \leqslant \frac{C}{|x|^{n-2}}$*, where* $C = \|u\|_{L^\infty(\mathbb{R}^n)}\rho_1^{n-2}$ *and* $x \in \mathbb{R}^n - B(0,\rho_1)$*.*

Proof. Let $v = \frac{C}{|x|^{n-2}}$, then by the special choice of C we have

$$-\Delta v(x) = 0 \quad \text{in } \mathbb{R}^n - B(0,\rho_1) \quad \text{and} \quad v(x) \geqslant u(x) \quad \text{on } \partial B(0,\rho_1).$$

Now consider $w = u - v$, then w satisfies:

$$-\Delta w(x) \leqslant 0 \quad \text{in } \mathbb{R}^n - B(0,\rho_1) \quad \text{and} \quad w(x) \leqslant 0 \quad \text{on } \partial B(0,\rho_1).$$

Moreover we see that $\lim_{|x|\to\infty} w(x) = 0$. We now claim that $w(x) \leqslant 0$ in $\mathbb{R}^n - B(0,\rho_1)$. Indeed, otherwise there would exist $x_0 \in \mathbb{R}^n - \overline{B(0,\rho_1)}$ such that $u(x_0) > v(x_0) > 0$. We also notice that $w(x) \leqslant 0$ on $\partial B(0,\rho_1)$ and $\lim_{|x|\to\infty} w(x) = 0$, we may assume w attains its maximum at x_0. Therefore we would have

$$0 \leqslant -\Delta w(x_0) = -\Delta u(x_0) < 0,$$

a contradiction. Hence, we have $u \leqslant v = \frac{C}{|x|^{n-2}}$ in $\mathbb{R}^n - B(0,\rho_1)$. $\square$

We must estimate the L^∞-norm of the solution. We prove:

Proposition 2.4. *Assume* (1.4)*. There exists a constant C so that for any solution u of* (1.1) *we have*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leqslant C\big(q, p, n, \|a^+\|_{L^\infty(\Omega^+)}, \Omega^+\big),$$

where this C tends to zero as $\|a^+\|_{L^\infty(\Omega^+)}$ tends to zero.

The maximum principle yields:

Lemma 2.5. *Assume* (1.2)*, then $\|u\|_{L^\infty(\mathbb{R}^n)}$ is attained in $\overline{\Omega^+}$, i.e. there exists $x_0 \in \overline{\Omega^+}$ such that $\|u\|_{L^\infty(\mathbb{R}^n)} = u(x_0)$.*

Proof. Since $\lim_{|x|\to\infty} u(x) = 0$, we may assume $\|u\|_{L^\infty(\mathbb{R}^n)}$ is attained at x_1, which is not in $\overline{\Omega^+}$. Let Ω be the connected component of $\mathbb{R}^n - \overline{\Omega^+}$, which contains x_1. By the strong maximum principle, $u(x) = u(x_1)$ in $\overline{\Omega}$. Since $\overline{\Omega^+} \cap \overline{\Omega}$ is not empty, we are done. $\square$

We next estimate the Dirichlet energy of the solutions. The essential step in the proof (see Claim below) will also be useful in our existence proofs in the next section:

Lemma 2.6. *Assume* (1.4)*, then we have*

$$\int_{\mathbb{R}^n} |\nabla u|^2\,dx \leqslant \int_{\mathbb{R}^n} a^+ u^{q+1}\,dx.$$

Proof. Since u satisfies the equation $-\Delta u = au^q + bu^p$, multiply both sides of the equation by u and integrate by parts. We have for $R \geqslant \rho_1$

$$\int_{B(0,R)} |\nabla u|^2\,dx - \int_{\partial B(0,R)} \frac{\partial u}{\partial n} u\,dS = \int_{B(0,R)} a^+ u^{q+1}\,dx + \int_{B(0,R)} b u^{p+1}\,dx$$
$$\leqslant \int_{\mathbb{R}^n} a^+ u^{q+1}\,dx. \tag{2.1}$$

Claim. *There exists a sequence* $\{R_n\}$ *and* $\lim_{n\to\infty} R_n = \infty$*, such that* $\int_{\partial B(0,R_n)} \frac{\partial u}{\partial n} u\,dS \to 0$ *as* $n \to \infty$*.*

Indeed, we have the following estimate:

$$\begin{aligned}\left|\int_{\partial B(0,R)} \frac{\partial u}{\partial n} u\,dS\right| &\leqslant \frac{\|u\|_{L^\infty(\mathbb{R}^n)}\rho_1^{n-2}}{R^{n-2}} \int_{\partial B(0,R)} |\nabla u|\,dS \\ &\leqslant \frac{\|u\|_{L^\infty(\mathbb{R}^n)}\rho_1^{n-2}}{R^{n-2}} \|1\|_{L^2(\partial B(0,R))} \|\nabla u\|_{L^2(\partial B(0,R))} \\ &\leqslant \frac{CR^{\frac{n-1}{2}}}{R^{n-2}} \|\nabla u\|_{L^2(\partial B(0,R))} \\ &\leqslant \frac{C}{R^{\frac{n-3}{2}}} \|\nabla u\|_{L^2(\partial B(0,R))}.\end{aligned}$$

Notice $\infty > \|\nabla u\|^2_{L^2(\mathbb{R}^n)} = \int_0^\infty \int_{\partial B(0,R)} |\nabla u|^2\,dS\,dr$, so there exists a sequence $\{R_n\}$ with $\lim_{n\to\infty} R_n = \infty$, such that $\|\nabla u\|_{L^2(\partial B(0,R_n))} \to 0$ as $n \to \infty$. Therefore

$$\int_{\partial B(0,R_n)} \frac{\partial u}{\partial n} u\,dS \to 0 \quad \text{as } n \to \infty,$$

and the claim is proven.

Applying this to (2.1), we have

$$\int_{\mathbb{R}^n} |\nabla u|^2\,dx \leqslant \int_{\mathbb{R}^n} a^+ u^{q+1}\,dx. \qquad \square$$

The next step is a bootstrap argument. Recall ρ_1 is chosen such that $\Omega^{0+} \Subset B(0,\rho_1)$.

Lemma 2.7. *Assume* (1.2)*. For any positive integer* $s \geqslant 2$*, there exists a constant*

$$C' = C'\big(q, p, n, s, \|a^+\|_{L^\infty(\Omega^+)}, \|u\|_{L^{2^*}(\mathbb{R}^n)}, \Omega^+\big)$$

so that

$$\|u\|_{L^{(s+1)\frac{n}{n-2}}(\mathbb{R}^n)} \leqslant C'.$$

Moreover, C' *tends to zero as* $\|a^+\|_{L^\infty(\Omega^+)}$ *and* $\|u\|_{L^{2^*}(\mathbb{R}^n)}$ *tend to zero.*

Proof. We rewrite Eq. (1.1) as

$$-\Delta u + a^- u^q - b u^p = a^+ u^q.$$

Then we just follow the steps in Appendix B in Struwe [24] (or Theorem 0 in Brezis [7]). The term $a^-u^q - bu^p$ are non-negative and so they may be neglected. Note that the existence of the integrals is assured *a priori* without truncation, since a^+ is continuous and has compact support. □

The uniform estimate on u will be derived by comparison with the solution $\omega \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ of

$$-\Delta\omega = a^+(x)\omega^q, \qquad \omega(x) \to 0 \quad \text{as } |x| \to \infty. \tag{2.2}$$

The existence and uniqueness of $\omega \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ follows from Theorem 2′ in Brezis and Kamin [8], using only the hypothesis that Ω^+ is bounded. The decay at infinity is also clear from Lemma 2.1. Note that the conclusions of Lemmas 2.5, 2.6, and 2.7 also hold with ω replacing u.

We are now ready to complete the proof of Proposition 2.4.

Proof of Proposition 2.4. As remarked above, ω satisfies the conclusions of the above Lemmas 2.5, 2.6 and 2.7. From Lemma 2.6, we know that

$$\int_{\mathbb{R}^n} |\nabla\omega|^2\,dx \leqslant \int_{\mathbb{R}^n} a^+\omega^{q+1}\,dx. \tag{2.3}$$

By the Sobolev embedding we also have

$$\int_{\mathbb{R}^n} |\nabla\omega|^2\,dx \geqslant C\|\omega\|^2_{L^{2^*}(\mathbb{R}^n)} \tag{2.4}$$

for some constant C independent of ω. Also by Hölder we obtain

$$\int_{\mathbb{R}^n} a^+\omega^{q+1}\,dx \leqslant \|a^+\|_{L^t(\mathbb{R}^n)} \left\|\omega^{q+1}\right\|_{L^{\frac{2^*}{q+1}}(\mathbb{R}^n)} = \|a^+\|_{L^t(\mathbb{R}^n)}\|\omega\|^{q+1}_{L^{2^*}(\mathbb{R}^n)}, \tag{2.5}$$

where t is the conjugate of $\frac{2^*}{q+1}$. Therefore combine (2.3), (2.4) and (2.5), we get

$$C\|\omega\|^2_{L^{2^*}(\mathbb{R}^n)} \leqslant \|a^+\|_{L^t(\mathbb{R}^n)}\|\omega\|^{q+1}_{L^{2^*}(\mathbb{R}^n)},$$

that is

$$\|\omega\|_{L^{2^*}(\mathbb{R}^n)} \leqslant C\big(\|a^+\|_{L^t(\mathbb{R}^n)}\big)^{\frac{1}{1-q}}.$$

Choosing s so that $(s+1)\frac{n}{n-2} \geqslant \frac{(n+1)}{2}$, from Lemma 2.7, we know that $\|\omega\|_{L^{(s+1)\frac{n}{n-2}}(\mathbb{R}^n)}$ is uniformly bounded. Apply the standard elliptic estimate (see [14]) on the domain $\Omega^+ \Subset \hat{O} = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \Omega^{0+}) < \epsilon\}$, where ϵ is chosen so small that $\|a^+\|_{L^\infty(\hat{O})} = \|a^+\|_{L^\infty(\Omega^+)}$ and volume $(\hat{O}) \leqslant$ 2volume(Ω^+), we have

$$\|\omega\|_{W^{2,\frac{n+1}{2}}(\Omega^+)} \leqslant C\big(\|\Delta\omega\|_{L^{\frac{n+1}{2}}(\hat{O})} + \|\omega\|_{L^{\frac{n+1}{2}}(\hat{O})}\big).$$

Then by Sobolev embedding theorem, in view of Lemma 2.5 we have shown that there exists a constant $C'' = C''(q, p, n, \|a^+\|_\infty, \Omega^+)$ so that $\|\omega\|_\infty \leqslant C''$.

To conclude, we use the following comparison result, which will be useful throughout the paper:

Lemma 2.8. *Assume* (1.7), *and let* $u_1, u_2 \in C^1_{loc}(\mathbb{R}^n) \cap W^{2,s}_{loc}(\mathbb{R}^n)$, $s > n$, *be two functions such that for some* $I \subset M$,

1. u_1, u_2 *are positive in* $\overline{\Omega^+_I}$;
2. $u_1 = 0$ *in* $\overline{\Omega^+ - \Omega^+_I}$;
3. $\lim_{|x|\to\infty} u_1(x) = \lim_{|x|\to\infty} u_2(x) = 0$;
4. *For a.e.* $x \in \mathbb{R}^n$,

$$-\Delta u_1 \leqslant a u_1^q + b u_1^p,$$
$$-\Delta u_2 \geqslant a u_2^q + b u_2^p.$$

Then we must have $u_1 \leqslant u_2$ *in* $\mathbb{R}^n$.

This lemma is proven for C^2 solutions in a bounded domain in [3]. Essentially the same proof shows that it is true for solutions in $C^1_{loc}(\mathbb{R}^n) \cap W^{2,s}_{loc}(\mathbb{R}^n)$ for $s > n$ by using Serrin's maximum principle [21], but for completeness we include the proof in Appendix A.

Applying Lemma 2.8 we thus obtain $0 \leqslant u(x) \leqslant \omega(x)$ holds for all $x \in \mathbb{R}^n$, for any solution u of (1.1), and the proposition is proven. □

Corollary 2.9. *Assume* (1.4), *then*

$$u(x) \leqslant \frac{\eta_1 \rho_1^{n-2}}{|x|^{n-2}}$$

for any $x \in \mathbb{R}^n - B(0, \rho_1)$ *and* η_1 *is a number depending on* q, p, n, Ω^+ *and* $\|a^+\|_{L^\infty(\mathbb{R}^n)}$.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. This is a comparison argument using the method from [11]. For positive numbers M, c, let $w(s)$ be the function defined implicitly by

$$\int_{w(s)}^{M} \frac{dt}{\sqrt{\frac{c}{q+1} t^{q+1}}} = \sqrt{2} s,$$

with constants M and c to be chosen later. Indeed we can write $w(s)$ explicitly in terms of s

$$w(s) = \left[M^{1-\frac{q+1}{2}} - \sqrt{2} s \left(1 - \frac{q+1}{2}\right) \sqrt{\frac{c}{q+1}} \right]^{\frac{2}{1-q}}.$$

Notice that since $0 < q < 1$, then $\frac{2}{1-q} > 2$. So $w(s)$ is at least twice continuously differentiable in $[0, B]$, where B is defined by

$$M^{1-\frac{q+1}{2}} = \sqrt{2} B \left(1 - \frac{q+1}{2}\right) \sqrt{\frac{c}{q+1}}.$$

It is easy to see that $w(s)$ satisfies

$$w''(s) - c w^q(s) = 0 \quad \text{in } (0, B).$$

Moreover $w(s)$ is a decreasing function in s, $w(B) = w'(B) = w''(B) = 0$. Therefore, by defining $w(s) \equiv 0$ for $s \in [B, \infty)$, we obtain a nonincreasing solution of

$$w''(s) - cw^q(s) = 0 \quad \text{in } (0, \infty)$$

with $w(0) = M$ and $\operatorname{supp}(w) = [0, B]$.

We know from the previous corollary that $u(x) \leqslant \frac{\eta_1 \rho_1^{n-2}}{|x|^{n-2}}$ for $|x| > \rho_1$. Consider the function $f(s) : [0, 1] \to \mathbb{R}$ defined by

$$f(s) = w^{\frac{1-q}{2}}(s) = M^{1-\frac{q+1}{2}} - \sqrt{2}s\left(1 - \frac{q+1}{2}\right)\sqrt{\frac{c}{q+1}},$$

where for some constant $\rho > \rho_1 + 1$, we choose our values of M, c as

$$M = \frac{\eta_1 \rho_1^{n-2}}{(\rho - 1)^{n-2}} \quad \text{and} \quad c = \frac{\eta}{\rho^{(n-2)(1-q)}},$$

and η is such that

$$\sqrt{2}\left(1 - \frac{q+1}{2}\right)\sqrt{\frac{\eta}{q+1}} = \left(\eta_1 \rho_1^{n-2}\right)^{1-\frac{q+1}{2}} + 1.$$

We now claim that there exists $\rho > \rho_1 + 1$ such that

$$a^- \geqslant \frac{\eta}{|x|^{(n-2)(1-q)}} \quad \text{for } |x| \geqslant \rho - 1 \quad \text{and} \quad f(1) < 0.$$

Actually we can rewrite $f(1)$ as the following form

$$f(1) = \frac{1}{(\rho - 1)^{\frac{(n-2)(1-q)}{2}}}\left[\left(\eta_1 \rho_1^{n-2}\right)^{1-\frac{q+1}{2}} - \sqrt{2}\left(1 - \frac{q+1}{2}\right)\sqrt{\frac{\eta}{q+1}}\left(\frac{\rho - 1}{\rho}\right)^{\frac{(n-2)(1-q)}{2}}\right].$$

By the assumption that $\liminf_{|x|\to\infty} a^-|x|^{(n-2)(1-q)} > \eta$ and the choice of η, the claim is true for some large ρ.

Since $f(0) > 0$ and $f(1) < 0$, then according to the Mean Value Theorem

$$0 < \sup_{0 \leqslant t \leqslant 1}\left\{t \;\middle|\; f(s) \geqslant 0 \text{ for } s \in [0,\ t]\right\} < 1.$$

Therefore for the choice of M and c, B is well defined and $0 < B < 1$.

Let $v(x) = w(|x| - (\rho - 1))$, then we see that v satisfies

$$\Delta v - cv^q \leqslant 0 \quad \text{in } \mathbb{R}^n - B(0, \rho - 1),$$
$$v = M \quad \text{on } \partial\left(\mathbb{R}^n - B(0, \rho - 1)\right).$$

Also notice that for $|x| \in [\rho - 1,\ \rho)$, $a^-(x) \geqslant \frac{\eta}{|x|^{(n-2)(1-q)}} > \frac{\eta}{\rho^{(n-2)(1-q)}} = c$.

For u we have

$$\Delta u + au^q + bu^p = 0 \quad \text{in } \mathbb{R}^n - B(0, \rho - 1),$$
$$u \leqslant M \quad \text{on } \partial\left(\mathbb{R}^n - B(0, \rho - 1)\right).$$

By subtracting them, we have

$$-\Delta(v-u) \geqslant -a(x)u^q - cv^q \quad \text{for } x \in \mathbb{R}^n - B(0,\rho-1).$$

We now claim that $v \geqslant u \geqslant 0$ for $x \in \mathbb{R}^n - B(0,\rho-1)$. Otherwise there would exist $x_0 \in (\mathbb{R}^n - \overline{B(0,\rho-1)})$ such that $u(x_0) > v(x_0) \geqslant 0$, which implies that $v-u$ attains its global minimal value at some point $x_0 \in \mathbb{R}^n - \overline{B(0,\rho-1)}$. At x_0,

$$\begin{aligned} 0 &\geqslant -\Delta(v-u)(x_0) \\ &\geqslant \big(-a(x_0)u^q(x_0) - cv^q(x_0)\big) \\ &\geqslant \begin{cases} -a(x_0)u^q(x_0) > 0 & \text{if } v(x_0)=0, \\ (-a(x_0)-c)v^q(x_0) > 0 & \text{if } v(x_0) > 0, \end{cases} \end{aligned}$$

a contradiction, and so the claim is proven.

So we must have $v \geqslant u \geqslant 0$ for $x \in (\mathbb{R}^n - B(0,\rho-1))$, which implies u has compact support. Therefore $\operatorname{supp} u \Subset B(0,\rho)$. □

In the end we note that the main ingredient in the above proof is the decay estimate on the solution in the exterior of $B(0,\rho_1)$. Any improvement on the required decay (1.5) of $a^-(x)$ would require a sharper estimate in Corollary 2.9.

Remark 2.10. As noted in the Introduction, the one- and two-dimensional cases must be handled differently. In particular, the decay estimates of Lemma 2.3 are no longer valid in $\mathbb{R}^n$, $n=1,2$, principally because the fundamental solution does not decay to zero at infinity. Nevertheless, Lemma 2.1 still holds for any classical solution in $L^t(\mathbb{R}^n)$ for $t \geqslant 1$, so we can prove in very similar way that all classical solutions of (1.1) in $L^t(\mathbb{R}^n)$ have compact supports under strong assumption $\liminf_{|x|\to\infty} a^-(x) > 0$. However, we cannot uniformly control the size of the support because the Sobolev inequalities are domain-dependent in dimensions $n=1,2$. The statement we can make in any dimension is the following:

Theorem 2.11. *Under* (1.4), *if* $\liminf_{|x|\to\infty} a^- > 0$, *all classical solutions in* $L^t(\mathbb{R}^n)$ *for* $t \geqslant 1$ *must have compact support.*

Proof. The proof is much simpler since we do not need to choose the place where we make the comparison. We may just pick $M = \|u\|_{L^\infty(\mathbb{R}^n)}$ and compare w with u outside $B(0,\rho_1)$. □

Remark 2.12. If $a^-(x)$ decays too fast at infinity, solutions may not have compact support. To see this, recall that in a bounded domain with Neumann condition there is a necessary condition for the existence of solutions (see [1,4]),

$$\int_{\operatorname{supp}(U)} a(x) + bU^{p-q}\,dx < 0. \tag{2.6}$$

When Eq. (1.1) has solutions with compact support, they also solve the Neumann problem inside a big ball, so this condition must hold *a posteriori*. However, if we choose $b \equiv 0$ and $a(x) \in L^1(\mathbb{R}^n)$ satisfying (1.4) with $\int_{\mathbb{R}^n} a(x) > 0$ then a compactly supported solution could never satisfy (2.6), and thus no solution can have compact support.

3. Existence and uniqueness

In this section we present the proof of the basic existence theorems, Theorems 1.1 and 1.7, and a more general form of the uniqueness result Theorem 1.5. Throughout we assume the dimension $n \geqslant 3$.

3.1. Existence of maximal and minimal solutions

We use the method of monotone iteration (see Hess [16]). Existence results of this type are well known in the setting of bounded domains (see [3], for example) and we adapt the technique here for entire solutions in $\mathbb{R}^n$.

Recall from (2.2) that $\omega \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ is the unique solution of the equation

$$-\Delta\omega = a^+\omega^q, \qquad \omega(x) \to 0 \quad \text{as } |x| \to \infty.$$

To begin the iteration, we start with the following equation:

$$-\Delta z + a^- z^q - bz^p = a^+ f^q \quad \text{in } \mathbb{R}^n,\ z \geqslant 0 \text{ and } z \in \mathcal{D}^{1,2}\big(\mathbb{R}^n\big), \tag{3.1}$$

where f is Hölder continuous and $0 \leqslant f \leqslant \omega$.

Lemma 3.1. *Assume* (1.2), *then* (3.1) *has a solution* $Z \in \mathcal{D}^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *and* $Z \leqslant \omega$.

Proof. We use monotone iteration to prove this result. Let us consider the following equation

$$-\Delta z_n + a^- z_n^q - bz_n^p = a^+ f^q \quad \text{in } B(0,n),\ z \geqslant 0 \text{ and } z \in H_0^1\big(B(0,\ n)\big). \tag{3.2}$$

It is easy to see that $\underline{z} = 0$ is a subsolution, $\bar{z} = \int_{\mathbb{R}^n} \Phi(x-y)a^+(y)f^q(y)\,dy$ is a supersolution, where Φ is the fundamental solution of Laplace's equation. Furthermore we see that $\bar{z} \leqslant \omega$ since $0 \leqslant f \leqslant \omega$. Therefore by sub-supersolution method the above equation has a solution $Z_n \leqslant \omega$.

Claim 1. *The solution* Z_n *is unique.*

Indeed if there is another solution $\overline{Z_n}$, then we have

$$-\Delta(Z_n - \overline{Z_n})\,dx + a^-\big(Z_n^q - \overline{Z_n}^q\big)\,dx - b\big(Z_n^p - \overline{Z_n}^p\big)\,dx = 0.$$

Then multiply both sides by $Z_n - \overline{Z_n}$ and integrate by parts, we have

$$\int\limits_{B(0,n)} \big|\nabla(Z_n - \overline{Z_n})\big|^2\,dx + \int\limits_{B(0,n)} a^-\big(Z_n^q - \overline{Z_n}^q\big)(Z_n - \overline{Z_n})\,dx - \int\limits_{B(0,\ n)} b\big(Z_n^p - \overline{Z_n}^p\big)(Z_n - \overline{Z_n})\,dx = 0.$$

Since they are all non-negative, we conclude that $Z_n = \overline{Z_n}$.

It is easy to see that Z_{n+1} is a supersolution for (3.2), so by the uniqueness $Z_{n+1} \geqslant Z_n$. Moreover we have the following estimate

$$\int\limits_{B(0,n)} |\nabla Z_n|^2\,dx + \int\limits_{B(0,n)} a^- Z_n^{q+1}\,dx - \int\limits_{B(0,n)} bZ_n^{p+1}\,dx = \int\limits_{B(0,n)} a^+ f^q Z_n\,dx \leqslant \int\limits_{\mathbb{R}^n} a^+\omega^{q+1}\,dx.$$

Therefore let $Z = \lim_{n\to\infty} Z_n$, then Z is a solution to (3.1) and it satisfies

$$\int\limits_{\mathbb{R}^n} |\nabla Z|^2\,dx + \int\limits_{\mathbb{R}^n} a^- Z^{q+1}\,dx - \int\limits_{\mathbb{R}^n} bZ^{p+1}\,dx \leqslant \int\limits_{\mathbb{R}^n} a^+\omega^{q+1}\,dx \tag{3.3}$$

and $Z \leqslant \omega$. $\square$

Next, in order to prove the uniqueness of the solution Z, we need to improve Lemma 2.3. Let

$$V = \frac{[n(n-2)]^{\frac{n-2}{4}}}{(1+|x|^2)^{\frac{n-2}{2}}}. \tag{3.4}$$

Of course $V(x)$ is (up to scaling and translation) the unique solution of the familiar critical Sobolev exponent equation in $\mathbb{R}^n$, $\Delta V + V^{\frac{n+2}{n-2}} = 0$ (see [10]). Under assumption (1.2), Ω^+ is bounded, we can pick $\rho_1 > 0$ so that $\Omega^+ \Subset B(0, \rho_1)$. We have the following lemma.

Lemma 3.2. *Assume Z in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is a non-negative smooth solution of $-\Delta Z \leqslant 0$ in $\mathbb{R}^n - B(0,\rho_1)$. Then, there exists a constant $C > 0$ such that $Z \leqslant CV$ in $\mathbb{R}^n - B(0,\rho_1)$. Moreover there exists an increasing sequence $\{R_n\}$ with $\lim_{n\to\infty} R_n = \infty$ so that $\int_{\partial B(0,\ R_n)} \frac{\partial Z}{\partial n} Z \to 0$ as $n \to \infty$.*

Proof. From Lemma 2.1, we have $\lim_{|x|\to\infty} Z(x) = 0$. Following the proof of Lemma 2.3, replace v in Lemma 2.3 by CV for some big constant C. Notice that $-\Delta(CV) > 0$, we can show that $Z \leqslant CV$ in $\mathbb{R}^n - B(0,\rho_1)$. Therefore we have $Z \leqslant C_1|x|^{-(n-2)}$ for some C_1, then the last part follows from the claim in Lemma 2.6. □

Lemma 3.3. *Under hypothesis* (1.2)*, the solution obtained from previous lemma is unique.*

Proof. Suppose there are two solutions $Z, \overline{Z}$, which satisfy the estimates (3.3), then we have

$$-\Delta(Z-\overline{Z})\,dx + a^-\big(Z^q - \overline{Z}^q\big)\,dx - b\big(Z^p - \overline{Z}^p\big)\,dx = 0.$$

Then multiply both sides by $Z - \overline{Z}$ and integrate by parts over $B(0,R)$, we have

$$\int\limits_{B(0,\ R)} \big|\nabla(Z-\overline{Z})\big|^2\,dx + \int\limits_{B(0,R)} a^-\big(Z^q - \overline{Z}^q\big)(Z-\overline{Z})\,dx - \int\limits_{B(0,R)} b\big(Z^p - \overline{Z}^p\big)(Z-\overline{Z})\,dx$$
$$+ \int\limits_{\partial B(0,\ R)} \frac{\partial(Z-\overline{Z})}{\partial n}(Z-\overline{Z})\,dS = 0.$$

From Lemma 3.2, there exists a sequence $\{R_n\}$ and $\lim_{n\to\infty} R_n = \infty$, such that $\int_{\partial B(0,\ R_n)} \frac{\partial(Z-\overline{Z})}{\partial n}(Z - \overline{Z}) \to 0$ as $n \to \infty$. Let $n \to \infty$, we have

$$\int\limits_{\mathbb{R}^n} \big|\nabla(Z-\overline{Z})\big|^2\,dx + \int\limits_{\mathbb{R}^n} a^-\big(Z^q - \overline{Z}^q\big)(Z-\overline{Z})\,dx - \int\limits_{\mathbb{R}^n} b\big(Z^p - \overline{Z}^p\big)(Z-\overline{Z})\,dx = 0.$$

So we can conclude $Z = \overline{Z}$. □

Next we go to the main iteration process. Let us consider the following iteration equation

$$-\Delta u_{n+1} + a^- u_{n+1}^q - b u_{n+1}^p = a^+ u_n^q \quad \text{in } \mathbb{R}^n,\ u_n \geqslant 0 \text{ and } u_n \in \mathcal{D}^{1,2}\big(\mathbb{R}^n\big), \tag{3.5}$$

where $u_1 = \underline{u}_\rho$ for small ρ such that $0 \leqslant \underline{u}_\rho \leqslant \omega$, and $\underline{u}_\rho$ is constructed in the following manner: take small ball $B \Subset \Omega^+$, let $\underline{a} = \inf_{x\in B} a(x)$, define

$$\underline{u}_\rho = \begin{cases} \rho\xi_i & \text{in } B, \\ 0, & \text{else}, \end{cases}$$

where $\xi > 0$ is the eigenfunction corresponding to the first eigenvalue of the following problem

$$-\Delta\xi = \lambda \underline{a}\xi \quad \text{in } B \quad \text{and} \quad \xi = 0 \quad \text{on } \partial B,$$

for details, see [3]. Notice that $\underline{u}_\rho$ is Lipschitz in $\mathbb{R}^n$ and satisfies

$$\int_{\mathbb{R}^n} \nabla \underline{u}_\rho \nabla\phi\, dx \leqslant \int_{\mathbb{R}^n} a\underline{u}_\rho^q \phi\, dx + \int_{\mathbb{R}^n} b\underline{u}_\rho^p \phi\, dx \quad \text{for } \phi \in C_0^\infty(\mathbb{R}^n),$$

since $\underline{u}_\rho$ is a subsolution.

Lemma 3.4. *Assume* (1.2), *then* $\omega \geqslant u_2 \geqslant u_1 = \underline{u}_\rho$ *and*

$$\int_{\mathbb{R}^n} |\nabla u_2|^2\, dx + \int_{\mathbb{R}^n} a^- u_2^{q+1}\, dx - \int_{\mathbb{R}^n} b u_2^{p+1}\, dx \leqslant \int_{\mathbb{R}^n} a^+ \omega^{q+1}\, dx. \tag{3.6}$$

Proof. The proof is simple, we just go back to proof of Lemma 3.1. It is easy to see that $\underline{u}_\rho$ is a subsolution to Eq. (3.2) and ω is a supersolution, therefore we have desired results. □

Base on the above lemma, we can start our induction process.

Lemma 3.5. *Assume* (1.2), *then* $\omega \geqslant u_{n+1} \geqslant u_n \geqslant \underline{u}_\rho$ *and*

$$\int_{\mathbb{R}^n} |\nabla u_n|^2\, dx + \int_{\mathbb{R}^n} a^- u_n^{q+1}\, dx - \int_{\mathbb{R}^n} b u_n^{p+1}\, dx \leqslant \int_{\mathbb{R}^n} a^+ \omega^{q+1}\, dx. \tag{3.7}$$

Proof. From the above lemma we know the initial step is true, now assume $u_n \geqslant u_{n-1}$, we have

$$-\Delta u_{n+1} + a^- u_{n+1}^q - b u_{n+1}^p \geqslant -\Delta u_n + a^- u_n^q - b u_n^p,$$

that is

$$-\Delta(u_n - u_{n+1}) + a^-\left(u_n^q - u_{n+1}^q\right) - b\left(u_n^p - u_{n+1}^p\right) \leqslant 0.$$

Then multiply both sides by $(u_n - u_{n+1})^+$, integrate over $B(0,\ R)$, and follow the steps in claim of Lemma 3.3, we will have

$$\int_{\mathbb{R}^n} \left|(u_n - u_{n+1})^+\right|^2 dx + \int_{\mathbb{R}^n} a^-\left(u_n^q - u_{n+1}^q\right)(u_n - u_{n+1})^+\, dx - \int_{\mathbb{R}^n} b\left(u_n^p - u_{n+1}^p\right)(u_n - u_{n+1})^+\, dx \leqslant 0.$$

From the above we conclude $(u_n - u_{n+1})^+ = 0$, which means $u_{n+1} \geqslant u_n$. □

Now we are in position to prove Theorem 1.1:

Proof of Theorem 1.1. We simply take $U = \lim_{n\to\infty} u_n$, then in view of the estimates from the previous we know U is the required solution of (1.1). □

Remark 3.6. Assumption (1.2) is not essential for the existence of U; for a more general existence result see [8].

Corollary 3.7. *Under hypothesis* (1.4), (1.1) *has a classical compactly supported solution U with its support contained in $B(0,\rho)$ if* $\lim_{|x|\to\infty} a^-(x)|x|^{(n-2)(1-q)} > \eta$, *where η and ρ are from Theorem* 1.2.

In terms of the solution class S_I (defined in (1.4)), we obtain the following existence result:

Corollary 3.8. *Assume* (1.7), *then $S_I \neq \emptyset$ for any non-empty $I \in M$.*

Proof. Construct a subsolution $\underline{u}_\rho$ as a superposition of disjointly supported subsolutions, one for each component of Ω_I^+, as in the proof of Theorem 1.1. Monotone iteration then produces a solution which is positive in each component. $\square$

With some minor modifications of these arguments we may now prove the existence of minimal and maximal elements in S_I as announced in Theorem 1.7.

Proof of Theorem 1.7. We first assume that hypothesis (1.7) holds. To prove assertion (a) we may argue as in Theorem 4 of [18], and define $u_I = \inf_{u\in S_I} u(x)$, which (by [18]) is the desired minimal solution.

To prove part (b) of Theorem 1.7, we refine our choice of supersolution. To achieve this, we bring in the function $V(x)$ from (3.4).

Claim. *Assuming only* (1.2), *there exists a constant $C > 0$ such that*

$$-\Delta(CV) + a^-(CV)^q - b(CV)^p \geqslant a^+(CV)^q \tag{3.8}$$

and $CV \geqslant u$ for any solution u of (1.1).

Indeed, for (3.8) to hold, we only need C big enough to achieve $C^{1-q}V^{2^*-1-q} \geqslant a^+$ since $-\Delta V = V^{2^*-1}$. Notice that by assuming only (1.2), we still have L^∞-estimate as in Lemma 2.4 because of Lemma 3.2, so for C big enough, we have $CV \geqslant u$.

We then restart our iteration process (3.5) as in the proof of Theorem 1.1, starting with the supersolution $u_1 = CV \in \mathcal{D}^{1,2}(\mathbb{R}^n)$. By similar proof to Lemma 3.5, we deduce that a decreasing sequence $\{u_n\}$ such that $u_{n+1} \leqslant u_n \leqslant u_1 = CV$ and (3.7) holds with CV in place of ω. Moreover for any solution u of (1.1), we must have $u \leqslant u_n$ for all n, by an induction process. Hence we conclude that $u_n \to U$, a solution of (1.1), and with $U \geqslant u$ holding for any solution u of (1.1). Therefore, U must be the maximal solution. Since $\omega > 0$ in $\mathbb{R}^n$, it is an easy consequence of case 1 in Lemma 2.8 that $U < \omega$ in $\mathbb{R}^n$. $\square$

Finally, we also conclude that, for the parametrized family of problems (1.9), the maximal solution $U_\lambda \in S_M$ is monotone.

Corollary 3.9. *Assuming* (1.7), *the maximal solution $U_\lambda \in S_M$ of* (1.9) *is increasing as λ increases.*

Proof. For $0 < \lambda_1 < \lambda_2$, we pick C big enough so that $CV \geqslant \max(U_{\lambda_1}, U_{\lambda_2})$. Then we iterate this supersolution CV for $(1.9)_{\lambda_2}$. As above, it results in a monotone decreasing sequence $\{u_n\}$ and $u_n \geqslant \max(U_{\lambda_1}, U_{\lambda_2})$. But then $U_{\lambda_1} \leqslant u_n \to U_{\lambda_2}$. $\square$

3.2. Uniqueness and support

We now turn to questions of uniqueness and the characterization of the solution space of (1.1) in terms of the supports of the solutions. First, we note that every solution of (1.1) must be positive in at least one connected component of Ω^+.

Lemma 3.10. *Assume* (1.7)*, if* $I = \emptyset$*, then* $N_I = \emptyset$*.*

Proof. If $I = \emptyset$, then for any $u \in N_I$ it is a subharmonic function. Thus, for any $x \in \mathbb{R}^n$, we have

$$0 \leqslant u(x) \leqslant \frac{1}{|\partial B(x,R)|} \int\limits_{\partial B(x,R)} u(y)\,dy \to 0 \quad \text{as } R \to \infty$$

since $\lim_{|x|\to\infty} u(x) = 0$, which implies $u(x) = 0$. $\square$

Lemma 3.11. *The classical solution u of* (1.1) *is either positive in* Ω_i^+ *or entirely zero in* Ω_i^+ *for any* $i \in M$*.*

Proof. Let us consider the set $S = \{x \in \Omega_i^+ \mid u(x) = 0\}$. First, we claim that S is open in Ω_i^+. Indeed, if $S \neq \emptyset$, then pick any $x_0 \in S$, we have $u(x_0) = 0$ and $a(x_0) > 0$. Therefore by continuity

$$a(x) + b(x)u^{p-q}(x) > 0 \quad \text{in } B(x_0, \epsilon),$$

for small $\epsilon > 0$. Hence we have

$$-\Delta u = a(x)u^q + b(x)u^p = \big(a(x) + b(x)u^{p-q}\big)u^q \geqslant 0 \quad \text{in } B(x_0,\epsilon), \qquad u(x) \geqslant 0 \quad \text{in } B(x_0,\epsilon).$$

So by maximum principle $u(x) \equiv 0$ in $B(x_0,\epsilon)$, which means S is open in Ω_i^+.

It is also clear by continuity that S is closed in Ω_i^+, therefore $u > 0$ in Ω_i^+ or $u \equiv 0$ in Ω_i^+ due to the connectivity of Ω_i^+. $\square$

For uniqueness questions it is not enough to know that any solution u of (1.1) is positive in Ω_i^+, we require u to be positive up to and including the boundary of Ω_i^+. In case $p \geqslant 1$, this follows from the classical Hopf Lemma (see below) but the question is more delicate for $p < 1$. Therefore we define a function class P which incorporates this property,

$$\begin{aligned} P = \Big\{ v \in C^1_{loc}\big(\mathbb{R}^n\big) \cap W^{2,s}_{loc}\big(\mathbb{R}^n\big) \;\Big|\; s > n,& \\ v \gneq 0 \text{ in } \mathbb{R}^n,\ v > 0 \text{ in } \overline{\Omega_i^+} \text{ if } v > 0 \text{ in } \Omega_i^+ \text{ for } i \in M \Big\}.& \end{aligned}$$

Lemma 3.12. *Assume* (1.7) *and solution u of* (1.1) *is positive in* Ω_i^+ *for some* $i \in M$*. Then if* $p \geqslant 1$*,* $u > 0$ *in* $\overline{\Omega_i^+}$*; if* $p < 1$ *and* $\frac{b(x)}{a(x)}$ *is uniformly bounded in* $\overline{\Omega^+}$*,* $u > 0$ *in* $\overline{\Omega_i^+}$*.*

Proof. For $p \geqslant 1$, hypothesis (1.7) ensures that an interior ball condition is satisfied by Ω_i^+, so we may directly apply Hopf's Lemma to the equation

$$-\Delta u - b(x)u^p = a(x)u^q \geqslant 0 \quad \text{in } \Omega_i^+, \qquad u \geqslant 0 \quad \text{in } \overline{\Omega_i^+}.$$

We conclude that $u > 0$ in $\overline{\Omega^+}$.

For $p < 1$, if $x_0 \in \partial\Omega_i^+$ and $u(x_0) = 0$. Since Ω_i^+ satisfies an interior ball condition, we take a small ball $B_\epsilon \subset \Omega_i^+$ with radius ϵ and $x_0 \in \partial B_\epsilon$. For ϵ small we have

$$-\Delta u = a(x)\left(1 + \frac{b(x)}{a(x)}u^{p-q}\right)u^q \geqslant 0 \quad \text{in } B_\epsilon.$$

Hopf's Lemma implies that $\nabla u(x_0) \neq 0$. Since u attains minimal value at x_0, $\nabla u(x_0) = 0$, a contradiction. $\square$

Definition 3.13. We say that $b(x)$ is compatible with $a(x)$ if any nonzero solution u of (1.1) lies in the function set P.

From the above lemma, we see that when $p \geqslant 1$, any non-positive $b(x)$ is compatible with $a(x)$, but for $p < 1$ an extra hypothesis must be imposed on b near $\partial\Omega_i^+$.

In Spruck [23], it was shown that if $\limsup_{|x|\to\infty} a(x) < 0$ and $\nabla a(x) \cdot x < 0$, Eq. (1.1) with $b(x) \equiv 0$ has a unique compactly supported solution. With a very mild assumption on $a(x)$, we show a uniqueness result, which generalizes the uniqueness result of Spruck and includes Theorem 1.5.

Theorem 3.14. *Assume* (1.7)*, if $b(x)$ is compatible with $a(x)$, then the number of elements in N_I is at most* 1 *for any non-empty I. In particular if $k = 1$, then the solution to* (1.1) *is unique and its support is connected.*

Proof. Let us take two element u_1 and u_2 from N_I and apply Lemma 2.8. We check all the conditions required by Lemma 2.8. Since $b(x)$ is compatible with $a(x)$, then $u_1, u_2 \in P$, which means that the first condition is satisfied. The second condition is automatic because $u_1, u_2 \in N_I$. The third condition is assured by Lemma 2.1 and the last condition is also automatic. We apply Lemma 2.8 twice and get $u_1 \leqslant u_2$, $u_1 \geqslant u_2$, hence $u_1 \equiv u_2$. $\square$

Immediately we combine the existence of a maximal solution from Theorem 1.7(b) with the uniqueness result obtained above to get the following corollary.

Corollary 3.15. *Assume* (1.7)*, if b is compatible with $a(x)$, the maximal solution U is the unique element in S_M.*

Here we present the proof for Proposition 1.6.

Proof of Proposition 1.6. Recall that in Proposition 1.6 we assume $p \geqslant 1$. For $b(x) \equiv -1$, define $a_i(x)$ by

$$a_i(x) = \begin{cases} \lambda, & \text{if } x \in \Omega_i^+, \\ -1, & \text{if } x \notin \Omega_i^+. \end{cases}$$

Then we consider the following equation:

$$-\Delta u = a_i(x)u^q - u^p \quad \text{in } \mathbb{R}^n,\ u \geqslant 0 \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^n). \tag{3.9}$$

By elliptic regularity and the strong maximum principle, we see that solution to (3.9) lies in the set P. By Lemma 2.8 the solution to (3.9) is unique, denoted by u_i, and compactly supported by Theorem 1.2. So if we choose δ^* big enough so that $\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \emptyset$ for any $i \neq j$, then by Theorem 3.14 N_I contains exactly one solution for each $I \neq \emptyset$. This proves the first assertion of Proposition 1.6.

For the second statement in the proposition, notice that $\Omega_i^+ \Subset \operatorname{supp}(u_i)$. For non-empty I, choose any $i \in I$, then the minimal solution u_I in S_I is supersolution for problem (3.9). By uniqueness we find that $u_I \geqslant u_i$ for any $i \in I$. So if we choose δ_* so small that $\operatorname{supp}(u_i) \cap \Omega_j^+ \neq \emptyset$ for any $i \neq j$, which implies that $\operatorname{supp}(u_I) \cap \Omega_j^+ \neq \emptyset$. Therefore by uniqueness result above, (1.1) has only one solution, which is positive in Ω^+. $\square$

In the case where $a(x)$ and $b(x)$ are radially symmetric we obtain more precise information, via uniqueness:

Corollary 3.16. *Assume* (1.7)*, if a and b are radially symmetric and b is compatible with a, then the unique element in N_I is radially symmetric. If in addition Ω^+ is a ball centered at the origin and $b(x) = 0$ in Ω^+, then the unique solution of* (1.1) *is radially decreasing.*

Proof. If a and b are radially symmetric, since Laplace's operator is invariant under rotation, Lemma 2.8 ensures that the element in N_I must be radially symmetric. If Ω^+ is a ball centered at the origin and $b(x) = 0$ in Ω^+, then b is compatible with a. The result follows from maximum principle and the fact that the unique solution uniformly converges to zero at the infinity. $\square$

Remark 3.17. Note that we do not need to apply the moving planes method in this setting since we have the uniqueness result. Furthermore, the moving plane process would require more stringent hypotheses on a and b.

4. The parametrized equation

In this section we consider the effect of the parameter λ on the shape and multiplicity of solutions to the parametrized family,

$$-\Delta u = a_\lambda(x)u^q + bu^p \quad \text{in } \mathbb{R}^n,\ u \geqslant 0 \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \tag{4.1}$$

where $a_\lambda(x) = \lambda a^+(x) - a^-(x)$.

We consider both asymptotic limits, $\lambda \to 0$ and $\lambda \to \infty$, for the maximal solution $U_\lambda \in L^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,2}(\mathbb{R}^n)$. Assuming that b is compatible with a, we show that as λ decreases to zero, U_λ decreases (see Corollary 3.9), and ultimately the support of U_λ breaks into several disjoint components. This is the content of Theorem 1.9, stated in the Introduction. Before proving the theorem we need two lemmas.

Lemma 4.1. *For any positive constant c, the equation $\Delta \bar{u} = c\bar{u}^q$ in $\mathbb{R}^n$ has a radial solution $\overline{U} = \theta r^{\frac{2}{1-q}}$, where $\theta^{1-q} = \frac{(1-q)^2 c}{2[n-q(n-2)]}$.*

Lemma 4.2. *For any ball $B \Subset \mathbb{R}^n - \Omega^{0+}$, any $g(x) \geqslant 0$, the following problem has at most one non-negative classical solution,*

$$-\Delta v = a(x)v^q + bv^p \quad \text{in } B, \quad v = g \quad \text{on } \partial B.$$

Proof. Lemma 4.1 follows from direct calculation. For Lemma 4.2, suppose there were two classical solutions v_1, v_2 and $v_1 \neq v_2$. Assuming that $(v_1 - v_2)$ achieves a positive maximum value at some $x_0 \in B$, we have

$$0 \leqslant -\Delta(v_1 - v_2)(x_0) = a(x_0)\big(v_1^q(x_0) - v_2^q(x_0)\big) + b(x_0)\big(v_1^p(x_0) - v_2^p(x_0)\big) < 0,$$

which is a contradiction. Hence we must have $v_1 \equiv v_2$, and Lemma 4.2 is verified. $\square$

With the help of the above two lemmas, we give the proof of Theorem 1.9.

Proof of Theorem 1.9. By assumption $a(x)$ is admissible, then $\operatorname{dist}(\Omega_i^{0+}, \Omega_j^{0+}) > 0$ for any $i \neq j$. Let $\delta = \frac{1}{16}\inf_{i\neq j}\operatorname{dist}(\Omega_i^{0+}, \Omega_j^{0+})$, then $\delta > 0$. As before, take $\rho_1 > 0$ so that $\Omega^{0+} \Subset B(0, \rho_1)$. Let

$$C_i = \big\{x \in B(0, \rho_1 + 48\delta) \bigm| \operatorname{dist}\big(x, \Omega_i^{0+}\big) \leqslant \delta\big\}.$$

It is easy to see that $C_i \cap C_j = \emptyset$ for any $i \neq j$. Let $C = \bigcup_{i\in M} C_i$. We define

$$N = \big\{x \in B(0, \rho_1 + 32\delta) \bigm| \operatorname{dist}\big(x, \Omega^{0+}\big) \geqslant 4\delta\big\}.$$

For any $x \in N$, $\overline{B(x,\delta)} \cap C_i = \emptyset$ for any $i \in M$. Finally set

$$\underline{a} = \inf_{x \in B(0, R+48\delta) - C} a^-(x).$$

By Lemma 4.1 we find that the following equation

$$\Delta \bar{u} = \underline{a} \bar{u}^q \quad \text{in } B(x,\delta), \qquad \bar{u} = \theta(\delta)^{\frac{2}{1-q}} \quad \text{on } \partial B(x, \ \delta)$$

has a solution $\overline{U} = \theta |y-x|^{\frac{2}{1-q}}$, where $x \in N$.

We now claim that the equation

$$-\Delta v = a^-(y) v^q + b(y) v^p \quad \text{in } B(x,\delta), \qquad v = U_\lambda \quad \text{on } \partial B(x,\delta)$$

has a unique solution $v = U_\lambda$ if $\|U_\lambda\|_{L^\infty(R^n)} \leqslant \theta(\delta)^{\frac{2}{1-q}}$, where U_λ is the maximal solution of (4.1). Indeed, from Lemma 2.4, we have $\lim_{\lambda \to 0} U_\lambda = 0$. Therefore there exists $\lambda_* > 0$ such that $\|U_\lambda\|_{L^\infty(R^n)} \leqslant \theta(\delta)^{\frac{2}{1-q}}$ for $\lambda \leqslant \lambda_*$. Hence $\overline{U}$ is a supersolution for the above equation and 0 is a subsolution, so the above equation has a solution v. But it is clear the maximal solution U_λ is also a solution, by Lemma 4.2 we know that the above equation has a unique solution $v = U_\lambda$. Therefore we have $0 \leqslant U_\lambda(x) \leqslant \overline{U}(x) = 0$, which means $U_\lambda(x) = 0$ for all $x \in N$. Hence we can write

$$U_\lambda = \sum_{i \in M} u_i \quad \text{and} \quad \operatorname{supp} u_i \cap \operatorname{supp} u_j = \emptyset \quad \text{for } i \neq j,$$

where $\Omega^+ \subset \operatorname{supp}(u_i)$, which means that $\sum_{i \in I} u_i \in N_I$. □

Remark 4.3. If we also assume that b is compatible with a, then solutions in N_I will be unique for all $I \neq \emptyset$. Thus, for λ small, the support of the maximal solution U_λ will break into k disjoint components, and these disjoint compactly supported solutions then generate the unique element in each N_I by superposition.

Next we consider the limit $\lambda \to \infty$. We will prove that for $p < 1$ and λ sufficiently large, (4.1) has a unique solution. In particular, it will be positive in each of the connected components of Ω^{0+}.

Under the assumption (1.7), fix $I \subset M$, by Theorem 1.7 the minimal element of S_I exists, denoted by u_λ. Let $v_\lambda = \lambda^{\frac{1}{q-1}} u_\lambda$, then v_λ satisfies the following equation in $\mathbb{R}^n$,

$$-\Delta v = a^+ v^q - \frac{a^-}{\lambda} v^q + b \lambda^{\frac{1-p}{q-1}} v^p, \quad v \geqslant 0 \text{ and } v \in \mathcal{D}^{1,2}(\mathbb{R}^n). \tag{4.2}$$

For each fixed λ, let $\overline{S}_I$ be the corresponding set of S_I, associated with the above equation (4.2). Let us begin the proof of Theorem 1.10 with a few lemmas. Recall that Theorem 1.10 concerns the case $p < 1$.

Lemma 4.4. *Assume* (1.7)*, then v_λ is the minimal element in $\overline{S}_I$. Moreover v_λ is increasing as λ increases and $v_\lambda \leqslant \omega$, where ω is as in* (2.2)*.*

Proof. Suppose $0 < \lambda_1 < \lambda_2$, since $-\frac{1}{\lambda}$ and $b\lambda^{\frac{1-p}{q-1}}$ are increasing in λ, then from Theorem 1.1 we see that v_{λ_2} is a supersolution for Eq. (4.2) at $\lambda = \lambda_1$, by choosing suitable $\underline{u}_\rho$ as subsolution, we get a solution $v \in \overline{S}_I$ of (4.2) at $\lambda = \lambda_1$ such that $v_{\lambda_1} \leqslant v \leqslant v_{\lambda_2}$. The last part is from Theorem 1.7. □

Lemma 4.5. *There exists a unique solution $\overline{\omega}$ to the following problem*:

$$-\Delta\overline{\omega} = a^{+}\overline{\omega}^{q} + b\overline{\omega}, \qquad \overline{\omega}(x) \to 0 \quad \text{as } |x| \to \infty. \tag{4.3}$$

Proof. Again the existence and uniqueness of $\overline{\omega}$ follows from [8]. $\square$

Lemma 4.6. *Assume* (1.7), *we have the following*:

$$\text{for } p < 1, \quad v_\lambda \text{ converges uniformly to } \omega, \quad \text{where } \omega \text{ is from (2.2)},$$

$$\text{for } p = 1, \quad v_\lambda \text{ converges uniformly to } \overline{\omega}, \quad \text{where } \overline{\omega} \text{ is from (4.3)}.$$

Proof. From the above lemma we see that v_λ is increasing in λ and uniformly bounded. Let $V = \lim_{\lambda\to\infty} v_\lambda$, then it is easy to see that $\lim_{|x|\to\infty} V(x) = 0$. Moreover from Eq. (4.2) we have $\|v_\lambda\|_{C^{1,\alpha}(B(0,R))}$ is uniformly bounded for fixed R, therefore we have v_λ uniformly converges to V. Hence V solves the following equation

$$-\Delta w = a^{+}w^{q} \quad \text{if } p > 1 \quad \text{or} \quad -\Delta w = a^{+}w^{q} + bw \quad \text{if } p = 1.$$

Since $\lim_{|x|\to\infty} V(x) = 0$, by uniqueness of the solution (in either case) we obtain our conclusion. $\square$

Now we are ready to prove Theorem 1.10.

Proof of Theorem 1.10. Since $u_\lambda = \lambda^{\frac{1}{1-q}} v_\lambda$ and v_λ uniformly converges to $V > 0$ in $\mathbb{R}^n$. Since u_λ is minimal element in S_I and the choices for I are finite, all solutions of $(4.1)_\lambda$ are positive in $\overline{\Omega^{+}}$ when λ is large. Therefore Lemma 2.8 implies that S_M has only one element. $\square$

In the case $p > 1$, it is easy to see that the above process could not work, partially because now $\lim_{\lambda\to\infty} \lambda^{\frac{1-p}{q-1}} = \infty$. To obtain some asymptotic results in this regime, we must impose some conditions on $b(x)$ and around the set Ω^{+}.

Let u_λ be any solution of (1.9). We prove:

Theorem 4.7. *Assume* (1.7) *and* $p > 1$.

1. *For any* $\sigma > 0$, $\lim_{\lambda\to\infty} \|u_\lambda\|_{L^\infty(\mathbb{R}^n)}^{p-q+\sigma} \lambda^{-1} = \infty$.
2. *If* $\inf_{x\in\Omega^{+}} |b(x)| > 0$, *then* $\|u_\lambda\|_{L^\infty(\mathbb{R}^n)}^{p-q} \leqslant C\lambda$ *for some constant* $C > 0$ *independent of* λ.
3. *If* $b(x) = 0$ *in a ball* $B \subset \Omega_i^{+}$ *for some* $i \in M$, *then* $\liminf_{\lambda\to\infty} \|u_\lambda\|_{L^\infty(\mathbb{R}^n)} \lambda^{\frac{1}{q-1}} > 0$, *for* $I = \{i\}$ *and* $u_\lambda \in S_I$.
4. *If* $\{x \in \mathbb{R}^n \mid b(x) = 0\} \cap \Omega^{+}$ *has an open connected component* O *such that* $\Omega_i^{+} \cap O \neq \emptyset$ *for any* $i \in M$, *then the problem* (1.9) *has a unique solution, which is positive in* Ω^{+}, *for* λ *large enough.*

We first require some lemmas. Let $v_\lambda = \lambda^{\frac{1}{q-1}} u_\lambda$, so that v_λ satisfies the following modified equation:

$$-\Delta v = a^{+}v^{q} - \frac{a^{-}}{\lambda} v^{q} + \frac{b}{\lambda^{1-\frac{\epsilon}{1-q}}} u_\lambda^{p-q-\epsilon} v^{q+\epsilon} \quad \text{in } \mathbb{R}^n, \tag{4.4}$$

where we pick $\epsilon > 0$ small so that $\epsilon < 1 - q$. Under the assumption (1.7) Eq. (4.4) admits a minimal solution $\overline{v_\lambda}$ in the class $\overline{S_I}$, where $\overline{S_I}$ is the corresponding set of S_I, associated with Eq. (4.4).

Lemma 4.8. *Assume* (1.7)*, if for some* $\sigma > 0$, $\liminf_{\lambda\to\infty} \|u_\lambda\|^{p-q+\sigma}_{L^\infty(\mathbb{R}^n)}\lambda^{-1} < \infty$*. Then there exists an increasing sequence* $\{\lambda_n\}$ *with* $\lim_{n\to\infty}\lambda_n = \infty$ *so that* $\|\overline{v_{\lambda_n}}\|_{L^\infty(\mathbb{R}^n)}$ *has a positive lower bound* $C(\Omega_I^+)$ *independent of* λ *for large* λ*.*

Proof. By assumption we can pick an increasing sequence $\{\lambda_n\}$ with $\lim_{n\to\infty}\lambda_n = \infty$ so that $\|u_{\lambda_n}\|^{p-q+\sigma}_{L^\infty(\mathbb{R}^n)}\lambda_n^{-1} \leqslant C$ for some $C > 0$. Hence we have $\|u_{\lambda_n}\|_{L^\infty(\mathbb{R}^n)} \leqslant C\lambda_n^{\frac{1}{p-q+\sigma}}$, so

$$\frac{u_{\lambda_n}^{p-q-\epsilon}}{\lambda_n^{1-\frac{\epsilon}{1-q}}} \leqslant C\lambda_n^{\frac{p-q-\epsilon}{p-q+\sigma}-(1-\frac{\epsilon}{1-q})}.$$

For this fixed $\sigma > 0$, we can choose $\epsilon > 0$ small so that $\frac{p-q-\epsilon}{p-q+\sigma} < 1 - \frac{\epsilon}{1-q}$. Therefore $\frac{b}{\lambda_n^{1-\frac{\epsilon}{1-q}}}u_{\lambda_n}^{p-q-\epsilon} \to 0$ uniformly in $\overline{\Omega^+}$. By the same proof for Lemma 4 in [18], the result follows. $\square$

We are ready to prove the first two assertions in Theorem 4.7.

Proof of Theorem 4.7. For the first, since under hypothesis (1.7) the choices for I are finite, we only need to show the theorem is true for u_λ, here u_λ is the minimal element in S_I for each fixed $I \subset M$.

We proceed by contradiction: suppose that for some $\sigma > 0$,

$$\liminf_{\lambda\to\infty} \|u_\lambda\|^{p-q+\sigma}_{L^\infty(\mathbb{R}^n)}\lambda^{-1} \leqslant C_0 < \infty. \tag{4.5}$$

Take ϵ and $\{\lambda_n\}$ as in the proof of the above lemma. There exists $C > 0$, for example $C = C(\Omega_I^+)/2$ from the above lemma, so that for n large

$$\|v_{\lambda_n}\|_{L^\infty(\mathbb{R}^n)} \geqslant \|\overline{v_{\lambda_n}}\|_{L^\infty(\mathbb{R}^n)} \geqslant C.$$

This implies $\|u_{\lambda_n}\|_{L^\infty(\mathbb{R}^n)} \geqslant C\lambda_n^{\frac{1}{1-q}}$, which contradicts (4.5). Therefore, we must have $\liminf_{\lambda\to\infty}\|u_\lambda\|^{p-q+\sigma}_{L^\infty(\mathbb{R}^n)}\lambda^{-1} = \infty$ for any $\sigma > 0$.

For the second part, let us assume $\inf_{x\in\Omega^+}|b(x)| = \bar{b} > 0$. Lemma 2.5 implies that

$$b(x_0)u_\lambda^p(x_0) + a(x_0)u_\lambda^q(x_0) = -\Delta u_\lambda(x_0) \geqslant 0,$$

where u_λ attains its global maximum at $x_0 \in \overline{\Omega^+}$. Hence we have

$$\lambda\|a^+\|_{L^\infty(\mathbb{R}^n)}\|u_\lambda\|^q_{L^\infty(\mathbb{R}^n)} \geqslant \big(-b(x_0)\big)\|u_\lambda\|^p_{L^\infty(\mathbb{R}^n)} \geqslant \bar{b}\|u_\lambda\|^p_{L^\infty(\mathbb{R}^n)},$$

that is $\|u_\lambda\|^{p-q}_{L^\infty(\mathbb{R}^n)} \leqslant \lambda\|a^+\|_{L^\infty(\mathbb{R}^n)}\bar{b}^{-1}$. This proves the second assertion of the theorem.

To prove the third assertion, we require the following lemma from [18]:

Lemma 4.9. *(See [18, Lemma 4].) Assume* (1.7)*, if* $b(x) = 0$ *in a ball* $B \subset \Omega_i^+$ *for some* $i \in M$*, then* $\|\overline{v_\lambda}\|_{L^\infty(B)}$ *has a positive lower bound* $C(B)$ *independent of* λ *for large* λ*.*

The proof of the third statement is then very simple. Take $c = \frac{1}{2}C(B)$, where $C(B)$ is from the above lemma. For large λ,

$$\|v_\lambda\|_{L^\infty(\mathbb{R}^n)} \geqslant \|\overline{v_\lambda}\|_{L^\infty(\mathbb{R}^n)} \geqslant c,$$

which leads to our conclusion.

Finally, we prove the fourth statement of Theorem 4.7 by contradiction. Since Ω^+ has finitely many connected components, we can assume that there exists an increasing sequence $\{\lambda_n\}$ with $\lim_{n\to\infty}\lambda_n=\infty$ so that $u_{\lambda_n}>0$ in Ω_i^+ for some $i\in M$ and $u_{\lambda_n}=0$ in Ω_j^+ for some $j\neq i$. Take $I=\{i\}$, assume u_{λ_n} is the minimal element in S_I, then like the proof for Theorem 1.10, restricting to a subsequence if necessary, we have $v_{\underline{\lambda_n}}\geqslant\overline{v_{\lambda_n}}\to\overline{V}$ in O, which solves $-\Delta w=a^+w^q$ in O, by the above lemma and maximum principle $\overline{V}>0$ in O, which contradicts our assumption. This concludes the proof of Theorem 4.7. □

5. Solutions not in $\mathcal{D}^{1,2}(\mathbb{R}^n)$

In this section we consider the possibility that u is an entire solution to the PDE $-\Delta u=a(x)u^q+b(x)u^p$ in $\mathbb{R}^n$, without requiring u to lie in the finite energy space $\mathcal{D}^{1,2}(\mathbb{R}^n)$. When $u\in\mathcal{D}^{1,2}(\mathbb{R}^n)$ we prove the uniform decay estimates in Lemmas 2.1 and 2.3. Since these estimates are crucial for the proof of compact support (Theorem 1.2), it is useful to have some discussion in this direction.

In general we do expect that there are solutions, which are not in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. For example, consider the following equation

$$\Delta z=c_1z^q+b_1(x)z^p\quad\text{in }\mathbb{R}^n-B(0,r_1),\tag{5.1}$$

where $c_1>0$, $0<q<p$ and $r_1>0$, moreover $b_1(x)=c_2(r-r_1)^{\frac{1+q-2p}{1-q}}r^{-1}$ for $r\geqslant r_1$ and $c_2>0$. We seek the form of solution $z=\theta(r-r_1)^{\frac{2}{1-q}}$ for some $\theta>0$. Plug it into Eq. (5.1) and calculate. It is easy to see that if the following is satisfied

$$c_1=\frac{2}{1-q}\left(\frac{2}{1-q}-1\right)\theta^{1-q}\quad\text{and}\quad c_2=\frac{2(n-1)}{1-q}\theta^{1-p}.$$

Hence $z=\theta(r-r_1)^{\frac{2}{1-q}}$ is a solution of Eq. (5.1). Let us look at the equation

$$-\Delta u=a(x)u^q+b(x)u^p\quad\text{in }\mathbb{R}^n,$$

where $a(x)=-c_1$ in $\mathbb{R}^n-B(0,R_1)$ for some $R_1>0$ and $b(x)\leqslant 0$.

From Theorem 1.2 we find that there exists $\rho>R_1$ such that $\operatorname{supp}(u)\subset B(0,\rho)$ for any compactly supported solution u of the above equation and ρ is independent of b. Now if we assume $r_1=\rho$ and $b(x)=-b_1(x)$ for $|x|\geqslant\rho$, then the above equation has a solution U of the form

$$U=\begin{cases}\theta(r-\rho)^{\frac{2}{1-q}} & \text{for } r\geqslant\rho,\\ u & \text{for } r\leqslant\rho\end{cases}$$

and it is clear that $U\notin\mathcal{D}^{1,2}(\mathbb{R}^n)$.

The first step to prove the compactness of the solution is to show that solution uniformly converges to zero at infinity. By Lemma 2.1, solutions in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ have this kind of property. But without assuming in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, we can still have this vanishing property. Here is a result in radial case.

Theorem 5.1. *Assume $a(x)=a(|x|)$, $b(x)=b(|x|)$ and $\liminf_{|x|\to\infty}a^-(x)=c>0$. Then any smooth radial solution $u(x)=u(|x|)$ has the property that*

$$\lim_{|x|\to\infty}u(x)=0$$

if $u\in L^\infty(\mathbb{R}^n)$ and $\lim_{r\to\infty}u_rr^{-1}=0$.

Proof. First pick $R>0$ big enough so that $a^-(x)\geqslant \frac{c}{2}$ for any $|x|\geqslant R$, then we discuss the behavior of $u(r)$ in the domain $[R,\ \infty)$, we divide into three cases.

Case 1. There exists $r_1\geqslant R$ such that $u_r(r_1)=0$.

First we see that $u(r)$ must attain local minimal value at $r=r_1$. Indeed if $u(r_1)=0$, then we are done! So assume $u(r_1)>0$, then we have

$$u_{rr}(r_1)=u_{rr}(r_1)+(n-1)u_r(r_1)r^{-1}=\Delta u(r_1)=-a(r_1)u^q(r_1)-b(r_1)u^p(r_1)>0.$$

Therefore $u(r)$ attains local minimal value at $r=r_1$.

We now claim that $u(r)=0$ for $r\geqslant r_1$. Otherwise, there would exist $r_2>r_1$ such that $u(r_2)>0$. So we should have that $u_r(r)>0$ for $r>r_2$. If not, there would exist $r_3>r_2$ such that $u_r(r_3)=0$, it is easy to see that $u(r)$ attains local minimal value at $r=r_3$, therefore $u(r)$ achieves local maximal value at some $r_4\in(r_1,r_3)$, but this is impossible because $u_{rr}(r_4)=-a(r_4)u^q(r_4)-b(r_4)u^p(r_4)>0$. So we must have $u_r(r)>0$ for $r>r_2$. Since $u\in L^\infty(\mathbb{R}^n)$ we have $u(r)\uparrow M$ as $r\to\infty$ for some positive constant M, this leads to a sequence $\{r_n\}$ and $\lim_{n\to\infty} r_n=\infty$ such that $\lim_{n\to\infty} u_{rr}(r_n)=0$. But since $\lim_{r\to\infty} u_r r^{-1}=0$, then we should have

$$\liminf_{n\to\infty} u_{rr}(r_n)=\liminf_{n\to\infty}\bigl(-a(r_n)u^q(r_n)-b(r_n)u^p(r_n)\bigr)>0.$$

This is a contradiction! So we have $u(r)=0$ for $r\geqslant r_1$.

Case 2. $u_r(r)>0$ for any $r\in[R,\infty)$.

From the above proof we see that this case is impossible.

Case 3. $u_r(r)<0$ for any $r\in[R,\infty)$.

Let $m=\lim_{r\to\infty} u(r)$, then we must have $m=0$. Otherwise $m>0$, then we can find a sequence $\{r_n\}$ and $\lim_{n\to\infty} r_n=\infty$ such that

$$\lim_{n\to\infty} u_{rr}(r_n)=0.$$

But

$$\liminf_{n\to\infty} u_{rr}(r_n)=\liminf_{n\to\infty}\left(-a(r_n)u^q(r_n)-b(r_n)u^p(r_n)-\frac{n-1}{r_n}u_r(r_n)\right)\geqslant \liminf_{n\to\infty}\bigl(-a(r_n)u^q(r_n)\bigr)>0.$$

This is a contradiction, and hence we must have $\lim_{r\to\infty} u(r)=0$! $\square$

It should be interesting and possible to prove some results in non-radial settings. In particular, Brezis and Kamin prove (Lemma 6 of [8]) that under hypothesis (1.2), if we could show that

$$\frac{1}{|\partial B(0,R)|}\int\limits_{\partial B(0,R)} u\to 0 \quad \text{as } R\to 0,$$

then any smooth solution $u\in L^\infty(\mathbb{R}^n)$ of (1.1) has the property that $\lim_{|x|\to\infty} u(x)=0$. Many open questions remain.

Appendix A

In this section we give a detailed proof of Lemma 2.8. The analogous result for bounded domains appeared in [3] and originated from [23] (see also [8]). Although the proof is almost the same as in [3], we provide a proof here since the lemma is so crucial to our methods.

For our version we will use Serrin's maximum principle for strong solutions in $W^{2,s}_{loc}$ [21]. Hence, we note that for solutions $u \in W^{2,s}_{loc}(\mathbb{R}^n)$, $s > n$, Δu exists a.e. pointwise and coincides with its distributional derivative. In particular, $-\Delta u \geqslant f$ a.e. pointwise is equivalent to $-\Delta u \geqslant f$ in the sense of distributions.

Following [3], we transform our solution u by $U = \frac{1}{1-q}u^{1-q}$, and require also the regularized form $U_\epsilon = \frac{1}{1-q}(u+\epsilon)^{1-q}$. For later use, we note that these transformed functions satisfy:

$$\nabla U_\epsilon = (u+\epsilon)^{-q}\nabla u,$$

$$\Delta U_\epsilon = (-q)(u+\epsilon)^{-q-1}|\nabla u|^2 + (u+\epsilon)^{-q}\Delta u,$$

$$\Delta U_\epsilon = (-q)(u+\epsilon)^{q-1}|\nabla U_\epsilon|^2 - a\frac{u^q}{(u+\epsilon)^q} - b\frac{u^p}{(u+\epsilon)^q},$$

and similarly for U by setting $\epsilon = 0$. We are now ready to prove the comparison result, Lemma 2.8.

Proof of Lemma 2.8. Suppose the contrary, $D := \{x \in \mathbb{R}^n \mid u_1 > u_2\} \neq \emptyset$. Let $U_1 = \frac{1}{1-q}u_1^{1-q}$, $U_2 = \frac{1}{1-q}u_2^{1-q}$, so that

$$U_1 > U_2 \quad \text{in } D.$$

Since $\lim_{|x|\to\infty} u_1(x) = \lim_{|x|\to\infty} u_2(x) = 0$, there exists a point $x_0 \in D$ where the difference $\delta := U_1 - U_2$ attains its maximal value. Let us now distinguish two cases.

Case 1. Suppose that $U_2(x_0) > 0$ for some $x_0 \in D$ where δ takes its maximal value. Denote by V the maximal connected component of the set $D_1 := \{x \in D\colon\ U_2(x) > 0\}$ containing x_0, then δ belongs to $C^2(V)$ and the above calculations show that

$$\Delta\delta = \Delta U_1 - \Delta U_2 \geqslant (-q)u_1^{q-1}|\nabla U_1|^2 + qu_2^{q-1}|\nabla U_2|^2 - b\big(u_1^{p-q} - u_2^{p-q}\big).$$

It is easy to see that from $u_1 > u_2$ in D we have

$$u_2^{q-1} > u_1^{q-1} \quad \text{in } D, \qquad u_1^{p-q} > u_2^{p-q} \quad \text{in } D.$$

So

$$\begin{aligned}
&\Delta\delta + qu_2^{q-1}\big((\nabla U_1 + \nabla U_1), \nabla\delta\big)\\
&\quad\geqslant (-q)u_1^{q-1}|\nabla U_1|^2 + qu_2^{q-1}|\nabla U_2|^2 + qu_2^{q-1}|\nabla U_1|^2 - qu_2^{q-1}|\nabla U_2|^2 - b\big(u_1^{p-q} - u_2^{p-q}\big)\\
&\quad\geqslant \big(qu_2^{q-1} - qu_1^{q-1}\big)|\nabla U_1|^2 - b\big(u_1^{p-q} - u_2^{p-q}\big)\\
&\quad\geqslant 0.
\end{aligned}$$

Since δ assumes its maximal value at an interior point of V, the weak maximum principle of Serrin [21] ensures that $\delta \equiv \mathit{constant}$ in V. It then follows that

$$0 = \nabla\delta = \nabla U_1 - \nabla U_2 \quad \text{and} \quad \Delta\delta \equiv 0 \quad \text{in } V.$$

But by assumption in V we have

$$0 \equiv \Delta\delta = q|\nabla U_1|^2\big(u_2^{q-1} - u_1^{q-1}\big) - b\big(u_1^{p-q} - u_2^{p-q}\big) \geqslant 0,$$

which implies $\nabla U_1 = 0$ in V, in turn we have $\nabla U_2 = 0$ in V, so u_1 and u_2 must be constant in V. But since $\lim_{|x|\to\infty} u_1(x) = \lim_{|x|\to\infty} u_2(x) = 0$, we must have $u_1 \equiv u_2$ in V. This is a contradiction, so Case 1 is impossible.

Case 2. Suppose $U_2(x_0) = 0$ for all x_0 where δ achieves its maximal value. Let

$$C := \big\{x \in D\colon\ \delta(x) = \delta(x_0)\big\}.$$

Note by assumption, $U_2 \equiv 0$ in C. Since $\delta = \delta(x_0) > 0$ in C, we have $U_1 > 0$ in C. On the other hand also by assumption $u_1 \equiv 0$ in $\overline{\Omega^+ - \Omega_I^+}$. Hence we have

$$C \cap \overline{\big(\Omega^+ - \Omega_I^+\big)} = \emptyset.$$

From definition of the set P, $U_2 > 0$ in $\overline{\Omega_I^+}$,

$$C \cap \overline{\Omega_I^+} = \emptyset.$$

So we have

$$C \cap \overline{\Omega^+} = \emptyset,$$

which means that C and $\overline{\Omega^+}$ are at a positive distance to each other. Therefore there exists a neighborhood U of C such that $\overline{U} \cap \overline{\Omega^+} = \emptyset$ and $\delta(x) > 0$ in $\overline{U}$. Then by monotonicity we obtain

$$\min_{\overline{W}}(u_1 - u_2) > 0,$$

where W is a connected component of U.

Thus there exists $b > 0$ such that $\delta(x) \leqslant b < \delta(x_0)$ for $\forall x \in \partial W$. For $\forall \epsilon > 0$, we define

$$U_{2\epsilon} := \frac{1}{1-q}(u_2 + \epsilon)^{1-q}, \qquad \delta_\epsilon := U_1 - U_{2\epsilon}.$$

Clearly $\delta_\epsilon \leqslant \delta$ in D. We can pick positive ϵ small enough such that

$$u_1 > u_2 + \epsilon \quad \text{and} \quad \delta_\epsilon(x_0) > 0.$$

It follows that

$$\delta_\epsilon(x) \leqslant \delta(x) \leqslant b < \delta_\epsilon(x_0) \quad \forall x \in \partial W.$$

Hence δ_ϵ attains its maximum value at some interior point in W and is not constant in $\overline{W}$. On the other hand, from assumption (1.7) we have $a \leqslant 0$ in $\overline{W}$, therefore

$$\begin{aligned}
\Delta\delta_\epsilon &\geqslant (-q)u_1^{q-1}|\nabla U_1|^2 + q(u_2+\epsilon)^{q-1}|\nabla U_{2\epsilon}|^2 + a\frac{u_2^q}{(u_2+\epsilon)^q} - a - b\left(u_1^{p-q} - \frac{u_2^p}{(u_2+\epsilon)^q}\right) \\
&\geqslant (-q)(u_2+\epsilon)^{q-1}\big(|\nabla U_1|^2 - |\nabla U_{2\epsilon}|^2\big) + a\left(\frac{u_2^q}{(u_2+\epsilon)^q} - 1\right) - b\left(u_1^{p-q} - \frac{(u_2+\epsilon)^p}{(u_2+\epsilon)^q}\right) \\
&\geqslant (-q)(u_2+\epsilon)^{q-1}\big(\nabla(U_1 + U_{2,\epsilon}) \cdot \nabla\delta_\epsilon\big)
\end{aligned}$$

in W.

$$-\Delta\delta_\epsilon + (-q)(u_2+\epsilon)^{q-1}\big(\nabla(U_1+U_{2,\epsilon})\cdot\nabla\delta_\epsilon\big) \leqslant 0.$$

But by the weak maximum principle of Serrin [21], δ_ϵ cannot achieve its maximum in W unless it is constant. This is a contradiction, whence the result follows. □

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